

SOLUTIONS MANUAL

T H I R D E D I T I O N

**MECHANICAL
VIBRATIONS**

S I N G I R E S U S . R A O

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P U R D U E U N I V E R S I T Y



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CONTENTS

Chapter 1	Fundamentals of Vibration	1
Chapter 2	Free Vibration of Single Degree of Freedom Systems	39
Chapter 3	Harmonically Excited Vibration	100
Chapter 4	Vibration Under General Forcing Conditions	147
Chapter 5	Two Degree of Freedom Systems	198
Chapter 6	Multidegree of Freedom Systems	249
Chapter 7	Determination of Natural Frequencies and Mode Shapes	335
Chapter 8	Continuous Systems	368
Chapter 9	Vibration Control	426
Chapter 10	Vibration Measurement and Applications	473
Chapter 11	Numerical Integration Methods in Vibration Analysis	489
Chapter 12	Finite Element Method	533
Chapter 13	Nonlinear Vibration	602
Chapter 14	Random Vibration	649

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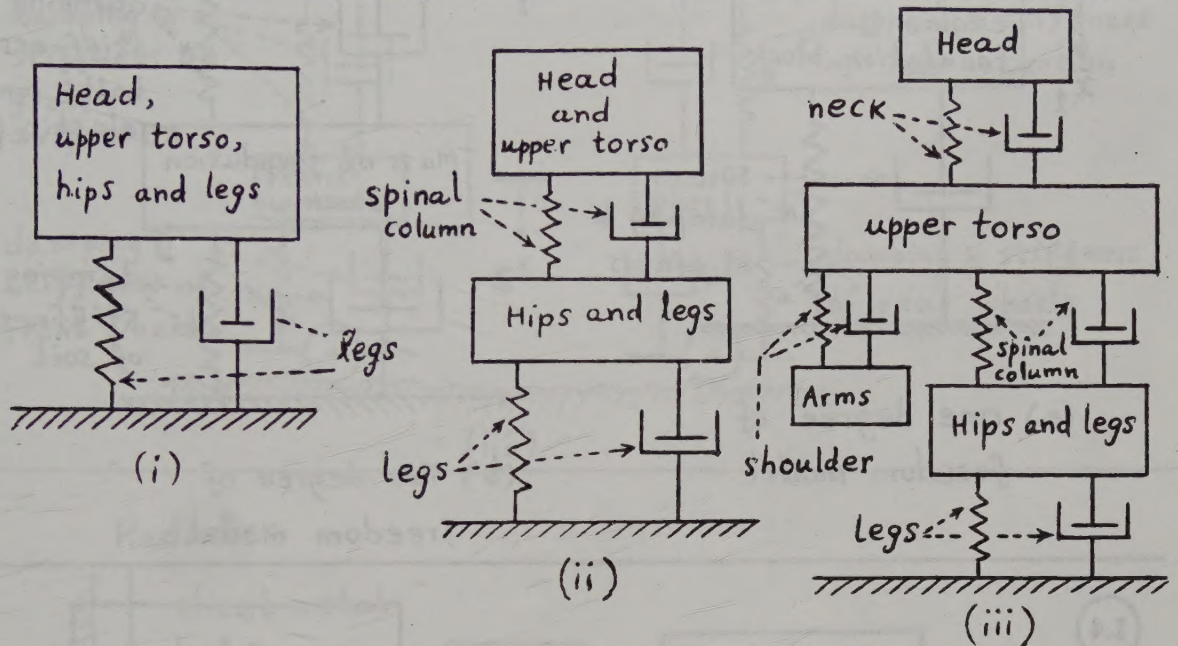
ISBN 0-201-52687-5

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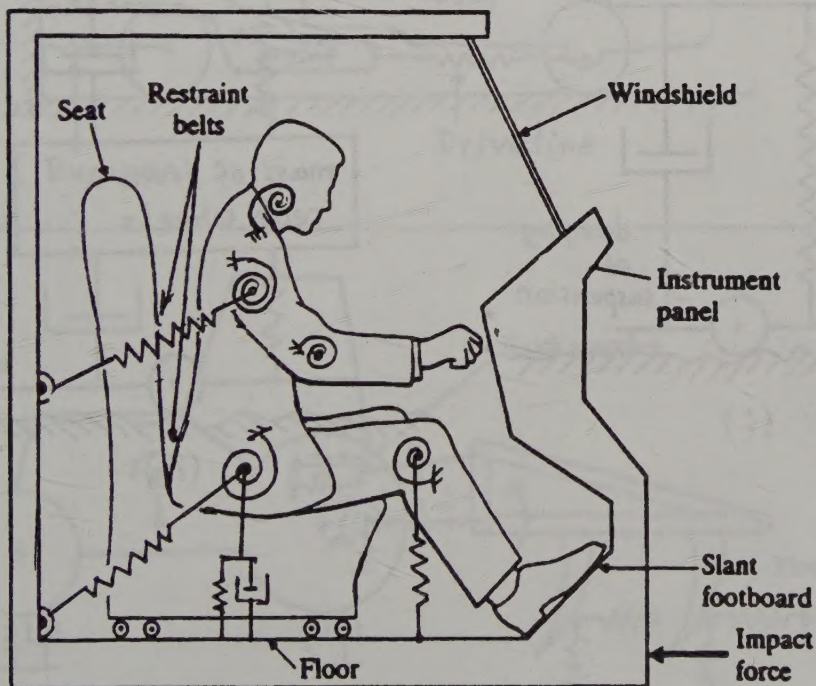
Chapter 1

Fundamentals of Vibration

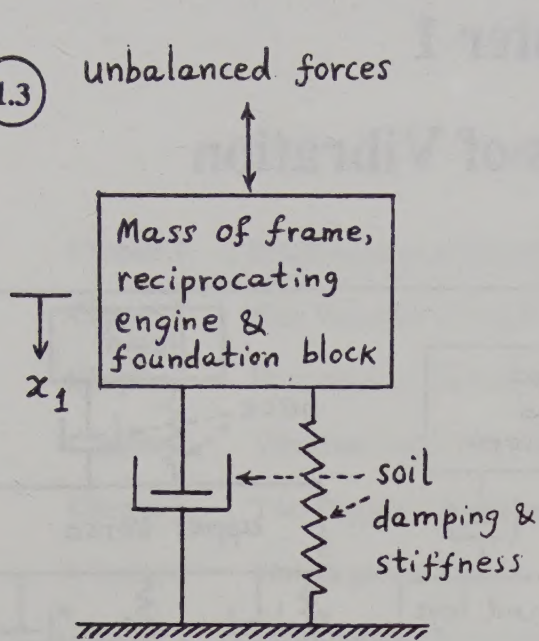
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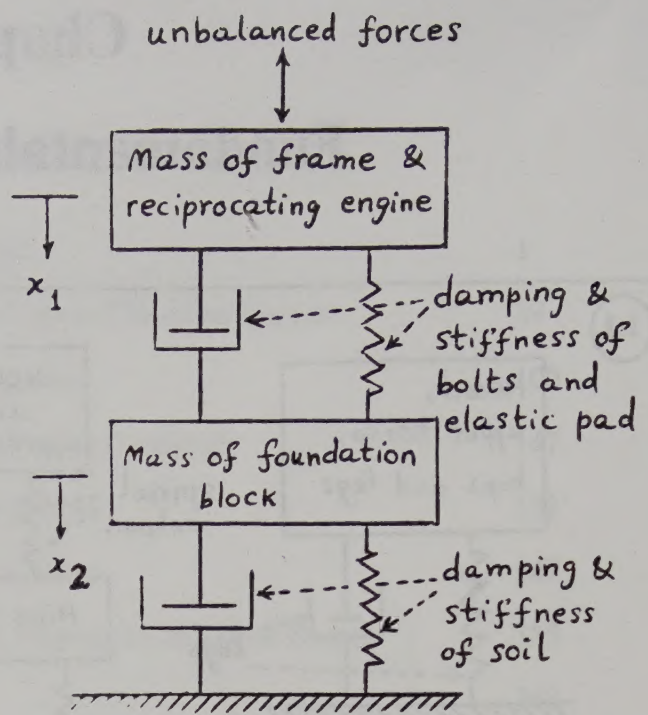
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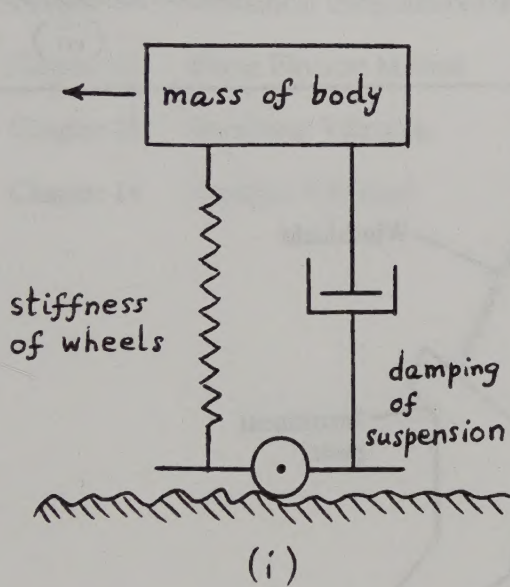


(a) one degree of freedom model

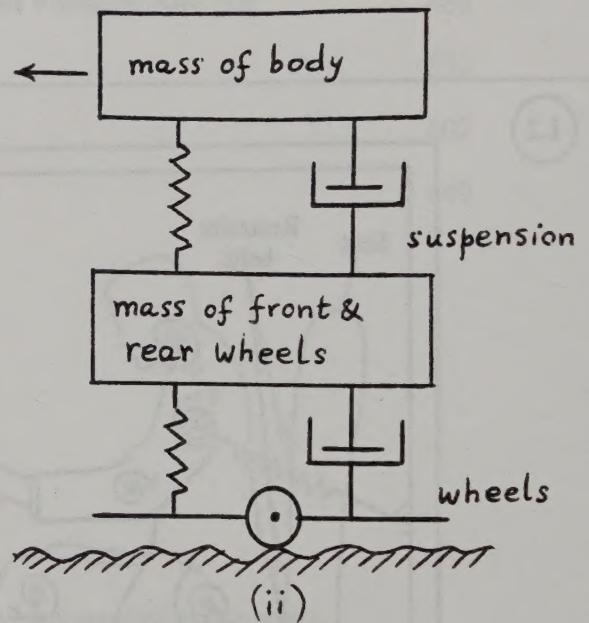


(b) Two degree of freedom model

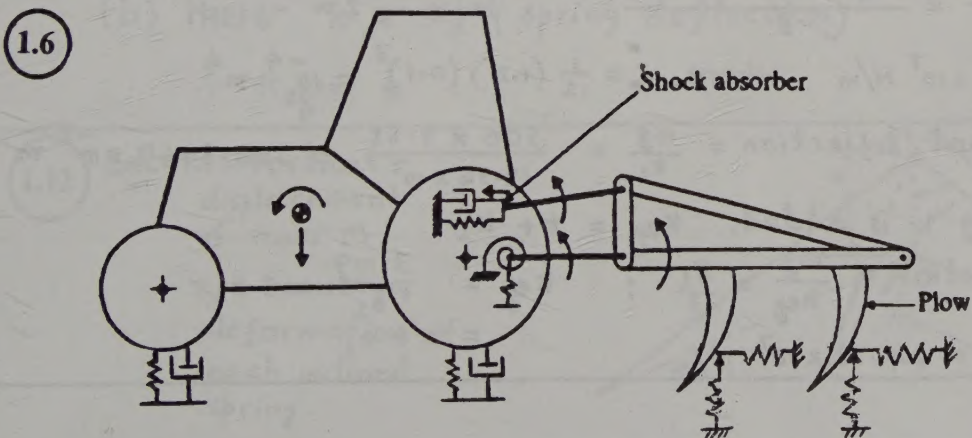
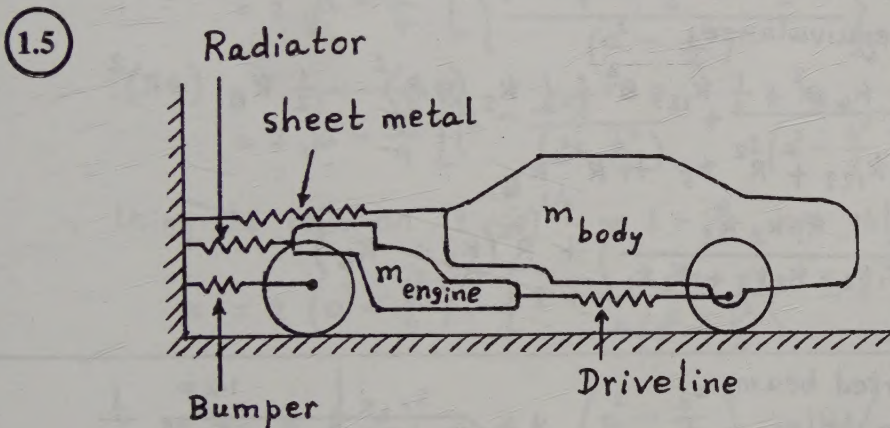
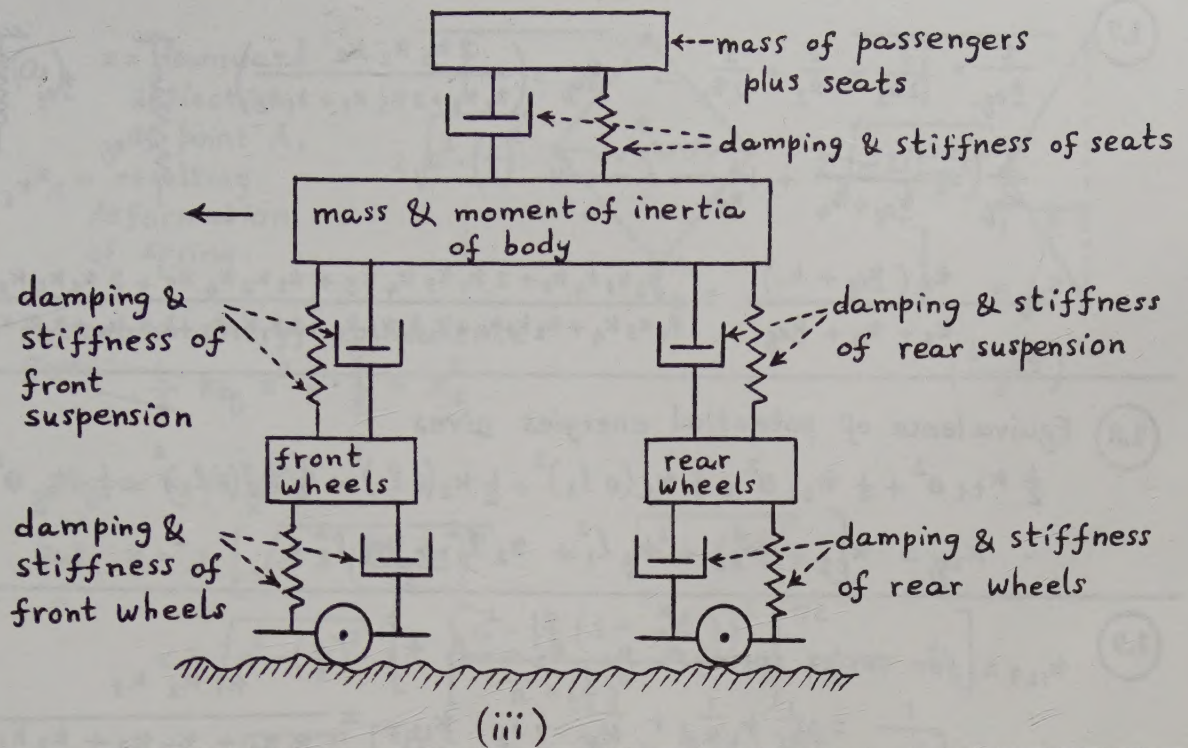
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(i)



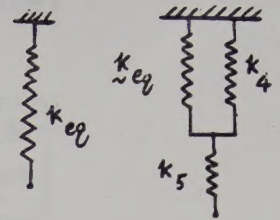
(ii)



1.7

$$\frac{1}{k_{eq}} = \frac{1}{2k_1} + \frac{1}{k_2} + \frac{1}{2k_3} ; \quad k_{eq} = \left(\frac{2k_1 k_2 k_3}{k_2 k_3 + 2k_1 k_3 + k_1 k_2} \right)$$

$$\frac{1}{k_{eq}} = \frac{1}{k_{eq} + k_4} + \frac{1}{k_5}$$



$$k_{eq} = \frac{k_5 (k_{eq} + k_4)}{k_5 + k_4 + k_{eq}} = \frac{k_2 k_3 k_4 k_5 + 2k_1 k_3 k_4 k_5 + k_1 k_2 k_4 k_5 + 2k_1 k_2 k_3 k_5}{k_2 k_3 k_4 + k_2 k_3 k_5 + 2k_1 k_3 k_4 + 2k_1 k_3 k_5 + k_1 k_2 k_4 + k_1 k_2 k_5 + 2k_1 k_2 k_3}$$

1.8

Equivalence of potential energies gives

$$\frac{1}{2} k_{t1} \theta^2 + \frac{1}{2} k_{t2} \theta^2 + \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_1)^2 + \frac{1}{2} k_3 (\theta l_2)^2 = \frac{1}{2} k_{eq} \theta^2$$

$$\therefore k_{eq} = k_{t1} + k_{t2} + k_1 l_1^2 + k_2 l_1^2 + k_3 l_2^2$$

1.9

k_{123} = for series springs k_1, k_2 and k_3 :

$$\frac{1}{k_{123}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} ; \quad k_{123} = \frac{k_1 k_2 k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1}$$

Using energy equivalence,

$$\frac{1}{2} k_{eq} \theta^2 = \frac{1}{2} k_4 \theta^2 + \frac{1}{2} k_{123} \theta^2 + \frac{1}{2} k_5 (\theta R)^2 + \frac{1}{2} k_6 (\theta R)^2$$

$$\therefore k_{eq} = k_4 + k_{123} + R^2 k_5 + R^2 k_6$$

$$= k_4 + \left(\frac{k_1 k_2 k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1} \right) + R^2 (k_5 + k_6)$$

1.10

For simply supported beam,
for load at middle,

$$k_1 = \frac{48 EI}{l^3} = \frac{48 (2.06 \times 10^{11}) (10^{-4})}{8}$$

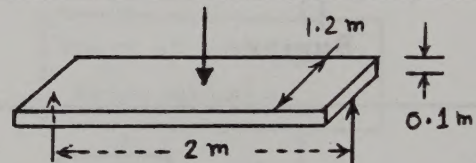
$$= 12.36 \times 10^7 \text{ N/m} \quad \text{where } I = \frac{1}{12} (1.2) (0.1)^3 = 10^{-4} \text{ m}^4$$

$$\delta_1 = \text{original deflection} = \frac{mg}{k_1} = \frac{500 \times 9.81}{12.36 \times 10^7} = 396.8447 \times 10^{-7} \text{ m}$$

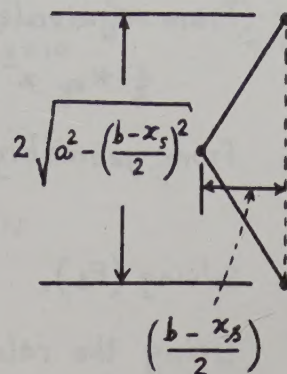
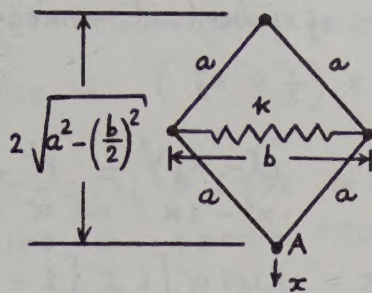
When spring k is added, $k_{eq} = k + k_1$

$$\text{New deflection} = \frac{mg}{k_{eq}} = \frac{\delta_1}{3} ; \quad k_{eq} = \frac{3 mg}{\delta_1} = 3 k_1$$

$$k = 2 k_1 = 24.72 \times 10^7 \text{ N/m}$$



- 1.11 (a) x = downward deflection of point A,
 x_s = resulting deformation of spring



Potential energy equivalence gives $\frac{1}{2} k_{eq} x^2 = \frac{1}{2} k x_s^2$

$$k_{eq} = k \left(\frac{x_s}{x} \right)^2$$

$$\begin{aligned} \text{But } x &= 2 \left[\sqrt{a^2 - \left(\frac{b - x_s}{2} \right)^2} - \sqrt{a^2 - \left(\frac{b}{2} \right)^2} \right] \\ &= 2 \sqrt{a^2 - \left(\frac{b}{2} \right)^2} \left[\left\{ \frac{a^2 - \left\{ \frac{b}{2} \left(1 - \frac{x_s}{b} \right) \right\}^2}{a^2 - \left(\frac{b}{2} \right)^2} \right\}^{1/2} - 1 \right] \\ &= 2 \sqrt{a^2 - \frac{b^2}{4}} \left[\left\{ \frac{\left(a^2 - \frac{b^2}{4} - \frac{x_s^2}{4} + \frac{b x_s}{2} \right)}{\left(a^2 - \frac{b^2}{4} \right)} \right\}^{1/2} - 1 \right] \\ &= 2 \sqrt{a^2 - \frac{b^2}{4}} \left[\left\{ 1 - \frac{x_s^2}{4 \left(a^2 - \frac{b^2}{4} \right)} + \frac{b x_s}{2 \left(a^2 - \frac{b^2}{4} \right)} \right\}^{1/2} - 1 \right] \end{aligned}$$

Using the relation $(1 + \theta)^{1/2} \approx 1 + \frac{\theta}{2}$, we obtain

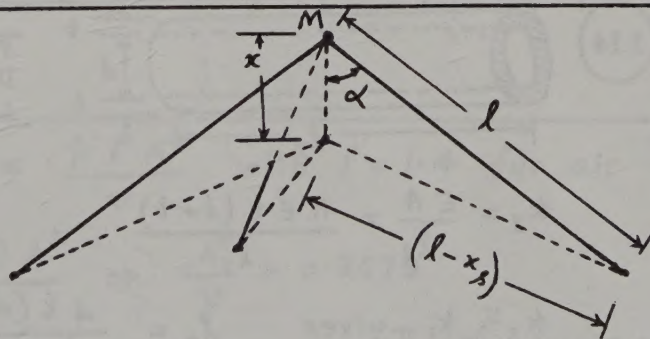
$$x = 2 \left(a^2 - \frac{b^2}{4} \right)^{1/2} \left[1 + \frac{b x_s}{4 \left(a^2 - \frac{b^2}{4} \right)} - 1 \right] = \frac{b x_s}{2 \left(a^2 - \frac{b^2}{4} \right)^{1/2}}$$

$$\therefore k_{eq} = k \left(\frac{x_s}{x} \right)^2 = 4k \left(\frac{a^2 - \frac{b^2}{4}}{b^2} \right) = k \left(\frac{4a^2 - b^2}{b^2} \right)$$

(b) Here $x = x_s$ (spring deflection)

$$\therefore k_{eq} = k$$

- 1.12 Let x = vertical displacement of mass M;
 x_s = resulting deformation of each inclined spring.



From equivalence of potential energy,

$$\frac{1}{2} k_{eq} x^2 = 3 \left(\frac{1}{2} k x_s^2 \right) ; \quad k_{eq} = 3 k \left(\frac{x_s}{x} \right)^2$$

From geometry,

$$(l - x_s)^2 = l^2 + x^2 - 2lx \cos \alpha$$

$$x^2 - 2x l \cos \alpha + 2l x_s - x_s^2 = 0 \quad (E_1)$$

$$\text{Solving } (E_1), \quad x = l \cos \alpha \left[1 \pm \left\{ 1 - \frac{(2l x_s - x_s^2)}{l^2 \cos^2 \alpha} \right\}^{1/2} \right] \quad (E_2)$$

Using the relation $\sqrt{1-\theta} \approx 1 - \frac{\theta}{2}$, (E_2) can be rewritten as

$$x = l \cos \alpha \left[1 \pm \left\{ 1 - \left(\frac{2l x_s - x_s^2}{2l^2 \cos^2 \alpha} \right) \right\} \right] \quad (E_3)$$

Assuming x to be small, we use minus sign and neglect x_s^2 compared to $2l x_s$ in (E_3) . This gives

$$x = \frac{x_s}{\cos \alpha}$$

$$\therefore k_{eq} = 3 k \cos^2 \alpha$$

In a similar manner, $c_{eq} = 3 c \cos^2 \alpha$

1.13 $k_{23} = \frac{k_2 k_3}{k_2 + k_3}$

$$k_4 = A \rho g = \frac{\pi d^2}{4} \rho g$$

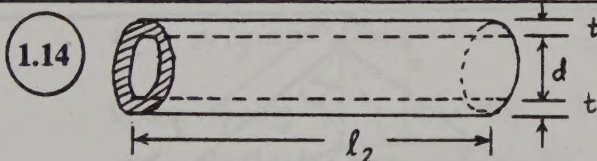
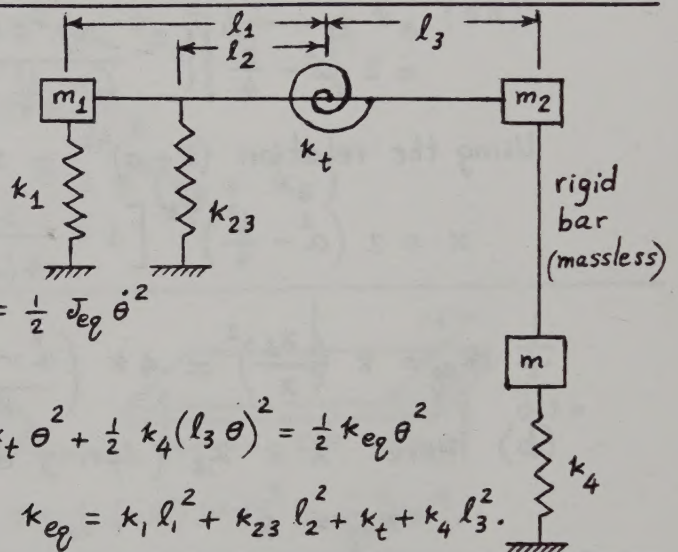
From kinetic energy,

$$\frac{1}{2} m_1 (l_1 \dot{\theta})^2 + \frac{1}{2} (m_2 + m) (l_3 \dot{\theta})^2 = \frac{1}{2} J_{eq} \dot{\theta}^2$$

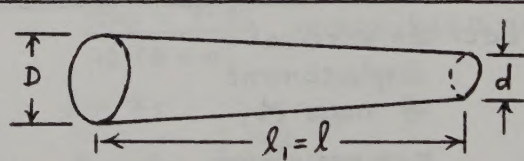
From potential energy,

$$\frac{1}{2} k_1 (l_1 \theta)^2 + \frac{1}{2} k_{23} (l_2 \theta)^2 + \frac{1}{2} k_t \theta^2 + \frac{1}{2} k_4 (l_3 \theta)^2 = \frac{1}{2} k_{eq} \theta^2$$

$$\therefore J_{eq} = m_1 l_1^2 + (m_2 + m) l_3^2 ; \quad k_{eq} = k_1 l_1^2 + k_{23} l_2^2 + k_t + k_4 l_3^2$$



$$k_2 = \frac{EA}{l_2} = \frac{\pi E t (d+t)}{l_2}$$



$$k_1 = \frac{\pi E D d}{4l}$$

$$k_2 = k_1 \text{ gives } l_2 = \frac{4 t (d+t)}{D d}$$

$$\textcircled{1.15} \quad (a) \quad F \approx F|_{x_0} + \left. \frac{dF}{dx} \right|_{x_0} (x - x_0) = \left(500x + 2x^3 \right)_{x=10} + (500 + 6x^2)_{x=10} \cdot (x - 10) \\ \approx 1100x - 4000$$

(b) at $x = 9 \text{ mm}$:

$$\text{Exact } F_9 = 500 \times 9 + 2(9)^3 = 5958 \text{ N}$$

$$\text{Approximate } F_9 = 1100 \times 9 - 4000 = 5900 \text{ N}$$

$$\text{Error} = -0.9735\%$$

(c) at $x = 11 \text{ mm}$:

$$\text{Exact } F_{11} = 500 \times 11 + 2(11)^3 = 8162 \text{ N}$$

$$\text{Approximate } F_{11} = 1100 \times 11 - 4000 = 8100 \text{ N}$$

$$\text{Error} = +0.7596\%$$

$$\textcircled{1.16} \quad p v^{\gamma} = \text{constant} \quad \dots (E_1) \quad ; \quad \text{Differentiation of } (E_1) \text{ gives} \\ dp v^{\gamma} + \gamma p v^{\gamma-1} dv = 0$$

$$dp = - \frac{\gamma p}{v} dv \quad \dots (E_2)$$

change in volume when mass moves by dx , $dv = -A \cdot dx \quad \dots (E_3)$

$$\text{Eqs. } (E_2) \text{ and } (E_3) \text{ give } dp = \frac{\gamma p A}{v} dx$$

$$\text{Force due to pressure change} = dF = dp \cdot A = \frac{\gamma p A^2}{v} dx$$

$$\text{spring constant of air spring} = k = \frac{dF}{dx} = \left(\frac{\gamma p A^2}{v} \right).$$

$\textcircled{1.17}$ Equivalent spring constants in different directions are

$$k_{e1} = \left(\frac{k_5 k_6 k_7}{k_5 k_6 + k_5 k_7 + k_6 k_7} \right), \quad k_{e2} = \left(\frac{k_8 k_9}{k_8 + k_9} \right),$$

$$k_{e3} = \left(\frac{k_1 k_2}{k_1 + k_2} \right), \quad k_{e4} = \left(\frac{k_3 k_4}{k_3 + k_4} \right)$$

If the force P moves by x , spring located at θ_i undergoes a displacement of $x_i = x \cos \theta_i$ (derivation as in problem 1.6)

$$\text{Equivalence of potential energy gives } \frac{1}{2} k_{eq} x^2 = \frac{1}{2} \sum_{i=1}^4 k_{ei} x_i^2$$

$$k_{eq} = \sum_{i=1}^4 (k_{ei} \cos^2 \theta_i)$$

$\textcircled{1.18}$ From Problem 1.16, $k = \frac{\gamma p A^2}{v}$ with $\gamma = 1.4$ for air
Let $p = 200 \text{ psi}$

$$k = 75 \text{ lb/in} = \frac{(200)(1.4) A^2}{v} \Rightarrow \frac{A^2}{v} = 0.2679$$

$$\text{Let diameter of piston} = d = 2 \text{ inch}; \quad A = \frac{\pi}{4} (2)^2 = 3.1416 \text{ in}^2$$

$$v = A^2 / 0.2679 = 36.8408 \text{ in}^3$$

$$\text{Let } h = 2 \text{ inch ; } \frac{\pi}{4} D^2 (2) = v \Rightarrow D = 4.8429 \text{ inch}$$

1.19 $F = a x + b x^3 = 2 (10^4) x + 4 (10^7) x^3$

$$\text{Around } x^* : F(x) \approx F(x^*) + \left. \frac{dF}{dx} \right|_{x^*} (x - x^*)$$

$$\text{When } x^* = 10^{-2} \text{ m, } F(x^*) = 2 (10^4) (10^{-2}) + 4 (10^7) (10^{-6}) = 240 \text{ N}$$

$$\left. \frac{dF}{dx} \right|_{x^*} = a + 3 b x^2 = 2 (10^4) + 3 (4) (10^7) (10^{-4}) = 32000$$

$$\text{Hence } F(x) = 240 + 32000 (x - 0.01) = (32000 x - 80) \text{ N}$$

Since the linearized spring constant is given by $F(x) = k_{eq} x$, we have $k_{eq} = 32,000 \text{ N/m}$.

1.20 $F_i = a_i x_i + b_i x_i^3 ; i = 1, 2$

Springs in series:

$$W = a_1 \delta_1 + b_1 \delta_1^3 \quad (1)$$

$$W = a_2 \delta_2 + b_2 \delta_2^3 \quad (2)$$

$$W = k_{eq} \delta_{st} \quad (3)$$

$$\delta_{st} = \delta_1 + \delta_2 \quad (4)$$

Solve Eqs. (1) and (2) for δ_1 and δ_2 , respectively. Substitute the result in Eq. (4) and then in Eq. (3) to find k_{eq} .

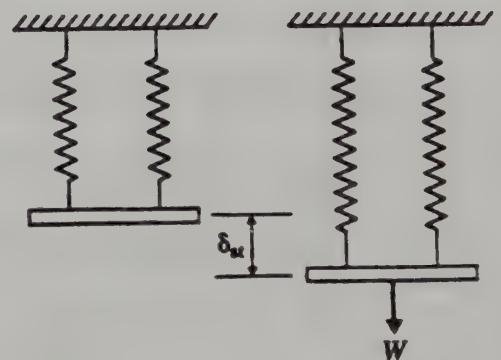
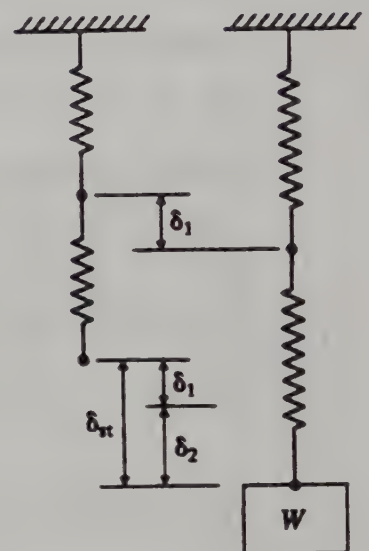
Springs in parallel:

$$W = F_1 + F_2$$

$$= a_1 \delta_{st} + b_1 \delta_{st}^3 + a_2 \delta_{st} + b_2 \delta_{st}^3$$

$$= k_{eq} \delta_{st}$$

$$k_{eq} = a_1 + b_1 \delta_{st}^2 + a_2 + b_2 \delta_{st}^2$$



$$(1.21) \quad k = \frac{G d^4}{8 D^3 N} \geq 8 \times 10^6 \text{ N/m} ; \quad \frac{D}{d} \geq 6 ; \quad N \geq 10$$

$$W = \pi D N \rho \left(\frac{\pi d^2}{4} \right) \quad \text{where } \rho = \text{weight per unit volume}$$

$$f_1 = \frac{1}{2} \sqrt{\frac{k g}{W}} = \frac{1}{2} \sqrt{\frac{G d^2 g}{2 \pi^2 D^4 N^2 \rho}} \geq 0.4 \text{ Hz}$$

Using $G = 73.1 \times 10^9 \text{ N/m}^2$, $\rho = 76000 \text{ N/m}^3$, $g = 9.81 \text{ m/sec}^2$,
 $\frac{D}{d} = 6, 8, 10$; $N = 10, 15, 20$; $d = 0.4, 0.6, \dots$, values of
 k and f_1 are computed.

Combination of $\frac{D}{d} = 6$, $N = 10$ and $d = 2.0 \text{ m}$, corresponding
to $k = 8.4606 \times 10^6 \text{ N/m}$ and $f_1 = 0.4801 \text{ Hz}$, can be
taken as an acceptable design.

(1.22) Total elongation (strain) is same in each material:

$$\epsilon_s = \epsilon_a = \frac{x}{\ell} \quad (1)$$

where x is the total elongation. Equation (1) can be expressed as

$$\frac{\sigma_s}{E_s} = \frac{\sigma_a}{E_a} = \frac{x}{\ell} \quad (2)$$

$$\text{or } \sigma_s = \frac{E_s x}{\ell} \quad (3)$$

$$\sigma_a = \frac{E_a x}{\ell} \quad (4)$$

Total axial force is:

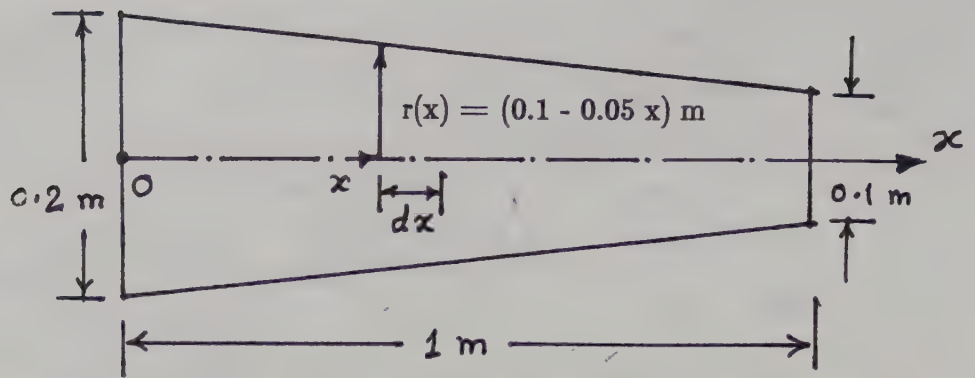
$$F = F_s + F_a = \sigma_s A_s + \sigma_a A_a \quad (5)$$

where F_s and F_a denote the axial forces acting on steel and aluminum, respectively, and A_s and A_a represent the cross-sectional areas of the two materials. Equating F to $k_{eq} x$ where k_{eq} denotes the equivalent spring constant of the bimetallic bar, we obtain from Eqs. (3) to (5):

$$F = k_{eq} x = \left(\frac{E_s x}{\ell} \right) A_s + \left(\frac{E_a x}{\ell} \right) A_a$$

$$\text{or } k_{eq} = \frac{E_s A_s}{\ell} + \frac{E_a A_a}{\ell} \quad (6)$$

1.23



$$J = \frac{\pi}{2} r^4 = \text{area polar moment of inertia at section } x = 1.5708 (0.1 - 0.05x)^4 \text{ m}^4$$

Knowing that the angle of twist, θ , between the ends of a uniform shaft of length ℓ under a torque T is given by $\theta = \frac{T \ell}{GJ}$, the angle of twist for an element of length dx can be expressed as

$$d\theta = \frac{T dx}{GJ} = \frac{T dx}{(80 (10^9)) 1.5708 (0.1 - 0.05x)^4} \quad (1)$$

The total angle of twist can be determined by integrating Eq. (1) from $x=0$ to 1 as:

$$\theta = \int_0^1 \frac{T dx}{(12.5664 (10^{10})) (0.1 - 0.05x)^4} = \left[\frac{T}{12.5664 (10^{10})} \right] \int_0^1 \frac{dx}{(0.1 - 0.05x)^4} \quad (2)$$

$$\begin{aligned} \text{But } \int_0^1 \frac{dx}{(0.1 - 0.05x)^4} &= -\frac{1}{0.05} \int_0^1 \frac{(-0.05 dx)}{(0.1 - 0.05x)^4} = -20 \int_0^{-0.05} \frac{dy}{(0.1 + y)^4} \\ &= 4.6667 (10^4) \text{ where } y = -0.05x \end{aligned}$$

$$\text{Hence } \theta = \frac{T (4.6667) (10^4)}{12.5664 (10^{10})} = T (0.3714 (10^{-6})) \text{ rad}$$

$$\text{This gives } k_t = \frac{T}{\theta} = 2.6925 (10^6) \text{ N-m/rad}$$

1.24

The steel and aluminum hollow shafts can be treated as two torsional springs in parallel. For a hollow shaft,

$$k_{eq} = \frac{\pi G}{32 \ell} (D^4 - d^4)$$

For the steel shaft, $G = 80 (10^9) \text{ Pa}$, $\ell = 5 \text{ m}$, $D = 0.25 \text{ m}$, $d = 0.15 \text{ m}$, and hence

$$k_{t_1} = \frac{\pi (8 (10^{10}))}{32 (5)} (0.25^4 - 0.15^4) = 5.34072 (10^6) \text{ N-m/rad}$$

For the aluminum shaft, $G = 26 (10^9) \text{ Pa}$, $\ell = 5 \text{ m}$, $D = 0.15 \text{ m}$, $d = 0.1 \text{ m}$, and hence

$$k_{t_2} = \frac{\pi (26 (10^9))}{32 (5)} (0.15^4 - 0.10^4) = 0.207395 (10^6) \text{ N-m/rad}$$

$$k_{eq} = k_{t_1} + k_{t_2} = 5.34072 (10^6) + 0.20739 (10^6) = 5.54811 (10^6) \text{ N-m/rad}$$

1.25 For helical spring: $k = \frac{G d^4}{64 n R^3}$

Spring 1: $k_1 = \frac{(12 \times 10^6)(2^4)}{64 (10)(6^3)} = 1,388.89 \text{ lb/in}$

Spring 2: $k_2 = \frac{(4 \times 10^6)(1^4)}{64 (10)(5^3)} = 50.00 \text{ lb/in}$

(a) Spring 2 inside spring 1 (parallel): $k_{eq} = k_1 + k_2 = 1,438.89 \text{ lb/in}$

(b) Spring 2 on top of spring 1 (series):

$$\frac{1}{k_{eq}} = \frac{1}{k_1} = \frac{1}{k_2} = \frac{k_2 + k_1}{k_1 k_2}$$

which gives $k_{eq} = 48.2625 \text{ lb/in}$.

1.26 Assume small angles θ_1 and θ_2 ; $\theta_2 = \left(\frac{p_1}{p_2}\right) \theta_1$

x_1 = horizontal displacement of C.G. of mass $m_1 = \theta_1 r_1$

x_2 = vertical displacement of C.G. of mass $m_2 = \theta_2 r_2 = p_1 \theta_1 r_2 / p_2$

y_1 = horizontal displacement of springs k_1 and $k_2 = \theta_1 (r_1 + l_1)$

y_2 = vertical displacement of springs k_3 and $k_4 = \theta_2 l_2 = p_1 l_2 \theta_1 / p_2$

Equivalence of kinetic energies gives

$$\frac{1}{2} J_{eq} (\dot{\theta}_1)^2 = \frac{1}{2} J_1 (\dot{\theta}_1)^2 + \frac{1}{2} J_2 (\dot{\theta}_2)^2 + \frac{1}{2} m_1 (\dot{x}_1)^2 + \frac{1}{2} m_2 (\dot{x}_2)^2$$

$$\therefore J_{eq} = J_1 + J_2 \left(\frac{p_1}{p_2}\right)^2 + m_1 r_1^2 + m_2 r_2^2 \left(\frac{p_1}{p_2}\right)^2$$

Equivalence of potential energies gives

$$\frac{1}{2} k_{eq} \theta_1^2 = \frac{1}{2} k_{12} y_1^2 + \frac{1}{2} k_{34} y_2^2 + \frac{1}{2} k_{t1} \theta_1^2 + \frac{1}{2} k_{t2} \theta_2^2$$

$$\text{with } k_{12} = k_1 + k_2, \quad k_{34} = k_3 k_4 / (k_3 + k_4)$$

$$y_1 = \theta_1 (r_1 + l_1), \quad y_2 = p_1 l_2 \theta_1 / p_2 \text{ and } \theta_2 = p_1 \theta_1 / p_2.$$

$$\therefore k_{eq} = (k_1 + k_2) (r_1 + l_1)^2 + \left(\frac{k_3 k_4}{k_3 + k_4}\right) \frac{p_1^2 l_2^2}{p_2^2} + k_{t1} + k_{t2} \frac{p_1^2}{p_2^2}.$$

1.27 $\theta = \frac{x}{b}, \quad x_1 = \frac{x \omega}{b}$

From equivalence of kinetic energies,

$$\frac{1}{2} m_{eq} \dot{x}^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

$$m_{eq} = m_1 \left(\frac{a}{b}\right)^2 + m_2 + J_0 \left(\frac{1}{b}\right)^2$$

1.28

Let $\dot{\theta}_i$ = angular velocity of the motor (input)

Angular velocities of different gear sets are:

J_{motor}, J_1	J_2, J_3	J_4, J_5	$\dots$	J_{2N}, J_{load}
$\dot{\theta}_i$	$\dot{\theta}_i \left(\frac{n_1}{n_2} \right)$	$\dot{\theta}_i \left(\frac{n_1}{n_2} \frac{n_3}{n_4} \right)$		$\dot{\theta}_i \left(\frac{n_1}{n_2} \frac{n_3}{n_4} \dots \frac{n_{2N-1}}{n_{2N}} \right)$

Equivalence of kinetic energies gives

$$\frac{1}{2} J_{\text{eq}} \dot{\theta}_i^2 = \frac{1}{2} J_{\text{motor}} \dot{\theta}_i^2 + \frac{1}{2} \sum_{k=1}^{2N} J_k \dot{\theta}_k^2 + \frac{1}{2} J_{\text{load}} \dot{\theta}_{\text{load}}^2$$

$$\therefore J_{\text{eq}} = (J_{\text{motor}} + J_1) + (J_2 + J_3) \left(\frac{n_1}{n_2} \right)^2 + (J_4 + J_5) \left(\frac{n_1}{n_2} \frac{n_3}{n_4} \right)^2 + \dots + (J_{2N} + J_{\text{load}}) \left(\frac{n_1}{n_2} \frac{n_3}{n_4} \dots \frac{n_{2N-1}}{n_{2N}} \right)^2$$

1.29

Equivalence of kinetic energies gives

$$\frac{1}{2} J_{\text{eq}} \dot{\theta}_1^2 = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2 \quad \text{where} \quad \dot{\theta}_2 = \dot{\theta}_1 \left(\frac{n_1}{n_2} \right)$$

$$J_{\text{eq}} = J_1 + J_2 \left(\frac{n_1}{n_2} \right)^2$$

1.30

When point A moves by distance $x = x_h$, the walking beam rotates by the angle

$$\theta_b = \frac{x_h}{\ell_3}$$

This corresponds to a linear motion of point B: $x_B = \theta_b \ell_2 = \frac{x_h \ell_2}{\ell_3}$

and the angular rotation of crank can be found from the relation:

$$x_B = r_c \sin \theta_c + \ell_4 \cos \phi = r_c \sin \theta_c + \ell_4 \sqrt{1 - \frac{r_c^2}{\ell_4^2} \sin^2 \theta_c}$$

For large values of ℓ_4 compared to r_c and for small values of x and θ_c , we have

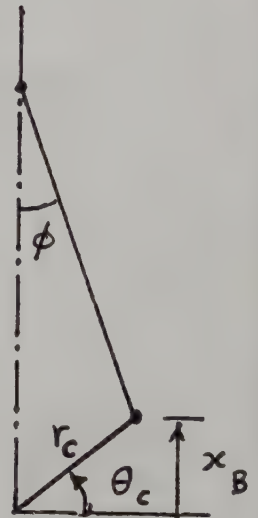
$$x_B \approx r_c \sin \theta_c = r_c \theta_c \quad \text{or} \quad \theta_c = \frac{x_B}{r_c} = \frac{x_h \ell_2}{\ell_3 r_c}$$

The kinetic energy of the system can be expressed as

$$T = \frac{1}{2} m_h \dot{x}_h^2 + \frac{1}{2} J_b \dot{\theta}_b^2 + \frac{1}{2} J_c \dot{\theta}_c^2$$

Equating this to $T = \frac{1}{2} m_{\text{eq}} \dot{x}^2 = \frac{1}{2} m_{\text{eq}} \dot{x}_h^2$, we obtain

$$m_{\text{eq}} = m_h + \frac{J_b}{\ell_3^2} + J_c \left(\frac{\ell_2}{\ell_3 r_c} \right)^2$$



1.31

When mass m is displaced by x , the bell crank lever rotates by the angle $\theta_b = \frac{x}{\ell_1}$. Thismakes the center of the sphere displace by $x_s = \theta_b \ell_2$. Since the sphere rotates with out slip, it rotates by an angle

$$\theta_s = \frac{x_s}{r_s} = \frac{\theta_b \ell_2}{r_s} = \frac{x \ell_2}{\ell_1 r_s}$$

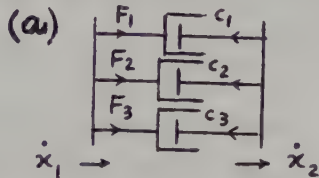
The kinetic energy of the system can be expressed as

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} J_s \dot{\theta}_s^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \left(\frac{\dot{x}}{\ell_1} \right)^2 + \frac{1}{2} \left(\frac{2}{5} m_s r_s^2 \right) \dot{x}^2 \left(\frac{\ell_2}{\ell_1 r_s} \right)^2$$

since for a sphere, $J_s = \frac{2}{5} m_s r_s^2$. Equating this to $T = \frac{1}{2} m_{eq} \dot{x}^2$, we obtain

$$m_{eq} = m + J_0 \frac{1}{\ell_1^2} + \frac{2}{5} m_s \frac{\ell_2^2}{\ell_1^2}$$

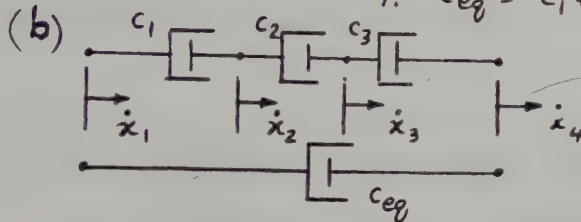
1.32



$F_i =$ damping force of $c_i = c_i (\dot{x}_2 - \dot{x}_1)$; $i=1,2,3$

$F_{eq} =$ damping force of $c_{eq} = c_{eq} (\dot{x}_2 - \dot{x}_1)$
 $\equiv F_1 + F_2 + F_3$

$$\therefore c_{eq} = c_1 + c_2 + c_3$$



$$F_1 = c_1 (\dot{x}_2 - \dot{x}_1)$$

$$F_2 = c_2 (\dot{x}_3 - \dot{x}_2)$$

$$F_3 = c_3 (\dot{x}_4 - \dot{x}_3)$$

$$\dot{x}_4 - \dot{x}_1 = \dot{x}_4 - \dot{x}_3 + \dot{x}_3 - \dot{x}_2 + \dot{x}_2 - \dot{x}_1$$

$$\frac{F_{eq}}{c_{eq}} = \frac{F_3}{c_3} + \frac{F_2}{c_2} + \frac{F_1}{c_1}$$

$$\text{Since } F_{eq} = F_1 = F_2 = F_3, \quad \frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}$$

(c) Equating the energies dissipated in a cycle,

$$\pi c_{eq} \omega x_1^2 = \pi c_1 \omega x_1^2 + \pi c_2 \omega x_2^2 + \pi c_3 \omega x_3^2$$

$$\text{where } x_1 = \theta l_1, \quad x_2 = \theta l_2 \text{ and } x_3 = \theta l_3$$

$$\therefore c_{eq} = c_1 + c_2 \left(\frac{l_2}{l_1} \right)^2 + c_3 \left(\frac{l_3}{l_1} \right)^2$$

(d) Equating the energies dissipated in a cycle,

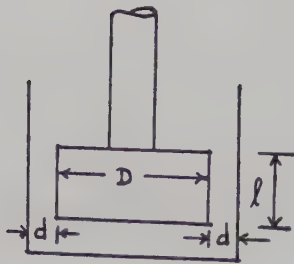
$$\pi c_{teq} \omega \theta_1^2 = \pi c_{t1} \omega \theta_1^2 + \pi c_{t2} \omega \theta_2^2 + \pi c_{t3} \omega \theta_3^2$$

$$\text{where } \theta_2 = \theta_1 \left(\frac{n_1}{n_2} \right) \text{ and } \theta_3 = \theta_1 \left(\frac{n_1}{n_3} \right).$$

$$\therefore c_{teq} = c_{t1} + c_{t2} \left(\frac{n_1}{n_2} \right)^2 + c_{t3} \left(\frac{n_1}{n_3} \right)^2.$$

1.33

Damping constant desired = $c = 1$ lb-sec/in, viscosity of the fluid = $\mu = 4 \mu \text{ reyn} = 4 (10^{-6}) \text{ lb-sec/in}^2$.



$$c = \mu \left\{ \frac{3 \pi D^3 \ell \left(1 + \frac{2d}{D}\right)}{4 d^3} \right\} \quad (1)$$

Assuming $x = D/d$ as the unknown with $\ell = 2$ in, Eq. (1) can be written as

$$c = \mu \left\{ \frac{3 \pi \ell x^3}{4} \right\} \left(1 + \frac{2}{x}\right) \quad \text{or} \quad 1 = (4 (10^{-6})) \left(\frac{3 \pi (2)}{4}\right) x^3 \left(1 + \frac{2}{x}\right) \quad (2)$$

This gives $x^3 + 2x^2 - 53,051.52 = 0$

Using a trial and error procedure, the solution of this cubic equation can be found as $x \approx 36.92$. Using $D = 3$ in, we get $d = 3/36.92 = 0.08126$ in.

1.34

$$c = \mu \left\{ \frac{3 \pi D^3 \ell}{4 d^3} \left(1 + 2 \frac{d}{D}\right) \right\};$$

D = diameter of piston

ℓ = axial length of piston

d = radial clearance

$\mu = 45 \mu \text{ reynolds}$

(from Shigley's Mechanical Engineering Design)

Let $d = 0.001''$, $D = 2.4''$ and above equation gives

$$10^5 = (45 \times 10^{-6}) \left\{ \frac{3 \pi (2.4)^3 \ell}{4 (0.001)^3} \left(1 + \frac{2 \times 0.001}{2.4}\right) \right\}$$

$$\therefore \ell = 0.6817''$$

1.35

Tangential velocity of inner cylinder = $\frac{D}{2} \omega$

For small d , rate of change of velocity of fluid is

$$\frac{dv}{dr} = \frac{\frac{D}{2} \omega}{d}$$

shear stress between cylinders is

$$\tau = \mu \frac{dv}{dr} = \mu \frac{\frac{D}{2} \omega}{2d}$$

and shear force is

$$F = \tau \cdot \text{Area} = \tau \pi D (\ell - h) = \frac{\pi \mu D^2 \omega (\ell - h)}{2d}$$

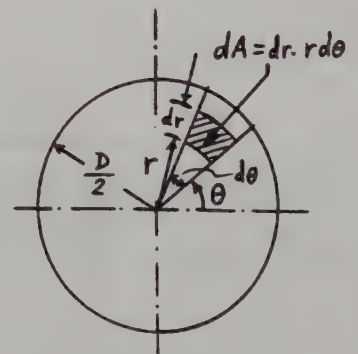
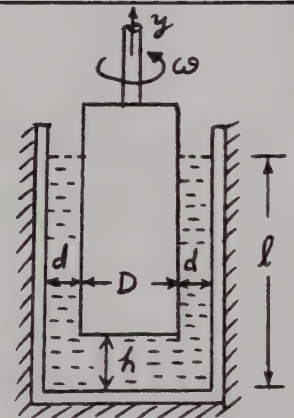
Torque developed = $M_{t1} = F \cdot \frac{D}{2}$

For small h , rate of change of velocity of fluid in vertical direction is

$$\frac{dv}{dy} = \frac{r \omega}{h}$$

Shear stress is $\tau = \mu \frac{dv}{dy} = \frac{\mu r \omega}{h}$

Force on area $dA = dF = \tau dA$



Torque between bottom surfaces of cylinders is

$$M_{t2} = \iint_{\text{area}} dM_{t2} \cdot dA \quad \text{where } dM_{t2} = dF \cdot r = \frac{\mu r^3 \omega}{h} dr d\theta$$

$$\text{i.e., } M_{t2} = \frac{\mu \omega}{h} \int_{r=0}^{D/2} \int_{\theta=0}^{2\pi} r^3 \cdot dr d\theta = \frac{\mu \omega \pi D^4}{64 h}$$

$$\text{Total torque} = M_t = M_{t1} + M_{t2} = \frac{\pi \mu D^3 \omega (l-h)}{4 d} + \frac{\pi \mu \omega D^4}{64 h}$$

Expressing M_t as $C_t v = C_t \omega D/2$, we get damping constant:

$$C_t = \frac{\pi \mu D^2 (l-h)}{2 d} + \frac{\pi \mu D^3}{32 h}$$

1.36

$$F = a \dot{x} + b \dot{x}^2 = 5 \dot{x} + 0.2 \dot{x}^2$$

$$F(\dot{x}) \approx F(\dot{x}_0) + \left. \frac{dF}{d\dot{x}} \right|_{\dot{x}_0} (\dot{x} - \dot{x}_0)$$

$$\text{At } \dot{x}_0 = 5 \text{ m/s, } F(\dot{x}_0) = 5(5) + 0.2(25) = 30 \text{ N, } \left. \frac{dF}{d\dot{x}} \right|_{\dot{x}_0} = (5 + 0.4 \dot{x})|_5 = 7 \text{ and hence}$$

$$F(\dot{x}) = 30 + 7(\dot{x} - 5) = 7\dot{x} - 5.$$

Thus the linearized damping constant is given by $F(\dot{x}) \approx 7\dot{x} = c_{eq} \dot{x}$ or $c_{eq} = 7 \text{ N-s/m}$.

1.37

Damping constant due to skin friction drag is:

$$c = 100 \mu \ell^2 d \quad (1)$$

Damping constant of a plate-type damper is:

$$c_p = \frac{\mu A}{h} \quad (2)$$

where A = area of plates and h = distance between the plates. If the area of plates (A) in Fig. 1.35 is taken to be same as the area of the plate shown in Fig. 1.82, we have $A = \ell d$. Equating (1) and (2) gives

$$100 \mu \ell^2 d = \frac{\mu \ell d}{h} \quad (3)$$

from which the clearance between the plates can be determined as $h = \frac{1}{100 \ell}$.

1.38

$$c = \frac{6 \pi \mu \ell}{h^3} \left\{ \left(a - \frac{h}{2} \right)^2 - r^2 \right\} \left[\frac{a^2 - r^2}{a - \frac{h}{2}} - h \right]$$

When $\mu = 0.3445 \text{ Pa-s}$, $\ell = 0.1 \text{ m}$, $h = 0.001 \text{ m}$, $a = 0.02 \text{ m}$, and $r = 0.005 \text{ m}$:

$$c = \frac{6 \pi (0.3445) (0.1)}{(10^{-3})^3} \left\{ (0.02 - 0.0005)^2 - 0.005^2 \right\} \left[\frac{0.02^2 - 0.005^2}{0.02 - 0.0005} - 0.001 \right]$$

$$= 4,205.6394 \text{ N-s/m}$$

$$\begin{aligned}
 \text{1.39} \quad \vec{x} &= 5 + 2i = A e^{i\theta} = A \cos \theta + i A \sin \theta \\
 A \cos \theta &= 5 \\
 A \sin \theta &= 2 \\
 A &= \sqrt{(A \cos \theta)^2 + (A \sin \theta)^2} = \sqrt{5^2 + 2^2} = 5.3852 \\
 \theta &= \tan^{-1} \left(\frac{A \sin \theta}{A \cos \theta} \right) = \tan^{-1} \left(\frac{2}{5} \right) = 21.8014^\circ
 \end{aligned}$$

$$\begin{aligned}
 \text{1.40} \quad \vec{x}_1 &= 1 + 2i = a_1 + a_2 i, \quad \vec{x}_2 = 3 - 4i = b_1 + b_2 i \\
 \vec{x} &= \vec{x}_1 + \vec{x}_2 = (a_1 + b_1) + i(a_2 + b_2) = 4 - 2i \\
 &= A e^{i\theta} = A \cos \theta + i A \sin \theta \\
 A &= \sqrt{4^2 + (-2)^2} = 4.4721 \\
 \theta &= \tan^{-1} \left(\frac{-2}{4} \right) = -26.5651^\circ
 \end{aligned}$$

$$\begin{aligned}
 \text{1.41} \quad z_1 &= (3 - 4i), \quad z_2 = (1 + 2i) \\
 z &= z_1 - z_2 = (3 - 4i) - (1 + 2i) = 2 - 6i = A e^{i\theta} \\
 \text{where } A &= \sqrt{2^2 + (-6)^2} = 6.3246 \text{ and } \theta = \tan^{-1} \left(\frac{-6}{2} \right) = \tan^{-1}(-3) = -1.2490 \text{ rad}
 \end{aligned}$$

$$\begin{aligned}
 \text{1.42} \quad z_1 &= 1 + 2i, \quad z_2 = 3 - 4i \\
 z &= z_1 z_2 = (1 + 2i)(3 - 4i) = 11 + 2i = A e^{i\theta} \\
 \text{where } A &= \sqrt{11^2 + 2^2} = 11.1803 \text{ and } \theta = \tan^{-1} (2/11) = 0.1798 \text{ rad}
 \end{aligned}$$

$$\begin{aligned}
 \text{1.43} \quad z &= \frac{z_1}{z_2} = \frac{1 + 2i}{3 - 4i} = \frac{(1 + 2i)(3 + 4i)}{(3 - 4i)(3 + 4i)} = \frac{-5 + 10i}{25} = -0.2 + 0.4i = A e^{i\theta} \\
 \text{where } A &= \sqrt{(-0.2)^2 + (0.4)^2} = 0.4472 \\
 \text{and } \theta &= \tan^{-1} \left(\frac{-0.4}{0.2} \right) = \tan^{-1}(-2) = -1.1071 \text{ rad}
 \end{aligned}$$

$$\begin{aligned}
 \text{1.44} \quad x(t) &= X \cos \omega t, \quad y(t) = Y \cos (\omega t + \phi) \\
 \text{(a)} \quad \frac{x^2}{X^2} &= \cos^2 \omega t, \quad \frac{y^2}{Y^2} = \cos^2 (\omega t + \phi), \\
 2 \frac{xy}{XY} \cos \phi &= 2 \cos \omega t \cos (\omega t + \phi) \cos \phi \\
 &= \cos^2 \omega t + \cos^2 (\omega t + \phi) - 2 \frac{xy}{XY} \cos \phi \\
 &= \cos^2 \omega t + \cos^2 (\omega t + \phi) - 2 \cos \omega t \cos \phi \cos (\omega t + \phi) \quad (1)
 \end{aligned}$$

Noting that $\cos^2 \alpha = \frac{1}{2} (1 + \cos 2\alpha)$, Eq. (1) can be rewritten as

$$\begin{aligned}
 &\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2 \frac{xy}{XY} \cos \phi \\
 &= \frac{1}{2} + \frac{1}{2} \cos 2\omega t + \frac{1}{2} + \frac{1}{2} \cos (2\omega t + 2\phi) - 2 \cos \omega t \cos \phi \cos (\omega t + \phi)
 \end{aligned}$$

$$\begin{aligned}
 &= 1 + \frac{1}{2} \left\{ 2 \cos \frac{2\omega t + 2\omega t + 2\phi}{2} \cos \frac{2\omega t - 2\omega t - 2\phi}{2} \right\} \\
 &\quad - 2 \cos \omega t \cos \phi \cos (\omega t + \phi) \\
 &= 1 + \cos (2\omega t + \phi) \cos \phi - 2 \cos \omega t \cos \phi \cos (\omega t + \phi) \\
 &= 1 + \cos (2\omega t + \phi) \cos \phi - 2 \cos \phi \left\{ \frac{1}{2} \left[\cos (\omega t + \phi - \omega t) + \cos (\omega t + \phi + \omega t) \right] \right\} \\
 &= 1 + \cos \phi \cos (2\omega t + \phi) - \cos \phi \left\{ \cos \phi + \cos (2\omega t + \phi) \right\} \\
 &= 1 - \cos^2 \phi = \sin^2 \phi \qquad (2)
 \end{aligned}$$

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2 \frac{xy}{XY} = \left(\frac{x}{X} - \frac{y}{Y} \right)^2 = 0$$

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} = 1$$

1.45

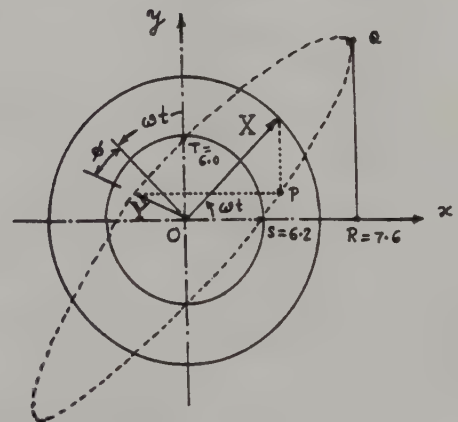
$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2 \frac{xy}{XY} \cos^2 \phi = \sin^2 \phi \quad (1)$$

$$x = \pm X \sin \phi = \pm 6.2 = OS \text{ in figure } (2)$$

$$y = \pm Y \sin \phi = \pm 6.0 = OT \text{ in figure} \quad (3)$$

$$OR = X \cos \phi = 7.6 \text{ in figure} \quad (4)$$

From Eqs. (2) and (4), we find



Equations (3) and (5) give

$$Y = \frac{6.0}{\sin \phi} = \frac{6.0}{\sin 39.2072^\circ} = 9.4918 \text{ mm}$$

1.46 (a) $x(t) = \frac{A}{1000} \cos(50t + \alpha) \text{ m}$ where A is in mm ---- (E₁)

$$x(0) = \frac{A}{1000} \cos \alpha = 0.003, \quad A \cos \alpha = 3 \quad \text{---- (E}_2\text{)}$$

$$\dot{x}(0) = -\frac{50A}{1000} \sin \alpha = 1, \quad A \sin \alpha = -20 \quad \text{---- (E}_3\text{)}$$

$$A = \{(A \cos \alpha)^2 + (A \sin \alpha)^2\}^{1/2} = 20.2237 \text{ mm}$$

$$\alpha = \tan^{-1} \left(\frac{A \sin \alpha}{A \cos \alpha} \right) = \tan^{-1}(-6.6667) = -81.4692^\circ = -1.4219 \text{ rad}$$

$$x(t) = 20.2237 \cos(50t - 1.4219) \text{ mm}$$

(b) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Eg. (E₁) can be expressed as $x(t) = A \cos 50t \cdot \cos \alpha - A \sin 50t \cdot \sin \alpha$
 $= A_1 \cos \omega t + A_2 \sin \omega t$

where $\omega = 50$, $A_1 = A \cos \alpha$, $A_2 = -A \sin \alpha$

$$\therefore x(t) = (3 \cos 50t + 20 \sin 50t) \text{ mm}$$

1.47 $x(t) = A_1 \cos \omega t + A_2 \sin \omega t$

$$\frac{dx}{dt}(t) = -A_1 \omega \sin \omega t + A_2 \omega \cos \omega t, \quad \frac{d^2x}{dt^2} = -A_1 \omega^2 \cos \omega t - A_2 \omega^2 \sin \omega t$$

$$\frac{d^2x}{dt^2} = -\omega^2 x(t) \text{ where } \omega^2 \text{ is a constant}$$

Hence $x(t)$ is a simple harmonic motion.

1.48 (a) Using trigonometric relations:

$$x_1(t) = 5 (\cos 3t \cos 1 - \sin 3t \sin 1)$$

$$x_2(t) = 10 (\cos 3t \cos 2 - \sin 3t \sin 2)$$

$$x(t) = x_1(t) + x_2(t) = \cos 3t (5 \cos 1 + 10 \cos 2) - \sin 3t (5 \sin 1 + 10 \sin 2)$$

If $x(t) = A \cos(\omega t + \alpha) = A \cos \omega t \cos \alpha - A \sin \omega t \sin \alpha$,

$$\omega = 3, \quad A \cos \alpha = 5 \cos 1 + 10 \cos 2 = -1.4599,$$

$$A \sin \alpha = 5 \sin 1 + 10 \sin 2 = 13.3003$$

$$A = \sqrt{(A \cos \alpha)^2 + (A \sin \alpha)^2} = 13.3802$$

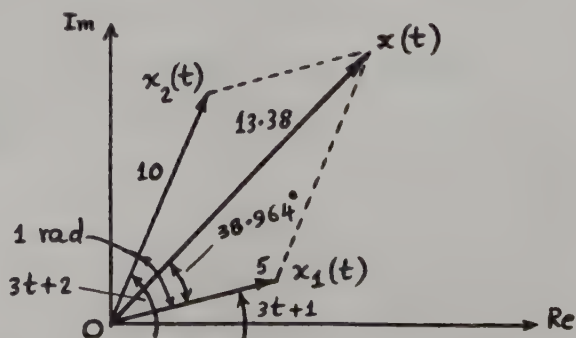
$$\alpha = \tan^{-1} \left(\frac{A \sin \alpha}{A \cos \alpha} \right) = \tan^{-1}(-9.1104) = 96.2640^\circ = 1.68 \text{ rad}$$

Angle between $x_1(t)$ and $x(t)$ is $96.2640^\circ - 57.3^\circ = 38.964^\circ$

(b) Using vector addition:

For an arbitrary value of $(\omega t + 1)$, harmonic motions $x_1(t)$ and $x_2(t)$ can be shown as in the figure. From vector addition, we find

$$x(t) \approx 13.38 \cos(\omega t + 1.68)$$



(C) Using complex numbers:

$$x_1(t) = \operatorname{Re} \{A_1 e^{i(\omega t + 1)}\} = \operatorname{Re} \{5 e^{i(\omega t + 1)}\}$$

$$x_2(t) = \operatorname{Re} \{A_2 e^{i(\omega t + 2)}\} = \operatorname{Re} \{10 e^{i(\omega t + 2)}\}$$

$$\text{If } x(t) = \operatorname{Re} \{A e^{i(\omega t + \alpha)}\},$$

$$A \cos(3t + \alpha) = A_1 \cos(3t + 1) + A_2 \cos(3t + 2)$$

$$\text{i.e. } A(\cos 3t \cos \alpha - \sin 3t \sin \alpha) = 5(\cos 3t \cos 1 - \sin 3t \sin 1) + 10(\cos 3t \cos 2 - \sin 3t \sin 2)$$

$$\text{i.e. } A \cos \alpha = 5 \cos 1 + 10 \cos 2, \quad A \sin \alpha = 5 \sin 1 + 10 \sin 2$$

$$A = 13.3802, \quad \alpha = 1.68 \text{ rad}$$

$$x(t) = \operatorname{Re} \{13.3802 e^{i(3t + 1.68)}\}$$

1.49

$$x(t) = 10 \sin(\omega t + 60^\circ) = x_1(t) + x_2(t)$$

$$\text{where } x_1(t) = 5 \sin(\omega t + 30^\circ) \text{ and } x_2(t) = A \sin(\omega t + \alpha^\circ)$$

$$10(\sin \omega t \cos 60^\circ + \cos \omega t \sin 60^\circ) = 5(\sin \omega t \cos 30^\circ + \cos \omega t \sin 30^\circ) + A(\sin \omega t \cos \alpha^\circ + \cos \omega t \sin \alpha^\circ)$$

$$10 \cos 60^\circ = 5 \cos 30^\circ + A \cos \alpha^\circ; \quad A \cos \alpha^\circ = 0.6699$$

$$10 \sin 60^\circ = 5 \sin 30^\circ + A \sin \alpha^\circ; \quad A \sin \alpha^\circ = 6.1603$$

$$A = \sqrt{0.6699^2 + 6.1603^2} = 6.1966$$

$$\alpha = \tan^{-1}(6.1603/0.6699) = 83.7938^\circ$$

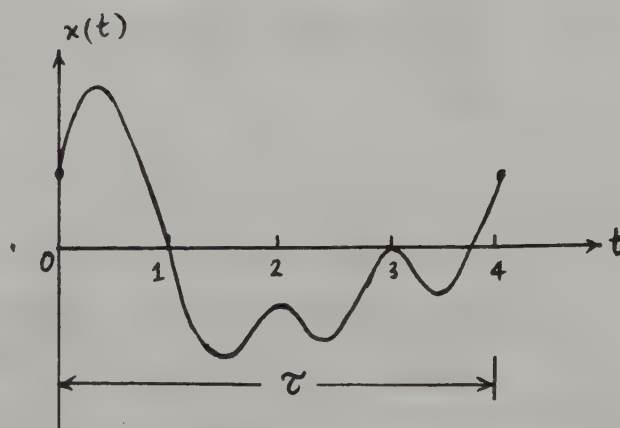
$$x_2(t) = 6.1966 \sin(\omega t + 83.7938^\circ)$$

1.50

$$x(t) = \frac{1}{2} \cos \frac{\pi}{2} t + \sin \pi t$$

$$= \frac{1}{2} \cos \frac{\pi}{2} t (1 + 4 \sin \frac{\pi}{2} t)$$

From the nature of the graph of $x(t)$, it can be seen that $x(t)$ is periodic with a time period of $\tau = 4$.



1.51

$$\text{If } x(t) \text{ is harmonic, } \ddot{x}(t) = -\omega^2 x(t)$$

$$\text{Here } x(t) = 2 \cos 2t + \cos 3t$$

$$\ddot{x}(t) = -8 \cos 2t - 9 \cos 3t \neq -\text{constant times } x(t)$$

$\therefore x(t)$ is not harmonic

1.52

$$x(t) = \frac{1}{2} \cos \frac{\pi}{2} t - \cos \pi t$$

$$\ddot{x}(t) = -\frac{\pi^2}{8} \cos \frac{\pi}{2} t + \pi^2 \cos \pi t \neq -\text{constant times } x(t)$$

$\therefore x(t)$ is not harmonic

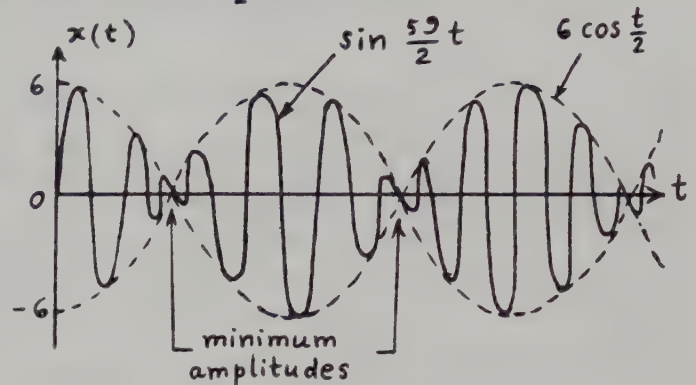
1.53

$$x(t) = x_1(t) + x_2(t) = 3 \sin 30t + 3 \sin 29t$$

Since $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$,

$$x(t) = \left(6 \cos \frac{t}{2}\right) \sin \frac{59}{2} t$$

This equation shows that the amplitude $\left(6 \cos \frac{t}{2}\right)$ varies with time between a maximum value of 6 and a minimum value of 0. The frequency of this oscillation (beat frequency) is $\omega_b = 1$.



Note: Beat frequency is twice the frequency of the term $6 \cos \frac{t}{2}$ since two peaks pass in each cycle of $\left(6 \cos \frac{t}{2}\right)$.

1.54

The resultant motion of two harmonic motions having identical amplitudes (X) but slightly different frequencies (ω and $\omega + \delta\omega$) is given by Eq. (1.67):

$$x(t) = 2X \sin \left[\omega t + \frac{\delta\omega t}{2} \right] \cos \left[\frac{\delta\omega t}{2} \right]$$

Thus the maximum amplitude of the resultant motion is equal to $2X$ and the beat frequency is equal to $\delta\omega$. From Fig. 1.86, we find that $2X \approx 5$ mm or $X = 2.5$ mm and

$$\frac{\delta\omega}{2} = \frac{2\pi}{\tau_{\text{beat}}} = \frac{2\pi}{\tau_{\text{larger}}} = \frac{2\pi}{2(12.6 - 4.2)} = 0.374 \text{ rad/sec}$$

or $\delta\omega = 0.748 \text{ rad/sec}$ and $\omega + \frac{\delta\omega}{2} = \frac{2\pi}{\tau_{\text{smaller}}} = \frac{2\pi}{1} = 6.2832 \text{ rad/sec}$

Hence $\omega = 6.2832 - 0.3740 = 5.9092 \text{ rad/sec}$. Thus the amplitudes of the two motions = $X = 2.5$ mm and their frequencies are $\omega = 5.9092 \text{ rad/sec}$ and $\omega + \delta\omega = 5.9092 + 0.7480 = 6.6572 \text{ rad/sec}$.

1.55

$$A = 0.05 \text{ m}, \quad \omega = 10 \text{ Hz} = 62.832 \text{ rad/sec}$$

$$\text{period} = \tau = \frac{2\pi}{\omega} = \frac{2\pi}{62.832} = 0.1 \text{ sec}$$

$$\text{maximum velocity} = A\omega = 0.05 \times 62.832 = 3.1416 \text{ m/s}$$

$$\text{maximum acceleration} = A\omega^2 = 0.05 (62.832)^2 = 197.393 \text{ m/s}^2$$

1.56

$$\omega = 15 \text{ cps} = 94.248 \text{ rad/sec}$$

$$\ddot{x}_{\max} = 0.5g = 0.5(9.81) = 4.905 \text{ m/s}^2 = A\omega^2$$

$$A = \text{amplitude} = 4.905 / (94.248)^2 = 0.0005522 \text{ m}$$

$$\dot{x}_{\max} = \text{max. velocity} = A\omega = 0.05204 \text{ m/s}$$

1.57

$$x = A \cos \omega t, \quad x_{\max} = A = 0.25 \text{ mm}, \quad \ddot{x} = -\omega^2 A \cos \omega t$$

$$\ddot{x}_{\max} = A\omega^2 = 0.4g = 3924 \text{ mm/s}^2; \quad \omega^2 = 3924/A = 15696 \text{ (rad/s)}^2$$

$$\text{operating speed of pump} = \omega = 125.2837 \text{ rad/s} = 19.9395 \text{ rpm}$$

1.58

$$x(t) = X \sin \frac{2\pi t}{\tau}; \quad x_{\text{rms}} = \left[\frac{1}{\tau} \int_0^{\tau} X^2 \sin^2 \frac{2\pi t}{\tau} dt \right]^{\frac{1}{2}}$$

$$\text{Using } \sin^2 \frac{2\pi t}{\tau} = \frac{1 - \cos \frac{4\pi t}{\tau}}{2}, \text{ we obtain}$$

$$\begin{aligned} x_{\text{rms}} &= \left[\frac{X^2}{\tau} \int_0^{\tau} \left(\frac{1}{2} - \frac{1}{2} \cos \frac{4\pi t}{\tau} \right) dt \right]^{\frac{1}{2}} = \left[\frac{X^2}{\tau} \left\{ \frac{t}{2} - \frac{1}{2} \frac{\tau}{4\pi} \sin \frac{4\pi t}{\tau} \right\} \Big|_0^{\tau} \right]^{\frac{1}{2}} \\ &= \left[\frac{X^2}{\tau} \left\{ \frac{\tau}{2} - \frac{\tau}{8\pi} \sin 4\pi - 0 + 0 \right\} \right]^{\frac{1}{2}} = \frac{X}{\sqrt{2}} \end{aligned}$$

1.59

$$x(t) = \frac{A}{\tau} t; \quad 0 \leq t \leq \tau$$

$$\begin{aligned} x_{\text{rms}} &= \left[\frac{1}{\tau} \int_0^{\tau} \frac{A^2}{\tau^2} t^2 dt \right]^{\frac{1}{2}} = \left[\frac{1}{\tau} \frac{A^2}{\tau^2} \left(\frac{t^3}{3} \right) \Big|_0^{\tau} \right]^{\frac{1}{2}} = \left[\frac{A^2}{\tau^3} \frac{\tau^3}{3} \right]^{\frac{1}{2}} = \left(\frac{A^2}{3} \right)^{\frac{1}{2}} = \frac{A}{\sqrt{3}} \end{aligned}$$

1.60

For even functions, $x(-t) = x(t)$.

$$\begin{aligned} \text{From Eq. (1.73), } b_n &= \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t \cdot dt = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \sin n\omega t \cdot dt \\ &= \frac{2}{\tau} \left[\int_{-\tau/2}^0 x(t) \sin n\omega t \cdot dt + \int_0^{\tau/2} x(t) \sin n\omega t \cdot dt \right] \quad \text{--- (E1)} \end{aligned}$$

Since $\sin(-n\omega t) = -\sin(n\omega t)$ = odd function of t , the product of $x(t)$ and $\sin n\omega t$ is an odd function.

Further, for an odd function $f(t)$, $f(-t) = -f(t)$, and

$$\begin{aligned}\int_{-a}^a f(t) dt &= \int_{-a}^0 f(t) dt + \int_0^a f(t) dt = \int_0^a f(-t) dt + \int_0^a f(t) dt \\ &= - \int_0^a f(t) dt + \int_0^a f(t) dt = 0 \quad \text{-----(E}_2\text{)}\end{aligned}$$

Equations (E₁) and (E₂) lead to $b_n = 0$.

Also, since $\cos n\omega t$ is an even function, we get

$$a_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \cos n\omega t dt = \frac{4}{\tau} \int_0^{\tau/2} x(t) \cos n\omega t dt$$

For odd functions, $x(-t) = -x(t)$.

From Eq. (1.72),
$$a_n = \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t dt = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \cos n\omega t dt$$

Since $\cos n\omega t$ is an even function, $\cos(-n\omega t) = \cos(n\omega t)$, the product of $x(t)$ and $\cos n\omega t$ is an odd function.

Hence $a_n = 0$.

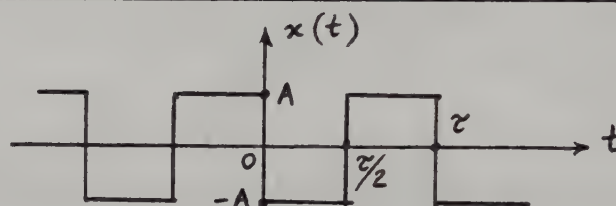
Further, since $\sin n\omega t$ is an odd function, $x(t) \sin n\omega t$ is an even function and hence

$$b_n = \frac{4}{\tau} \int_0^{\tau/2} x(t) \sin n\omega t dt$$

1.61

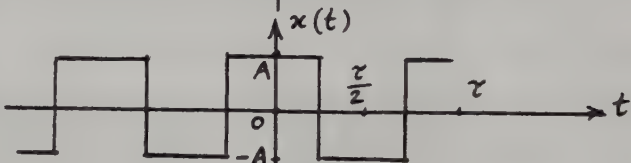
$$x(t) = \begin{cases} -A, & 0 \leq t \leq \frac{\tau}{2} \\ A, & \frac{\tau}{2} \leq t \leq \tau \end{cases}$$

(a)



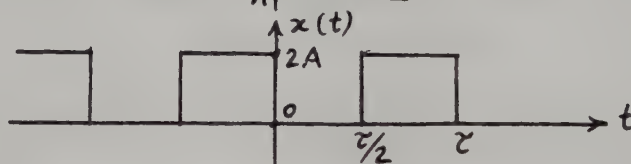
$$x(t) = \begin{cases} A, & 0 \leq t \leq \frac{\tau}{4} \\ -A, & \frac{\tau}{4} \leq t \leq \frac{3\tau}{4} \\ A, & \frac{3\tau}{4} \leq t \leq \tau \end{cases}$$

(b)



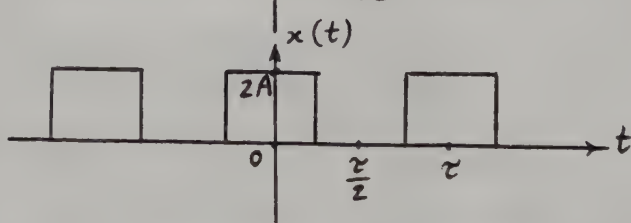
$$x(t) = \begin{cases} 0, & 0 \leq t \leq \frac{\tau}{2} \\ 2A, & \frac{\tau}{2} \leq t \leq \tau \end{cases}$$

(c)



$$x(t) = \begin{cases} 2A, & 0 \leq t \leq \frac{\tau}{4} \\ 0, & \frac{\tau}{4} \leq t \leq \frac{3\tau}{4} \\ 2A, & \frac{3\tau}{4} \leq t \leq \tau \end{cases}$$

(d)



(a) $x(-t) = -x(t)$, odd function, hence $a_0 = a_n = 0$

$$\begin{aligned} b_n &= \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t \cdot dt = \frac{2}{\tau} \left[-A \int_0^{\tau/2} \sin n\omega t \cdot dt + A \int_{\tau/2}^{\tau} \sin n\omega t \cdot dt \right] \\ &= -\frac{2A}{\tau} \left(-\frac{\cos n\omega t}{n\omega} \right)_0^{\tau/2} + \frac{2A}{\tau} \left(-\frac{\cos n\omega t}{n\omega} \right)_{\tau/2}^{\tau} \\ &= \frac{2A}{\tau n\omega} (2 \cos n\pi - \cos 0 - \cos 2n\pi) \end{aligned}$$

$$\therefore x(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin (2n-1) \omega t$$

(b) $x(-t) = x(t)$, even function, hence $b_n = 0$

$$a_0 = \frac{2}{\tau} \int_0^{\tau} x(t) dt = \frac{2}{\tau} \left[A \cdot (t)_0^{\tau/4} - A (t)_{\tau/4}^{3\tau/4} + A (t)_{3\tau/4}^{\tau} \right] = 0$$

$$\begin{aligned} a_n &= \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t \cdot dt \\ &= \frac{2A}{\tau n\omega} \left[\sin n\omega t \Big|_0^{\tau/4} - \sin n\omega t \Big|_{\tau/4}^{3\tau/4} + \sin n\omega t \Big|_{3\tau/4}^{\tau} \right] \\ &= \frac{A}{n\pi} \left[2 \sin \frac{n\pi}{2} - 2 \sin \frac{3n\pi}{2} + \sin 2\pi n \right] = \begin{cases} 4A/n\pi & \text{for } n=1,5,9,\dots \\ -4A/n\pi & \text{for } n=3,7,11,\dots \end{cases} \end{aligned}$$

$$\therefore x(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \cos \frac{2\pi(n-1)t}{\tau}$$

$$(c) a_0 = \frac{2}{\tau} \int_0^{\tau} x(t) dt = \frac{2}{\tau} \left[0 + 2A (t)_{\tau/2}^{\tau} \right] = 2A$$

$$a_n = \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t dt = \frac{4A}{n\omega\tau} (\sin n\omega t)_{\tau/2}^{\tau} = 0$$

$$b_n = \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t dt = -\frac{4A}{n\omega\tau} (\cos n\omega t)_{\tau/2}^{\tau}$$

$$= -\frac{4A}{n\omega\tau} (\cos 2\pi n - \cos n\pi)$$

$$\therefore x(t) = -\frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin (2n-1) \omega t \quad \text{with } \omega = 2\pi/\tau.$$

(d) $x(-t) = x(t)$, even function, hence $b_n = 0$

$$a_0 = \frac{2}{\tau} \int_0^{\tau} x(t) dt = \frac{2}{\tau} \left[2A \left(\frac{\tau}{4} - 0 \right) + 2A \left(\tau - \frac{3\tau}{4} \right) \right] = 2A$$

$$a_n = \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t dt = \frac{4A}{n\omega\tau} \left[(\sin n\omega t)_0^{\tau/4} + (\sin n\omega t)_{\frac{3\tau}{4}}^{\tau} \right]$$

$$= \frac{4A}{n\omega\tau} \left(\sin \frac{n\pi}{2} + \sin 2n\pi - \sin \frac{3n\pi}{2} \right)$$

$$\therefore x(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \cos \frac{2\pi(2n-1)t}{\tau} \quad \text{with } \omega = 2\pi/\tau.$$

1.62 $x(t) = \begin{cases} A \sin \frac{2\pi t}{\tau} & , \quad 0 \leq t \leq \frac{\tau}{2} \\ 0 & , \quad \frac{\tau}{2} \leq t \leq \tau \end{cases}$

$$a_0 = \frac{2}{\tau} \int_0^{\tau} x(t) dt = \frac{2A}{\tau} \int_0^{\tau/2} \sin \frac{2\pi t}{\tau} dt = \frac{2A}{\tau} \left(-\frac{\tau}{2\pi} \cos \frac{2\pi t}{\tau} \right)_0^{\tau/2}$$

$$= \frac{2A}{\pi}$$

$$a_n = \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t dt = \frac{2A}{\tau} \int_0^{\tau/2} \sin \frac{2\pi t}{\tau} \cos n\omega t dt \quad \dots (E_1)$$

Using the relation $\sin m\omega t \cdot \cos n\omega t = \frac{\sin(m+n)\omega t + \sin(m-n)\omega t}{2}$,

Eg. (E₁) can be rewritten as

$$a_n = \frac{A}{\tau} \int_0^{\pi/\omega} [\sin(1+n)\omega t + \sin(1-n)\omega t] dt$$

when $n=1$, $a_1 = \frac{A}{\tau} \int_0^{\pi/\omega} \sin 2\omega t \cdot dt = 0$

when $n=2, 3, 4, \dots$,
$$a_n = \frac{A}{\tau} \left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega} \right]_0^{\pi/\omega}$$

$$= \frac{A}{2\pi} \left[\frac{1 - \cos(1+n)\pi}{1+n} + \frac{1 - \cos(1-n)\pi}{1-n} \right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{2A}{(n-1)(n+1)\pi} & \text{if } n \text{ is even} \end{cases}$$

Similarly
$$b_n = \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t dt = \frac{2A}{\tau} \int_0^{\tau/2} \sin \frac{2\pi t}{\tau} \cos n\omega t dt$$

$$= \frac{A}{\tau} \int_0^{\tau/2} [\cos(1-n)\omega t - \cos(1+n)\omega t] dt$$

when $n=1$, $b_1 = \frac{A}{\tau} \int_0^{\pi/\omega} (\sin t - \sin 2t) dt = \frac{A}{2}$

when $n=2, 3, 4, \dots$,
$$b_n = \frac{A}{\tau} \left[\frac{\sin(1-n)\omega t}{(1-n)\omega} - \frac{\sin(1+n)\omega t}{(1+n)\omega} \right]_0^{\pi/\omega} = 0$$

$$\therefore x(t) = \frac{A}{\pi} + \frac{A}{2} \sin \omega t - \frac{2A}{\pi} \sum_{n=2,4,6,\dots}^{\infty} \frac{\cos n\omega t}{(n^2-1)}$$

$$1.63 \quad x(t) = \begin{cases} \frac{2At}{\tau} & , \quad 0 \leq t \leq \frac{\tau}{2} \\ -\frac{2At}{\tau} + 2A & , \quad \frac{\tau}{2} \leq t \leq \tau \end{cases}$$

$$\begin{aligned} a_0 &= \frac{2}{\tau} \int_0^{\tau} x(t) dt = \frac{2}{\tau} \left[\int_0^{\tau/2} \frac{2At}{\tau} dt + \int_{\tau/2}^{\tau} \left(-\frac{2At}{\tau} + 2A \right) dt \right] \\ &= \frac{2}{\tau} \left[\frac{2A}{\tau} \cdot \frac{t^2}{2} \Big|_0^{\tau/2} - \frac{2A}{\tau} \cdot \frac{t^2}{2} \Big|_{\tau/2}^{\tau} + 2A \cdot t \Big|_{\tau/2}^{\tau} \right] \\ &= \frac{2}{\tau} \left[\frac{A\tau}{4} - \frac{3A\tau}{4} + A\tau \right] = A \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t dt \\ &= \frac{2}{\tau} \left[\int_0^{\tau/2} \frac{2A}{\tau} t \cos n\omega t dt + \int_{\tau/2}^{\tau} \left(-\frac{2A}{\tau} t + 2A \right) \cos n\omega t dt \right] \\ &= \frac{2}{\tau} \left[\frac{2A}{\tau} \left\{ t \frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2} \right\}_0^{\tau/2} \right. \\ &\quad \left. - \frac{2A}{\tau} \left\{ t \frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2} \right\}_{\tau/2}^{\tau} + 2A \left(-\frac{\sin n\omega t}{n\omega} \right) \Big|_{\tau/2}^{\tau} \right] \end{aligned}$$

$$\text{As } \tau = \frac{2\pi}{\omega},$$

$$\begin{aligned} a_n &= \frac{\omega}{\pi} \left[\frac{A\omega}{\pi n^2\omega^2} \cos n\pi - \frac{A\omega}{\pi n^2\omega^2} - \frac{A\omega}{\pi n^2\omega^2} \cos 2\pi n + \frac{A\omega}{\pi n^2\omega^2} \cos n\pi \right] \\ &= \frac{2A}{n^2\pi^2} (\cos n\pi - 1) = \begin{cases} -\frac{4A}{n^2\pi^2} & , \quad n = 1, 3, 5, \dots \\ 0 & , \quad n = 2, 4, 6, \dots \end{cases} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t dt = \frac{2}{\tau} \left[\int_0^{\tau/2} \frac{2A}{\tau} t \sin n\omega t dt \right. \\ &\quad \left. + \int_{\tau/2}^{\tau} \left(-\frac{2A}{\tau} t + 2A \right) \sin n\omega t dt \right] \\ &= \frac{2}{\tau} \left[\frac{2A}{\tau} \left\{ -\frac{t \cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2} \right\}_0^{\tau/2} \right. \\ &\quad \left. - \frac{2A}{\tau} \left\{ -\frac{t \cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2} \right\}_{\tau/2}^{\tau} + 2A \left(-\frac{\cos n\omega t}{n\omega} \right) \Big|_{\tau/2}^{\tau} \right] \\ &= \frac{\omega}{\pi} \left[-\frac{A}{n\omega} \cos n\pi + \frac{2A}{n\omega} \cos 2\pi n - \frac{A}{n\omega} \cos n\pi - \frac{2A}{n\omega} \cos 2\pi n + \frac{2A}{n\omega} \cos n\pi \right] = 0 \end{aligned}$$

$$\therefore x(t) = \frac{A}{2} - \frac{4A}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos n\omega t$$

1.64

$$x(t) = \begin{cases} \frac{4At}{\tau} & , 0 \leq t \leq \frac{\tau}{4} \\ -\frac{4At}{\tau} + 2A & , \frac{\tau}{4} \leq t \leq \frac{3\tau}{4} \\ \frac{4At}{\tau} - 4A & , \frac{3\tau}{4} \leq t \leq \tau \end{cases}$$

$$a_0 = \frac{2}{\tau} \int_0^{\tau} x(t) dt = \frac{2}{\tau} \left[\frac{4A}{\tau} \left. \frac{t^2}{2} \right|_0^{\tau/4} + \left(-\frac{4A}{\tau} \frac{t^2}{2} + 2At \right) \Big|_{\tau/4}^{3\tau/4} + \left(\frac{4A}{\tau} \frac{t^2}{2} - 4At \right) \Big|_{3\tau/4}^{\tau} \right] = 0$$

$$\begin{aligned} a_n &= \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t dt = \frac{2}{\tau} \left[\frac{4A}{\tau} \int_0^{\tau/4} t \cos n\omega t dt - \frac{4A}{\tau} \int_{\tau/4}^{3\tau/4} t \cos n\omega t dt + 2A \int_{\tau/4}^{3\tau/4} \cos n\omega t dt \right. \\ &\quad \left. + \frac{4A}{\tau} \int_{3\tau/4}^{\tau} t \cos n\omega t dt - 4A \int_{3\tau/4}^{\tau} \cos n\omega t dt \right] \\ &= \frac{2}{\tau} \left[\frac{4A}{\tau} \left\{ t \frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2 \omega^2} \right\} \Big|_0^{\tau/4} - \frac{4A}{\tau} \left\{ t \frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2 \omega^2} \right\} \Big|_{\tau/4}^{3\tau/4} \right. \\ &\quad \left. + 2A \left(\frac{\sin n\omega t}{n\omega} \right) \Big|_{\tau/4}^{3\tau/4} + \frac{4A}{\tau} \left\{ t \frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2 \omega^2} \right\} \Big|_{3\tau/4}^{\tau} - 4A \left(\frac{\sin n\omega t}{n\omega} \right) \Big|_{3\tau/4}^{\tau} \right] \\ &= \frac{\omega}{\pi} \left[\sin \frac{n\pi}{2} \left(\frac{A}{n\omega} + \frac{A}{n\omega} - \frac{2A}{n\omega} \right) + \cos \frac{n\pi}{2} \left(\frac{2A}{\pi n^2 \omega} + \frac{2A}{\pi n^2 \omega} \right) \right. \\ &\quad \left. + \sin \frac{3n\pi}{2} \left(-\frac{3A}{n\omega} + \frac{2A}{n\omega} - \frac{3A}{n\omega} + \frac{4A}{n\omega} \right) + \cos \frac{3n\pi}{2} \left(-\frac{2A}{\pi n^2 \omega} - \frac{2A}{\pi n^2 \omega} \right) \right. \\ &\quad \left. + \cos 2\pi n \left(\frac{2A}{\pi n^2 \omega} \right) - \cos 0 \left(\frac{2A}{\pi n^2 \omega} \right) \right] = 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t dt = \frac{2}{\tau} \left[\frac{4A}{\tau} \int_0^{\tau/4} t \sin n\omega t dt - \frac{4A}{\tau} \int_{\tau/4}^{3\tau/4} t \sin n\omega t dt \right. \\ &\quad \left. + 2A \int_{\tau/4}^{3\tau/4} \sin n\omega t dt + \frac{4A}{\tau} \int_{3\tau/4}^{\tau} t \sin n\omega t dt - 4A \int_{3\tau/4}^{\tau} \sin n\omega t dt \right] \\ &= \frac{2}{\tau} \left[\frac{4A}{\tau} \left\{ \frac{1}{n^2 \omega^2} \sin n\omega t - \frac{t}{n\omega} \cos n\omega t \right\} \Big|_0^{\tau/4} - \frac{4A}{\tau} \left\{ \frac{1}{n^2 \omega^2} \sin n\omega t \right. \right. \\ &\quad \left. \left. - \frac{t}{n\omega} \cos n\omega t \right\} \Big|_{\tau/4}^{3\tau/4} + 2A \left(-\frac{\cos n\omega t}{n\omega} \right) \Big|_{\tau/4}^{3\tau/4} + \frac{4A}{\tau} \left\{ \frac{1}{n^2 \omega^2} \sin n\omega t \right. \right. \\ &\quad \left. \left. - \frac{t}{n\omega} \cos n\omega t \right\} \Big|_{3\tau/4}^{\tau} - 4A \left(-\frac{\cos n\omega t}{n\omega} \right) \Big|_{3\tau/4}^{\tau} \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{t}{n\omega} \cos n\omega t \left\{ \begin{array}{l} \tau = \frac{2\pi}{\omega} \\ \frac{3\tau}{4} = \frac{3\pi}{2\omega} \end{array} \right. - 4A \left(-\frac{\cos n\omega t}{n\omega} \right) \left\{ \begin{array}{l} \tau = \frac{2\pi}{\omega} \\ \frac{3\tau}{4} = \frac{3\pi}{2\omega} \end{array} \right. \Bigg] \\
& = \frac{4A}{\pi^2 n^2} \left(\sin \frac{n\pi}{2} - \sin \frac{3n\pi}{2} \right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8A}{\pi^2 n^2} (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases} \\
\therefore x(t) = \frac{8A}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} (-1)^{\frac{n-1}{2}} \frac{\sin n\omega t}{n^2}
\end{aligned}$$

1.65 $x(t) = A \left(1 - \frac{t}{\tau} \right), \quad 0 \leq t \leq \tau$

$$a_0 = \frac{2}{\tau} \int_0^{\tau} x(t) dt = \frac{2A}{\tau} \int_0^{\tau} \left(1 - \frac{t}{\tau} \right) dt = \frac{2A}{\tau} \left(t - \frac{t^2}{2\tau} \right)_0^{\tau} = A$$

$$\begin{aligned}
a_n &= \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t dt = \frac{2A}{\tau} \left(\frac{\sin n\omega t}{n\omega} - \frac{t}{\tau} \frac{\sin n\omega t}{n\omega} - \frac{\cos n\omega t}{\tau n^2 \omega^2} \right)_0^{\tau} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t dt = \frac{2A}{\tau} \left(-\frac{\cos n\omega t}{n\omega} + \frac{t}{\tau} \frac{\cos n\omega t}{n\omega} - \frac{\sin n\omega t}{\tau n^2 \omega^2} \right)_0^{\tau} \\
&= \frac{A}{\pi n}
\end{aligned}$$

$$\therefore x(t) = \frac{A}{2} + \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\omega t}{n}$$

1.66 The truncated series of k terms can be denoted as

$$\bar{x}(t) = \frac{\bar{a}_0}{2} + \sum_{n=1}^k \bar{a}_n \cos n\omega t + \sum_{n=1}^k \bar{b}_n \sin n\omega t \quad (1)$$

with $\bar{x}(t)$ denoting an approximation to the exact $x(t)$ given by Eq. (1.70). The error to be minimized is given by

$$E = \int_{-\pi/\omega}^{\pi/\omega} e^2(t) dt \quad (2)$$

$$\text{where } e(t) = x(t) - \bar{x}(t) \quad (3)$$

and $x(t)$ is the exact value (with infinite series on the right hand side of Eq. (1)). Treating E as a function of the unknowns $\bar{a}_n$ and $\bar{b}_n$, it can be minimized by setting:

$$\frac{\partial E}{\partial \bar{a}_n} = 2 \int_{-\pi/\omega}^{\pi/\omega} \left\{ x(t) - \bar{x}(t) \right\} \left(-\cos n\omega t \right) dt = 0 \quad (4)$$

$$\frac{\partial E}{\partial \bar{b}_n} = 2 \int_{-\pi/\omega}^{\pi/\omega} \left\{ x(t) - \bar{x}(t) \right\} \left(-\sin n\omega t \right) dt = 0 \quad (5)$$

Rearranging Eq. (4) gives

$$\int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n \omega t dt = \int_{-\pi/\omega}^{\pi/\omega} \bar{x}(t) \cos n \omega t dt \quad (6)$$

Using orthogonality property, the right hand side of Eq. (6) can be expressed as

$$\int_{-\pi/\omega}^{\pi/\omega} \bar{x}(t) \cos n \omega t dt = \begin{cases} 0 & \text{for } m \neq n \\ \frac{\bar{a}_n \pi}{\omega} & \text{for } m = n \end{cases} \quad (7)$$

This leads to

$$\int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n \omega t dt = \frac{\bar{a}_n \pi}{\omega} \quad (8)$$

$$\text{or } \bar{a}_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n \omega t dt ; n = 0, 1, 2, \dots, k \quad (9)$$

In a similar manner, we can derive:

$$\bar{b}_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} x(t) \sin n \omega t dt ; n = 1, 2, \dots, k \quad (10)$$

It can be observed that Eqs. (9) and (10) are similar to those of Eqs. (E.3) and (E.4).

1.67

i	t _i	x _i	n=1		n=2		n=3	
			x _i cos $\frac{2\pi t_i}{0.32}$	x _i sin $\frac{2\pi t_i}{0.32}$	x _i cos $\frac{4\pi t_i}{0.32}$	x _i sin $\frac{4\pi t_i}{0.32}$	x _i cos $\frac{6\pi t_i}{0.32}$	x _i sin $\frac{6\pi t_i}{0.32}$
1	0.02	9	8.3149	3.4442	6.3639	6.3640	3.4441	8.3149
2	0.04	13	9.1924	9.1924	0.0000	13.0000	-9.1924	9.1923
3	0.06	17	6.5056	15.7060	-12.0209	12.0208	-15.7059	-6.5057
4	0.08	29	0.0000	29.0000	-29.0000	0.0000	0.0000	-29.0000
5	0.10	43	-16.4556	39.7267	-30.4053	-30.4059	39.7271	-16.4548
6	0.12	59	-41.7195	41.7191	0.0000	-59.0000	41.7187	41.7199
7	0.14	63	-58.2045	24.1087	44.5482	-44.5472	-24.1101	58.2040
8	0.16	57	-57.0000	0.0000	57.0000	0.0000	-57.0000	0.0000
9	0.18	49	-45.2700	-18.7518	34.6477	34.6487	-18.7505	-45.2705
10	0.20	35	-24.7485	-24.7489	0.0000	35.0000	24.7493	-24.7482
11	0.22	35	-13.3936	-32.3359	-24.7493	24.7482	32.3354	13.3950
12	0.24	41	0.0000	-41.0000	-41.0000	0.0000	0.0000	41.0000
13	0.26	47	17.9866	-43.4221	-33.2333	-33.2347	-43.4229	17.9847
14	0.28	41	28.9917	-28.9911	0.0000	-41.0000	-28.9905	-28.9923
15	0.30	13	12.0105	-4.9747	9.1927	-9.1921	4.9755	-12.0102
16	0.32	7	7.0000	0.0000	7.0000	0.0000	7.0000	0.0000

$$\sum_{i=1}^{16} () \quad 558 \quad -166.7897 \quad -31.3278 \quad -11.6552 \quad -91.5984 \quad -43.2234 \quad 26.8281$$

$$\frac{1}{8} \sum_{i=1}^{16} () \quad 69.75 \quad -20.8487 \quad -3.9160 \quad -1.4569 \quad -11.4498 \quad -5.4029 \quad 3.3535$$

1.68

Speed = 100 rpm

In a minute, a point will be subjected to the

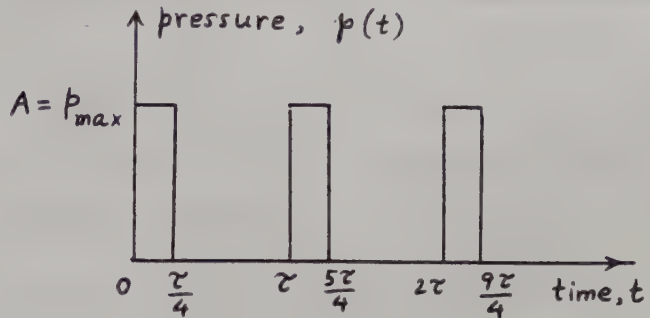
maximum pressure, $A =$

$p_{\max} = 100 \text{ psi}$, $100 \times 4 =$

400 times. Hence

$$\text{period} = \tau = \frac{60}{400} = 0.15 \text{ sec.}$$

$$p(t) = \begin{cases} A & , 0 \leq t \leq \tau/4 \\ 0 & , \tau/4 \leq t \leq \tau \end{cases}$$



$$a_0 = \frac{2}{\tau} \int_0^{\tau} p(t) dt = \frac{2}{\tau} A \left(t \right)_0^{\tau/4} = \frac{A}{2} = 50 \text{ psi}$$

$$a_m = \frac{2}{\tau} \int_0^{\tau} p(t) \cos m\omega t dt = \frac{2A}{\tau} \left(\frac{\sin m\omega t}{m\omega} \right)_0^{\tau/4} = \frac{A}{\pi m} \sin \frac{m\pi}{2}$$

$$b_m = \frac{2}{\tau} \int_0^{\tau} p(t) \sin m\omega t dt = -\frac{2A}{\tau} \left(\frac{\cos m\omega t}{m\omega} \right)_0^{\tau/4} = -\frac{A}{\pi m} (\cos \frac{m\pi}{2} - 1)$$

Evaluation of a_m and b_m :

$m=1$	$m=2$	$m=3$
$a_1 = \frac{A}{\pi} \sin \frac{\pi}{2} = \frac{A}{\pi}$	$a_2 = \frac{A}{2\pi} \sin \pi = 0$	$a_3 = \frac{A}{3\pi} \sin \frac{3\pi}{2}$
$= 31.8309 \text{ psi}$		$= -10.6103 \text{ psi}$
$b_1 = -\frac{A}{\pi} (\cos \frac{\pi}{2} - 1)$	$b_2 = -\frac{A}{2\pi} (\cos \pi - 1)$	$b_3 = -\frac{A}{3\pi} (\cos \frac{3\pi}{2} - 1)$
$= 31.8309 \text{ psi}$	$= 31.8309 \text{ psi}$	$= 10.6103 \text{ psi}$

$$\therefore p(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos m\omega t + b_m \sin m\omega t) \text{ psi}$$

1.69

i	t_i	M_{t_i}	$n=1$		$n=2$		$n=3$	
			$M_{t_i} \cos \frac{2\pi t_i}{0.012}$	$M_{t_i} \sin \frac{2\pi t_i}{0.012}$	$M_{t_i} \cos \frac{4\pi t_i}{0.012}$	$M_{t_i} \sin \frac{4\pi t_i}{0.012}$	$M_{t_i} \cos \frac{6\pi t_i}{0.012}$	$M_{t_i} \sin \frac{6\pi t_i}{0.012}$
1	0.0005	770	743.7627	199.2912	666.8391	385.0010	544.4712	544.4731
2	0.0010	810	701.4802	405.0007	404.9988	701.4812	0.0000	810.0000
3	0.0015	850	601.0398	601.0417	0.0000	850.0000	-601.0442	601.0373
4	0.0020	910	454.9978	788.0845	-455.0041	788.0808	-910.0000	0.0000

5	0.0025	1010	261.4043	975.5859	-874.689	504.995	-714.171	-714.184
6	0.0030	1170	0.0000	1170.0000	-1170.000	0.000	0.000	-1170.000
7	0.0035	1370	-354.5874	1323.3169	-1186.449	-685.010	968.748	-968.725
8	0.0040	1610	-805.0073	1394.2966	-804.987	-1394.309	1610.000	0.000
9	0.0045	1890	-1336.4407	1336.4229	0.000	-1890.000	1336.410	1336.454
10	0.0050	1750	-1515.5491	874.7922	875.019	-1515.534	0.000	1750.000
11	0.0055	1630	-1574.4619	421.8647	1411.634	-814.979	-1152.608	1152.560
12	0.0060	1510	-1510.0000	0.0000	1510.000	0.000	-1510.000	0.000
13	0.0065	1390	-1342.6345	-359.7671	1203.767	695.014	-982.858	-982.898
14	0.0070	1290	-1117.1677	-645.0088	644.982	1117.183	0.000	-1290.000
15	0.0075	1190	-841.4492	-841.4648	0.000	1190.000	841.479	-841.435
16	0.0080	1110	-554.9897	-961.2942	-555.021	961.276	1110.000	0.000
17	0.0085	1050	-271.7498	-1014.2249	-909.337	524.982	742.440	742.485
18	0.0090	990	0.0000	-990.0000	-990.000	0.000	0.000	990.000
19	0.0095	930	240.7123	-898.3081	-805.393	-465.018	-657.633	657.586
20	0.0100	890	445.0095	-770.7571	-444.981	-770.773	-890.000	0.000
21	0.0105	850	601.0478	-601.0337	0.000	-850.000	-601.022	-601.060
22	0.0110	810	701.4868	-404.9895	405.022	-701.468	0.000	-810.000
23	0.0115	770	743.7659	-199.2798	666.851	-384.980	544.500	-544.444
24	0.0120	750	750.0000	0.0000	750.000	0.000	750.000	0.000

$\sum_{i=1}^{24} ()$	27,300	-4,979.3242	1,803.7673	343.270	-1,754.097	428.734	661.855
$\frac{1}{12} \sum_{i=1}^{24}$	2,275	-414.9436	150.3139	28.606	-146.171	35.728	55.155

i	t_i	x_i	n=1		n=2		n=3	
			$x_i \cos \frac{2\pi t_i}{0.6}$	$x_i \sin \frac{2\pi t_i}{0.6}$	$x_i \cos \frac{4\pi t_i}{0.6}$	$x_i \sin \frac{4\pi t_i}{0.6}$	$x_i \cos \frac{6\pi t_i}{0.6}$	$x_i \sin \frac{6\pi t_i}{0.6}$
1	0.025	9.00	8.69	2.33	7.79	4.50	6.36	6.36
2	0.050	17.00	14.72	8.50	8.50	14.72	0.00	17.00
3	0.075	23.00	16.26	16.26	0.00	23.00	-16.26	16.26
4	0.100	25.00	12.50	21.65	-12.50	21.65	-25.00	0.00
5	0.125	26.00	6.73	25.11	-22.52	13.00	-18.38	-18.38
6	0.150	28.00	0.00	28.00	-28.00	0.00	0.00	-28.00
7	0.175	33.00	-8.54	31.88	-28.58	-16.50	23.33	-23.33
8	0.200	35.00	-17.50	30.31	-17.50	-30.31	35.00	0.00
9	0.225	34.00	-24.04	24.04	0.00	-34.00	24.04	24.04
10	0.250	29.00	-25.11	14.50	14.50	-25.11	0.00	29.00
11	0.275	24.00	-23.18	6.21	20.78	-12.00	-16.97	16.97
12	0.300	26.00	-26.00	0.00	26.00	0.00	-26.00	0.00
13	0.325	32.00	-30.91	-8.28	27.71	16.00	-22.63	-22.63
14	0.350	40.00	-34.64	-20.00	20.00	34.64	0.00	-40.00
15	0.375	18.00	-12.73	-12.73	0.00	18.00	12.73	-12.73
16	0.400	8.00	-4.00	-6.93	-4.00	6.93	8.00	0.00
17	0.425	-5.00	1.29	4.83	4.33	-2.50	-3.54	-3.54
18	0.450	-14.00	0.00	14.00	14.00	0.00	0.00	-14.00
19	0.475	-28.00	-7.25	27.05	24.25	14.00	19.80	-19.80
20	0.500	-37.00	-18.50	32.04	18.50	32.04	37.00	0.00
21	0.525	-33.00	-23.33	23.33	0.00	33.00	23.33	23.34
22	0.550	-29.00	-25.11	14.50	-14.50	25.11	0.00	29.00

23	0.575	-22.00	-21.25	5.69	-19.05	11.00	-15.56	15.56
24	0.600	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\sum_{i=1}^{24} ()$		239.00	-241.90	282.30	39.72	147.18	45.26	-4.88
$\frac{1}{12} \sum_{i=1}^{24} ()$		19.92	-20.16	23.53	3.31	12.26	3.77	-0.41

1.71 The main program and the output are shown below.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE FORIER
C
C =====
C FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION X(24), T(24), XSIN(24), XCOS(24), A(5), B(5)
  DATA N, M, TIME /24, 5, 0.012/
  DATA X/770, 810, 850, 910, 1010, 1170, 1370, 1610, 1890, 1750, 1630, 1510,
2 1390, 1290, 1190, 1110, 1050, 990, 930, 890, 850, 810, 770, 750/
  DATA T /.0005, .001, .0015, .002, .0025, .003, .0035, .004, .0045, .005,
2 .0055, .006, .0065, .007, .0075, .008, .0085, .009, .0095, .01, .0105,
3 .011, .0115, .012/
C END OF PROBLEM-DEPENDENT DATA
  CALL FORIER (N, M, TIME, X, T, AZERO, A, B, XSIN, XCOS)
  PRINT 100
100  FORMAT (///, 46H FOURIER SERIES EXPANSION OF THE FUNCTION X(T), //)
  PRINT 200, N, M, TIME

200  FORMAT (6H DATA: , //, 37H NUMBER OF DATA POINTS IN ONE CYCLE =, I5,
2  /, 42H NUMBER OF FOURIER COEFFICIENTS REQUIRED =, I5, /,
3 14H TIME PERIOD =, E15.8)
  PRINT 300, (T(I), I=1, N)
300  FORMAT (/, 33H TIME AT VARIOUS STATIONS, T(I) =, /, (4E15.8, 1X))
  PRINT 400, (X(I), I=1, N)
400  FORMAT (/, 31H KNOWN VALUES OF X(I) AT T(I) =, /, (4E15.8, 1X))
  PRINT 500
500  FORMAT (///, 29H RESULTS OF FOURIER ANALYSIS: , /)
  PRINT 600, AZERO
600  FORMAT (8H AZERO =, 2X, E15.8, //, 31H VALUES OF I, A(I) AND B(I) ARE
2 , /)
  DO 700, I =1, M
700  PRINT 800, I, A(I), B(I)
800  FORMAT (I5, 2X, E15.8, 2X, E15.8)
  STOP
  END

```

FOURIER SERIES EXPANSION OF THE FUNCTION X(T)

DATA:

NUMBER OF DATA POINTS IN ONE CYCLE = 24
NUMBER OF FOURIER COEFFICIENTS REQUIRED = 5
TIME PERIOD = 0.11999998E-01

TIME AT VARIOUS STATIONS, T(I) =

0. 50000008E-03 0. 99999993E-03 0. 15000000E-02 0. 20000001E-02
0. 24999999E-02 0. 30000000E-02 0. 35000001E-02 0. 40000007E-02
0. 45000017E-02 0. 49999990E-02 0. 55000000E-02 0. 60000010E-02
0. 64999983E-02 0. 69999993E-02 0. 75000003E-02 0. 80000013E-02
0. 84999986E-02 0. 89999996E-02 0. 95000006E-02 0. 10000002E-01
0. 10499999E-01 0. 11000000E-01 0. 11500001E-01 0. 11999998E-01

KNOWN VALUES OF X(I) AT T(I) =

0. 77000000E+03 0. 81000000E+03 0. 85000000E+03 0. 91000000E+03
0. 10100000E+04 0. 11700000E+04 0. 13700000E+04 0. 16100000E+04
0. 18900000E+04 0. 17500000E+04 0. 16300000E+04 0. 15100000E+04
0. 13900000E+04 0. 12900000E+04 0. 11900000E+04 0. 11100000E+04
0. 10500000E+04 0. 99000000E+03 0. 93000000E+03 0. 89000000E+03
0. 85000000E+03 0. 81000000E+03 0. 77000000E+03 0. 75000000E+03

RESULTS OF FOURIER ANALYSIS:

AZERO = 0. 22750000E+04

VALUES OF I, A(I) AND B(I) ARE

1	-0. 41494360E+03	0. 15031395E+03
2	0. 28605835E+02	-0. 14617058E+03
3	0. 35727844E+02	0. 55154602E+02
4	-0. 40830078E+02	0. 14440117E+01
5	0. 11577332E+02	-0. 16659973E+02

1.72

The main program and the output are given below.

```
C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE FORIER
C
C =====
C FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION X(16),T(16),XSIN(16),XCOS(16),A(5),B(5)
  DATA N,M,TIME /16,5,0.32/
  DATA X /9.,13.,17.,29.,43.,59.,63.,57.,49.,35.,35.,41.,47.,41.,
2 13.,7./
  DATA T /.02.,.04.,.06.,.08.,.1.,.12.,.14.,.16.,.18.,.2.,.22.,.24.,.26.,.28,
2 .3.,.32/
C END OF PROBLEM-DEPENDENT DATA
  CALL FORIER (N,M,TIME,X,T,AZERO,A,B,XSIN,XCOS)
  PRINT 100
100  FORMAT (//,46H FOURIER SERIES EXPANSION OF THE FUNCTION X(T),//)
  PRINT 200, N,M,TIME
200  FORMAT (6H DATA://,37H NUMBER OF DATA POINTS IN ONE CYCLE =,15,
2 /,42H NUMBER OF FOURIER COEFFICIENTS REQUIRED =,15,/,
3 14H TIME PERIOD =,E15.8)
  PRINT 300, (T(I);I=1,N)
300  FORMAT (/,33H TIME AT VARIOUS STATIONS, T(I) =,/, (4E15.8,1X))
  PRINT 400, (X(I);I=1,N)
400  FORMAT (/,31H KNOWN VALUES OF X(I) AT T(I) =,/, (4E15.8,1X))
  PRINT 500
```

```

500  FORMAT (///,29H RESULTS OF FOURIER ANALYSIS:,,)
      PRINT 600, AZERO
600  FORMAT (8H AZERO =,2X,E15.8,///,31H VALUES OF I, A(I) AND B(I) ARE
      2 ,/)
      DO 700, I =1,M
700  PRINT 800, I, A(I), B(I)
800  FORMAT (I5,2X,E15.8,2X,E15.8)
      STOP
      END

```

FOURIER SERIES EXPANSION OF THE FUNCTION X(T)

DATA:

```

NUMBER OF DATA POINTS IN ONE CYCLE = 16
NUMBER OF FOURIER COEFFICIENTS REQUIRED = 5
TIME PERIOD = 0.31999999E+00

```

TIME AT VARIOUS STATIONS, T(I) =

```

0.20000000E-01 0.39999999E-01 0.59999999E-01 0.79999983E-01
0.10000002E+00 0.12000000E+00 0.13999999E+00 0.16000003E+00
0.18000001E+00 0.19999999E+00 0.22000003E+00 0.24000001E+00
0.25999999E+00 0.27999997E+00 0.30000001E+00 0.31999999E+00

```

KNOWN VALUES OF X(I) AT T(I) =

```

0.90000000E+01 0.13000000E+02 0.17000000E+02 0.29000000E+02
0.43000000E+02 0.59000000E+02 0.63000000E+02 0.57000000E+02
0.49000000E+02 0.35000000E+02 0.35000000E+02 0.41000000E+02
0.47000000E+02 0.41000000E+02 0.13000000E+02 0.70000000E+01

```

RESULTS OF FOURIER ANALYSIS:

AZERO = 0.69750000E+02

VALUES OF I, A(I) AND B(I) ARE

```

1  -0.20848709E+02  -0.39159737E+01
2  -0.14568996E+01  -0.11449797E+02
3  -0.54029312E+01  0.33535175E+01
4  -0.17500381E+01  0.24999523E+01
5  -0.26129246E-01  0.10607624E+01

```

1.73

The main program and the output are shown below.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE FORIER
C
C =====
C FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION X(24),T(24),XSIN(24),XCOS(24),A(6),B(6)
  DATA N,M,TIME /24,6,0.6/
  DATA X/9,17,23,25,26,28,33,35,34,29,24,26,32,40,18,8,-5,-14,
2  -28,-37,-33,-29,-22,0/

```

```

DATA T /.025,.05,.075,.1,.125,.15,.175,.2,.225,.25,.275,.3,.325,
2 .35,.375,.4,.425,.45,.475,.5,.525,.55,.575,.6/
C END OF PROBLEM-DEPENDENT DATA
CALL FORIER (N,M,TIME,X,T,AZERO,A,B,XSIN,XCOS)
PRINT 100
100 FORMAT (///,46H FOURIER SERIES EXPANSION OF THE FUNCTION X(T),/)
PRINT 200, N,M,TIME
200 FORMAT (6H DATA: ,//,37H NUMBER OF DATA POINTS IN ONE CYCLE =,1),
2 /,42H NUMBER OF FOURIER COEFFICIENTS REQUIRED =,15,/,
3 14H TIME PERIOD =,E15.8)
PRINT 300, (T(I),I=1,N)
300 FORMAT (/,33H TIME AT VARIOUS STATIONS, T(I) =,/, (4E15.8,1X))
PRINT 400, (X(I),I=1,N)
400 FORMAT (/,31H KNOWN VALUES OF X(I) AT T(I) =,/, (4E15.8,1X))
PRINT 500
500 FORMAT (///,29H RESULTS OF FOURIER ANALYSIS: ,/)
PRINT 600, AZERO
600 FORMAT (8H AZERO =,2X,E15.8,///,31H VALUES OF I, A(I) AND B(I) ARE:
2 ,/)
DO 700, I =1,M
700 PRINT 800, I, A(I), B(I)
800 FORMAT (15,2X,E15.8,2X,E15.8)
STOP
END
FOURIER SERIES EXPANSION OF THE FUNCTION X(T)

```

DATA:

NUMBER OF DATA POINTS IN ONE CYCLE = 24
NUMBER OF FOURIER COEFFICIENTS REQUIRED = 6
TIME PERIOD = 0.60000002E+00

TIME AT VARIOUS STATIONS, T(I) =

0.24999999E-01 0.50000001E-01 0.74999988E-01 0.10000002E+00
0.12500000E+00 0.14999998E+00 0.17500001E+00 0.19999999E+00
0.22500002E+00 0.25000000E+00 0.27499998E+00 0.30000001E+00
0.32499999E+00 0.35000002E+00 0.37500000E+00 0.39999998E+00
0.42500001E+00 0.44999999E+00 0.47500002E+00 0.50000000E+00
0.52499998E+00 0.55000001E+00 0.57499999E+00 0.60000002E+00

KNOWN VALUES OF X(I) AT T(I) =

0.90000000E+01 0.17000000E+02 0.23000000E+02 0.25000000E+02
0.26000000E+02 0.28000000E+02 0.33000000E+02 0.35000000E+02
0.34000000E+02 0.29000000E+02 0.24000000E+02 0.26000000E+02
0.32000000E+02 0.40000000E+02 0.18000000E+02 0.80000000E+01
-0.50000000E+01 -0.14000000E+02 -0.28000000E+02 -0.37000000E+02
-0.33000000E+02 -0.29000000E+02 -0.22000000E+02 0.00000000E+00

RESULTS OF FOURIER ANALYSIS:

AZERO = 0.19916672E+02

VALUES OF I, A(I) AND B(I) ARE

1	-0.20158676E+02	0.23525284E+02
2	0.33099222E+01	0.12264638E+02
3	0.37719278E+01	-0.40640426E+00
4	-0.95843577E+00	0.32476425E+01
5	0.11377630E+01	-0.18716125E+01
6	-0.11667604E+01	0.12500324E+01

1.74

The main program and the output are given below.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE FORIER
C
C =====
C FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION X(14), T(14), XSIN(14), XCOS(14), A(10), B(10)
  DATA N, M, TIME /14, 10, 0.35/
  DATA X / .45, -.8, -.9, -.6, -.75, -.7, .55, 1.75, 1.65, .25, -1.1,
2 -1.4, -1.05, 0.0/
  DATA T / .025, .05, .075, .1, .125, .15, .175, .2, .225, .25, .275, .3, .325,
2 .35/
C END OF PROBLEM-DEPENDENT DATA
  CALL FORIER (N, M, TIME, X, T, AZERO, A, B, XSIN, XCOS)
  PRINT 100
100  FORMAT (//, 46H FOURIER SERIES EXPANSION OF THE FUNCTION X(T), //)
  PRINT 200, N, M, TIME

200  FORMAT (6H DATA: , //, 37H NUMBER OF DATA POINTS IN ONE CYCLE =, 15,
2 /, 42H NUMBER OF FOURIER COEFFICIENTS REQUIRED =, 15, /,
3 14H TIME PERIOD =, E15.8)
  PRINT 300, (T(I), I=1, N)
300  FORMAT (/, 33H TIME AT VARIOUS STATIONS, T(I) =, /, (4E15.8, 1X))
  PRINT 400, (X(I), I=1, N)
400  FORMAT (/, 31H KNOWN VALUES OF X(I) AT T(I) =, /, (4E15.8, 1X))
  PRINT 500
500  FORMAT (//, 29H RESULTS OF FOURIER ANALYSIS: , //)
  PRINT 600, AZERO
600  FORMAT (8H AZERO =, 2X, E15.8, //, 31H VALUES OF I, A(I) AND B(I) ARE:
2 , //)
  DO 700, I =1, M
700  PRINT 800, I, A(I), B(I)
800  FORMAT (15, 2X, E15.8, 2X, E15.8)
  STOP
END

```

FOURIER SERIES EXPANSION OF THE FUNCTION X(T)

DATA:

NUMBER OF DATA POINTS IN ONE CYCLE = 14
 NUMBER OF FOURIER COEFFICIENTS REQUIRED = 10
 TIME PERIOD = 0.35000002E+00

TIME AT VARIOUS STATIONS, $T(I) =$

0. 24999999E-01 0. 50000001E-01 0. 74999988E-01 0. 10000002E+00
 0. 12500000E+00 0. 14999998E+00 0. 17500001E+00 0. 19999999E+00
 0. 22500002E+00 0. 25000000E+00 0. 27499998E+00 0. 30000001E+00
 0. 32499999E+00 0. 35000002E+00

KNOWN VALUES OF $X(I)$ AT $T(I) =$

0. 44999999E+00 -0. 80000001E+00 -0. 89999998E+00 -0. 60000002E+00
 -0. 75000000E+00 -0. 69999999E+00 0. 55000001E+00 0. 17500000E+01
 0. 16499996E+01 0. 25000000E+00 -0. 11000004E+01 -0. 13999996E+01
 -0. 10500002E+01 0. 00000000E+00

RESULTS OF FOURIER ANALYSIS:

AZERO = -0. 37857169E+00

VALUES OF I , $A(I)$ AND $B(I)$ ARE

1	-0. 61951119E+00	-0. 35046142E+00
2	0. 46242946E+00	0. 92408919E+00
3	0. 41494656E+00	-0. 17127156E+00
4	0. 22273250E-01	0. 24691588E+00
5	-0. 45435190E-01	0. 10430527E+00
6	-0. 20434089E-01	0. 56009147E-01
7	-0. 49999803E-01	-0. 35824633E-05
8	-0. 20436604E-01	-0. 55996872E-01
9	-0. 45427930E-01	-0. 10429442E+00
10	0. 22274703E-01	-0. 24690533E+00

1.75

$$x_p = r + l - r \cos \theta - l \cos \phi = r + l - r \cos \omega t - l \sqrt{1 - \sin^2 \phi} \quad (E_1)$$

$$\text{But } l \sin \phi = r \sin \theta, \quad \cos \phi = \left(1 - \frac{r^2}{l^2} \sin^2 \omega t\right)^{\frac{1}{2}} \quad (E_2)$$

$$\text{Using (E}_2\text{) in (E}_1\text{), } x_p = r + l - r \cos \omega t - l \left(1 - \frac{r^2}{l^2} \sin^2 \omega t\right)^{\frac{1}{2}} \quad (E_3)$$

Let $\frac{r}{l} = \text{small } (< \frac{1}{4})$. Using $\sqrt{1 - \epsilon} \approx 1 - \frac{1}{2} \epsilon$, (E₃) becomes

$$x_p \approx r \left(1 + \frac{r}{2l}\right) - r \left(\cos \omega t + \frac{r}{4l} \cos 2\omega t\right) \quad (E_4)$$

(a) Eg. (E₄) gives $y_p = x_p - r \left(1 + \frac{r}{2l}\right) \approx -r \left(\cos \omega t + \frac{1}{4} \frac{r}{l} \cos 2\omega t\right)$ --- (E₅)

If $\frac{r}{l}$ is very small, $y_p \approx -r \cos \omega t \Rightarrow$ harmonic motion.

(b) To have amplitude of second harmonic smaller than that of first harmonic in Eg. (E₅), we need to have

$$\frac{1}{4} \frac{r}{l} \leq \frac{1}{25}, \quad \text{i.e., } \frac{r}{l} \leq \frac{4}{25}, \quad \text{i.e., } \frac{l}{r} \geq 6.25$$

Once the amplitude of second harmonic is smaller by a factor of 25, the amplitudes of higher harmonics arising from the expansion of square-root-term in (E₃) are expected to be still smaller.

1.76 Unbalanced force developed = $P = 2 m \omega^2 r \cos \omega t$, range of force = 0 - 100 N, range of frequency = 25 - 50 Hz = 157.08 - 314.16 rad/sec.

Parameters to be determined: m , r , ω .

Let $r = 0.1$ m. To generate 100 N force at 25 Hz, set:

$$P_{\max} = 100 = 2 m (157.08)^2 (0.1)$$

which gives

$$m = \frac{100}{2 (157.08)^2 (0.1)} = 0.0202641 \text{ kg} = 20.2641 \text{ g}$$

To generate 100 N force at 50 Hz, set:

$$P_{\max} = 100 = 2 m (314.16)^2 (0.1)$$

which yields

$$m = \frac{100}{2 (314.16)^2 (0.1)} = 0.0050660 \text{ kg} = 5.0660 \text{ g}$$

1.77 Goal: Weight to be maintained at 10 ± 0.1 lb/min

Parameters to be determined: Angular velocity of crank (ω), lengths of crank and connecting rod, dimensions of the wedge, dimensions of the orifice in the hopper, dimensions of the actuating rod, and dimensions of the lever arrangement.

Given: Density of the material in the hopper.

Procedure:

Select ω based on available motor. Determine the dimensions of the orifice in the hopper which delivers approximately 10 lb/min (assuming continuous flow of material). For trial dimensions of the wedge, determine the increase/decrease in the size (diameter) of the orifice. Choose the final dimensions of the wedge such that the material flow rate delivered by the orifice lies within the specified range.

1.78 Force to be applied = 200 lb, frequency = 50 Hz = 314.16 rad/sec.

Procedure:

1. Select a motor that provides, either directly or through a gear system, the desired frequency. Assume that it is connected to the cam.
 2. Determine the sizes and dimensions of the plate cam and the roller.
 3. Choose the dimensions of the follower.
 4. Select the weight as 200 lb. From the geometry, determine the range of displacement (vertical motion) of the weight.
 5. Determine the force exerted due to the falling weight.
-

1.79

Considerations to be taken in the design of vibratory bowl feeders:

1. Suitable design of the electromagnet and its coil.
2. Radius of the bowl and the pitch of the spiral (helical) delivery track.
3. Tooling to be fixed along the spiral track to reject the defective or out-of-tolerance or incorrectly oriented parts.
4. Design of elastic supports.
5. Size and location of the outlet.

1.80

Axial spring constant of each tube $= k = \frac{A E}{\ell}$.

Let diameter of each tube be 0.01 m (1 cm) with thickness 0.001 m (1 mm). Then

$$A = \frac{\pi}{4} (D^2 - d^2) = \frac{\pi}{4} (0.01^2 - 0.008^2) = 28.27 (10^{-6}) \text{ m}^2$$

This gives

$$k = \frac{(28.27 (10^{-6})) (2.07 (10^{11}))}{2} = 29.26 (10^5) \text{ N/m}$$

Since 76 tubes are in parallel, we have the total axial stiffness as:

$$k_{eq} = 76 k = (76) (29.26 (10^5)) = 222.38 (10^6) \text{ N/m}$$

The polar area moment of inertia of each tube is

$$J = \frac{\pi}{32} (D^4 - d^4) = \frac{\pi}{32} (0.01^4 - 0.008^4) = 580 (10^{-8}) \text{ m}^4$$

Torsional stiffness of each tube is given by

$$\frac{G J}{\ell} = \frac{(79.6154 (10^9)) (580 (10^{-8}))}{2} = 231 (10^3) \text{ N-m/rad}$$

For 76 tubes in parallel, equivalent torsional stiffness will be:

$$k_{teq} = (76) (231 (10^3)) = 17.56 (10^6) \text{ N-m/rad}$$

Chapter 2

Free Vibration of Single Degree of Freedom Systems

2.1 $\delta_{st} = 5 \times 10^{-3} \text{ m}$
 $\omega_n = \left(\frac{g}{\delta_{st}} \right)^{1/2} = \left(\frac{9.81}{5 \times 10^{-3}} \right)^{1/2} = 44.2945 \text{ rad/sec} = 7.0497 \text{ Hz}$

2.2 $\tau_n = 0.21 \text{ sec} = 2\pi \sqrt{\frac{m}{k}}$, $\sqrt{m} = 0.21 \sqrt{k} / 2\pi$

(i) $(\tau_n)_{\text{new}} = \frac{2\pi \sqrt{m}}{\sqrt{k_{\text{new}}}} = \frac{2\pi \sqrt{m}}{\sqrt{1.5k}} = \frac{2\pi \left(\frac{0.21 \sqrt{k}}{2\pi} \right)}{\sqrt{1.5k}} = 0.1715 \text{ sec.}$

(ii) $(\tau_n)_{\text{new}} = \frac{2\pi \sqrt{m}}{\sqrt{k_{\text{new}}}} = \frac{2\pi \sqrt{m}}{\sqrt{0.5k}} = 2\pi \left(\frac{0.21 \sqrt{k}}{2\pi} \right) \frac{1}{\sqrt{0.5k}} = 0.2970 \text{ sec.}$

2.3 $\omega_n = 62.832 \text{ rad/sec} = \sqrt{\frac{k}{m}}$, $\sqrt{m} = \sqrt{k} / 62.832$

When spring constant is reduced, ω_n decreases.

$(\omega_n)_{\text{new}} = 0.55 \omega_n = 34.5576 \text{ rad/sec} = \sqrt{\frac{k_{\text{new}}}{m_{\text{new}}}} = \sqrt{\frac{k-800}{m}}$

$\sqrt{\frac{k-800}{k}} \times 62.836 = 34.5576$, $\sqrt{\frac{k-800}{k}} = 0.55$

$\frac{k-800}{k} = (0.55)^2 = 0.3025$

$k = 1146.9534 \text{ N/m}$

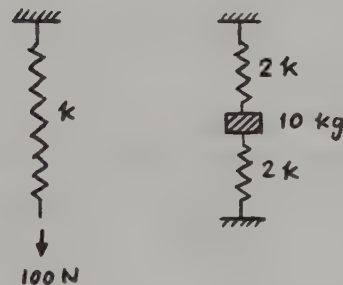
$\sqrt{m} = \sqrt{k} / 62.832$; $m = k / 62.832^2 = \frac{1146.9534}{3947.8602}$

$m = 0.2905 \text{ kg}$

2.4 $k = 100 / \left(\frac{10}{1000} \right) = 10000 \text{ N/m}$

$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m}} = \sqrt{\frac{4k}{m}} = \left(\frac{4 \times 10^4}{10} \right)^{1/2}$
 $= 63.2456 \text{ rad/sec}$

$\tau_n = \frac{2\pi}{\omega_n} = \frac{6.2832}{63.2456} = 0.0993 \text{ sec}$



2.5 $m = \frac{2000}{386.4}$
 Let $\omega_n = 7.5 \text{ rad/sec}$.

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

$$k_{eq} = m \omega_n^2 = \left(\frac{2000}{386.4} \right) (7.5)^2 = 291.1491 \text{ lb/in} = 4 k$$

where k is the stiffness of the air spring.

$$\text{Thus } k = \frac{291.1491}{4} = 72.7873 \text{ lb/in.}$$

2.6 $x = A \cos(\omega_n t - \phi_0)$, $\dot{x} = -\omega_n A \sin(\omega_n t - \phi_0)$,
 $\ddot{x} = -\omega_n^2 A \cos(\omega_n t - \phi_0)$

(a) $\omega_n A = 0.1 \text{ m/sec}$; $\tau_n = \frac{2\pi}{\omega_n} = 2 \text{ sec}$, $\omega_n = 3.1416 \text{ rad/sec}$

$$A = 0.1 / \omega_n = 0.03183 \text{ m}$$

(d) $x_0 = x(t=0) = A \cos(-\phi_0) = 0.02 \text{ m}$

$$\cos(-\phi_0) = \frac{0.02}{A} = 0.6283$$

$$\phi_0 = 51.0724^\circ$$

(b) $\dot{x}_0 = \dot{x}(t=0) = -\omega_n A \sin(-\phi_0) = -0.1 \sin(-51.0724^\circ)$
 $= 0.07779 \text{ m/sec}$

(c) $\ddot{x}|_{\max} = \omega_n^2 A = (3.1416)^2 (0.03183) = 0.314151 \text{ m/sec}^2$

2.7 For small angular rotation of bar PQ about P,

$$\frac{1}{2} (k_{12})_{eq} (\theta l_3)^2 = \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_2)^2$$

$$\text{i.e., } (k_{12})_{eq} = (k_1 l_1^2 + k_2 l_2^2) / l_3^2$$

Let k_{eq} = overall spring constant at Q.

$$\frac{1}{k_{eq}} = \frac{1}{(k_{12})_{eq}} + \frac{1}{k_3}$$

$$k_{eq} = \frac{(k_{12})_{eq} k_3}{(k_{12})_{eq} + k_3} = \frac{\left\{ k_1 \left(\frac{l_1}{l_3} \right)^2 + k_2 \left(\frac{l_2}{l_3} \right)^2 \right\} k_3}{k_1 \left(\frac{l_1}{l_3} \right)^2 + k_2 \left(\frac{l_2}{l_3} \right)^2 + k_3}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 l_1^2 + k_2 k_3 l_2^2}{m (k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$$

2.8 $m = 2000 \text{ kg}$, $\delta_{st} = 0.02 \text{ m}$

$$\omega_n = (g/\delta_{st})^{1/2} = \left(\frac{9.81}{0.02}\right)^{1/2} = 22.1472 \text{ rad/sec}$$

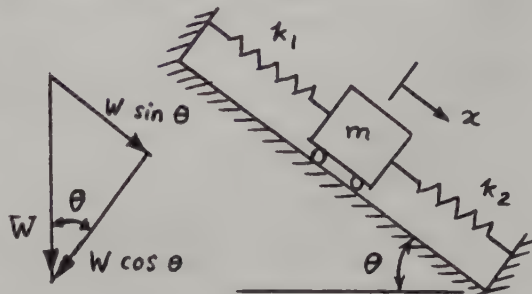
- 2.9 Let x be measured from the position of mass at which the springs are unstretched.

Equation of motion is

$$m \ddot{x} = -k_1(x + \delta_{st}) - k_2(x + \delta_{st}) + W \sin \theta \quad \text{--- (E}_1\text{)}$$

where $\delta_{st} (k_1 + k_2) = W \sin \theta$.

Thus Eq. (E₁) becomes $m \ddot{x} + (k_1 + k_2) x = 0 \Rightarrow \omega_n = \sqrt{\frac{k_1 + k_2}{m}}$.



2.10 $k_1 = \frac{A_1 E_1}{\ell_1} = \frac{\frac{\pi}{4} (0.05)^2 (30 \cdot 10^6)}{30 (12)}$
 $= 163.6250 \text{ lb/in}$

$$k_2 = \frac{A_2 E_2}{\ell_2} = \frac{163.625 (25)}{30} = 136.3542 \text{ lb/in}$$

$$k_{eq} = k_1 + k_2 = 163.6250 + 136.3542 = 299.9792 \text{ lb/in}$$

Let x be measured from the unstretched length of the springs. The equation of motion is:

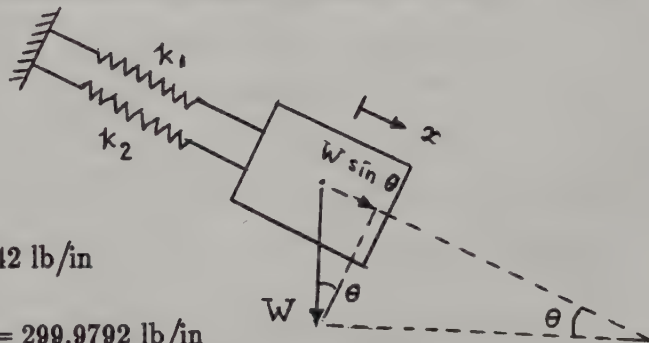
$$m \ddot{x} = -(k_1 + k_2) (x + \delta_{st}) + W \sin \theta$$

where $(k_1 + k_2) \delta_{st} = W \sin \theta$

$$\text{i.e., } m \ddot{x} + (k_1 + k_2) x = 0$$

Thus the natural frequency of vibration of the cart is given by

$$\omega_n = \sqrt{\frac{k_1 + k_2}{m}} = \sqrt{\frac{299.9792 (386.4)}{5000}} = 4.8148 \text{ rad/sec}$$



- 2.11 Weight of electronic chassis = 500 N. To be able to use the unit in a vibratory environment with a frequency range of 0 - 5 Hz, its natural frequency must be away from the frequency of the environment. Let the natural frequency be $\omega_n = 10 \text{ Hz} = 62.832 \text{ rad/sec}$. Since

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = 62.832$$

we have

$$k_{eq} = m \omega_n^2 = \left(\frac{500}{9.81} \right) (62.832)^2 = 20.1857 (10^4) \text{ N/m} \equiv 4 \text{ k}$$

so that k = spring constant of each spring = 50,464.25 N/m. For a helical spring,

$$k = \frac{G d^4}{8 n D^3}$$

Assuming the material of springs as steel with $G = 80 (10^9) \text{ Pa}$, $n = 5$ and $d = 0.005 \text{ m}$, we find

$$k = 50,464.25 = \frac{80 (10^9) (0.005)^4}{8 (5) D^3}$$

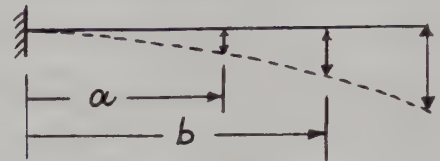
This gives

$$D^3 = \frac{1250 (10^{-3})}{50464.25} = 24,770.0 (10^{-9}) \text{ or } D = 0.0291492 \text{ m} = 2.91492 \text{ cm}$$

2.12

(i) with springs k_1 and k_2 :

Let y_a, y_b, y_l be deflections of beam at distances a, b, l from fixed end.



$$\frac{1}{2} (k_{12})_{eq} y_l^2 = \frac{1}{2} k_1 y_a^2 + \frac{1}{2} k_2 y_b^2$$

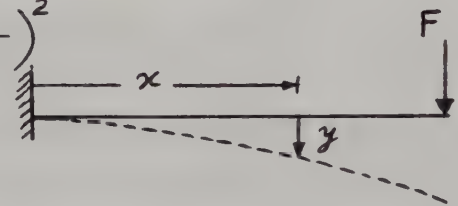
$$\text{i.e., } (k_{12})_{eq} = k_1 \left(\frac{y_a}{y_l} \right)^2 + k_2 \left(\frac{y_b}{y_l} \right)^2$$

$$y = \frac{F x^2}{6EI} (3l - x)$$

$$@ x = a, \quad y_a = \frac{F a^2}{6EI} (3l - a)$$

$$@ x = b, \quad y_b = \frac{F b^2}{6EI} (3l - b)$$

$$@ x = l, \quad y_l = \frac{F l^3}{3EI}$$



$$\omega_n = \left[\frac{k_1 k_3 \left(\frac{y_a}{y_l} \right)^2 + k_2 k_3 \left(\frac{y_b}{y_l} \right)^2}{m \left\{ k_1 \left(\frac{y_a}{y_l} \right)^2 + k_2 \left(\frac{y_b}{y_l} \right)^2 + k_{beam} \right\}} \right]^{\frac{1}{2}} \quad \text{where } k_{beam} = \frac{3EI}{l^3}$$

$$= \left[\frac{k_1 (3EI) a^4 (3l - a)^2 + k_2 (3EI) b^4 (3l - b)^2}{m l^3 \left\{ k_1 a^4 (3l - a)^2 + k_2 b^4 (3l - b)^2 + 12 EI l^3 \right\}} \right]^{\frac{1}{2}}$$

(ii) without springs k_1 and k_2 :

$$\omega_n = \sqrt{\frac{k_{beam}}{m}} = \sqrt{\frac{3EI}{m l^3}}$$

2.13 Let x_1, x_2 = displacements of pulleys 1, 2

$$x = 2x_1 + 2x_2 \quad \text{--- (E}_1\text{)}$$

Let P = tension in rope.

For equilibrium of pulley 1,

$$2P = k_1 x_1 \quad \text{--- (E}_2\text{)}$$

For equilibrium of pulley 2,

$$2P = k_2 x_2 \quad \text{--- (E}_3\text{)}$$

where $\frac{1}{k_1} = \frac{1}{4k} + \frac{1}{4k} = \frac{1}{2k}$; $k_1 = 2k$

and $k_2 = k + k = 2k$

Combining Eqs. (E₁) to (E₃):

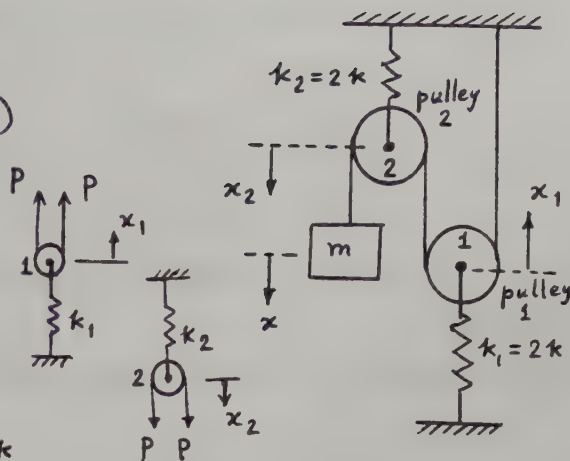
$$x = 2x_1 + 2x_2 = 2\left(\frac{2P}{k_1}\right) + 2\left(\frac{2P}{k_2}\right) = 4P\left(\frac{1}{2k} + \frac{1}{2k}\right) = \frac{4P}{k}$$

Let k_{eq} = equivalent spring constant of the system:

$$k_{eq} = \frac{P}{x} = \frac{k}{4}$$

Equation of motion of mass m : $m\ddot{x} + k_{eq}x = 0$

$$\therefore \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{4m}}$$



2.14 For a displacement of x of mass m , pulleys 1, 2 and 3 undergo displacements of $2x$, $4x$ and $8x$, respectively. The equation of motion of mass m can be written as

$$m\ddot{x} + F_0 = 0 \quad (1)$$

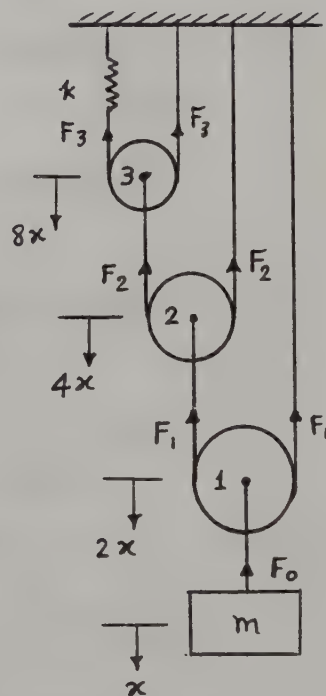
where $F_0 = 2F_1 = 4F_2 = 8F_3$ as shown in figure.

Since $F_3 = (8x)k$, Eq. (1) can be rewritten as

$$m\ddot{x} + 8F_3 = 8(8kx) = 0 \quad (2)$$

from which we can find

$$\omega_n = \sqrt{\frac{64k}{m}} = 8\sqrt{\frac{k}{m}} \quad (3)$$



2.15

$$(a) \quad \omega_n = \sqrt{4k/M}$$

$$(b) \quad \omega_n = \sqrt{4k/(M+m)}$$

Initial conditions:

$$\text{velocity of falling mass } m = v = \sqrt{2gl} \quad (\because v^2 - \overset{\uparrow}{u}^2 = 2gl)$$

$x=0$ at static equilibrium position.

$$x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{mg}{4k}$$

$$\text{Conservation of momentum: } (M+m) \dot{x}_0 = m v = m \sqrt{2gl}$$

$$\dot{x}_0 = \dot{x}(t=0) = \frac{m}{M+m} \sqrt{2gl}$$

$$\text{Complete solution: } x(t) = A_0 \sin(\omega_n t + \phi_0)$$

$$\text{where } A_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = \sqrt{\frac{m^2 g^2}{16 k^2} + \frac{m^2 gl}{2k(M+m)}}$$

$$\text{and } \phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(\frac{-\sqrt{g}}{\sqrt{8l k (M+m)}}\right)$$

2.16

(a) Velocity of anvil = $v = 50 \text{ ft/sec} = 600 \text{ in/sec}$. $x = 0$ at static equilibrium position.

$$x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{mg}{4k}$$

Conservation of momentum:

$$(M+m) \dot{x}_0 = m v \quad \text{or} \quad \dot{x}_0 = \dot{x}(t=0) = \frac{m v}{M+m}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{4k}{M+m}}$$

Complete solution:

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{m^2 g^2}{16 k^2} + \frac{m^2 v^2}{(M+m) 4 k} \right\}^{\frac{1}{2}}$$

and

$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(-\frac{mg}{4k} \sqrt{\frac{4k}{(M+m)}} \frac{(M+m)}{m v}\right) = \tan^{-1}\left(-\frac{g \sqrt{M+m}}{v \sqrt{4k}}\right)$$

Since $v = 600$, $m = 12/386.4$, $M = 100/386.4$, $k = 100$, we find

$$A_0 = \left\{ \left(\frac{12 (386.4)}{4 (100) (386.4)} \right)^2 + \left(\frac{12 (600)}{386.4} \right)^2 \frac{386.4}{112 (400)} \right\}^{\frac{1}{2}} = 1.7308 \text{ in}$$

$$\phi_0 = \tan^{-1} \left(- \frac{386.4 \sqrt{112}}{\sqrt{386.4} (600) \sqrt{400}} \right) = \tan^{-1} (-0.01734) = -0.9934 \text{ deg}$$

(b) $x = 0$ at static equilibrium position: $x_0 = x(t=0) = 0$. Conservation of momentum gives:

$$M \dot{x}_0 = m v \quad \text{or} \quad \dot{x}_0 = \dot{x}(t=0) = \frac{m v}{M}$$

Complete solution:

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{m^2 v^2 (M)}{M^2 4 k} \right\}^{\frac{1}{2}} = \frac{m v}{\sqrt{4 k M}} = \frac{12 (600) \sqrt{386.4}}{386.4 \sqrt{4 (100) (100)}} = 1.8314 \text{ in}$$

$$\phi_0 = \tan^{-1} \left(\frac{x_0 \omega_n}{\dot{x}_0} \right) = \tan^{-1} (0) = 0$$

$$(2.17) \quad k_1 = \frac{3 E_1 I_1}{l_1^3} \quad (\text{at tip}) ; \quad k_2 = \frac{48 E_2 I_2}{l_2^3} \quad (\text{at middle})$$

$$k_{eq} = k_1 + k_2$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\left(\frac{3 E_1 I_1}{l_1^3} + \frac{48 E_2 I_2}{l_2^3} \right) \frac{g}{W}}$$

$$(2.18) \quad k = \frac{AE}{l} = \frac{\left\{ \frac{\pi}{4} (0.01)^2 \right\} \{ 2.07 \times 10^{11} \}}{20} = 0.8129 \times 10^6 \text{ N/m}$$

$$m = 1000 \text{ kg}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{0.8129 \times 10^6}{1000} \right)^{\frac{1}{2}} = 28.5114 \text{ rad/sec}$$

$$\dot{x}_0 = 2 \text{ m/s}, \quad x_0 = 0 \quad (\text{suddenly stopped while it has velocity})$$

$$\text{Period of ensuing vibration} = \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{28.5114} = 0.2204 \text{ sec}$$

$$\text{Amplitude} = A = \dot{x}_0 / \omega_n = 2 / 28.5114 = 0.07015 \text{ m}$$

$$(2.19) \quad \omega_n = 2 \text{ Hz} = 12.5664 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

$$\sqrt{k} = 12.5664 \sqrt{m}$$

$$\omega'_n = \sqrt{\frac{k'}{m'}} = \sqrt{\frac{k}{m+1}} = 6.2832 \text{ rad/sec}$$

$$\begin{aligned}\sqrt{k} &= 6.2832 \sqrt{m+1} \\ &= 12.5664 \sqrt{m}\end{aligned}$$

$$\sqrt{m+1} = 2 \sqrt{m}, \quad m = \frac{1}{3} \text{ kg}$$

$$k = (12.5664)^2 m = 52.6381 \text{ N/m}$$

(2.20) Cable stiffness = $k = \frac{A E}{\ell} = \frac{1}{4} \left(\frac{\pi}{4} (0.01)^2 \right) 2.07 (10^{11}) = 4.0644 (10^8) \text{ N/m}$

$$\tau_n = 0.1 = \frac{1}{f_n} = \frac{2 \pi}{\omega_n}$$

$$\omega_n = \frac{2 \pi}{0.1} = 20 \pi = \sqrt{\frac{k}{m}}$$

Hence

$$m = \frac{k}{\omega_n^2} = \frac{4.0644 (10^8)}{(20 \pi)^2} = 1029.53 \text{ kg}$$

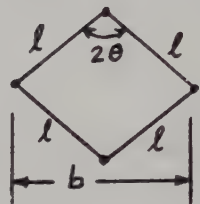
(2.21) $b = 2 \ell \sin \theta$
Neglect masses of links.

$$\begin{aligned}(a) \quad k_{eq} &= k \left(\frac{4 \ell^2 - b^2}{b^2} \right) = k \left(\frac{4 \ell^2 - 4 \ell^2 \sin^2 \theta}{4 \ell^2 \sin^2 \theta} \right) \\ &= k \left(\frac{\cos^2 \theta}{\sin^2 \theta} \right)\end{aligned}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k g \operatorname{cosec}^2 \theta}{W}}$$

(from solution of problem 1.8)

$$(b) \quad \omega_n = \sqrt{\frac{k g}{W}} \quad \text{since } k_{eq} = k.$$



(2.22)
$$\begin{aligned}y &= \sqrt{\ell^2 - (\ell \sin \theta - x)^2} - \ell \cos \theta = \sqrt{\ell^2 \cos^2 \theta - x^2 + 2 \ell x \sin \theta} - \ell \cos \theta \\ &= \ell \cos \theta \sqrt{1 - \frac{x^2}{\ell^2 \cos^2 \theta} + \frac{2 \ell x \sin \theta}{\ell^2 \cos^2 \theta}} - \ell \cos \theta \\ \frac{1}{2} k_{eq} x^2 &= \frac{1}{2} k_1 y^2 + \frac{1}{2} k_2 y^2\end{aligned}$$

where

$$\begin{aligned}y &\approx \ell \cos \theta \left(1 - \frac{1}{2} \frac{x^2}{\ell^2 \cos^2 \theta} + \frac{1}{2} \frac{2 \ell x \sin \theta}{\ell^2 \cos^2 \theta} \right) - \ell \cos \theta \\ &\approx \frac{x \sin \theta}{\cos \theta} = x \tan \theta\end{aligned}$$

Thus k_{eq} can be expressed as

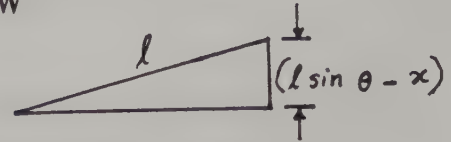
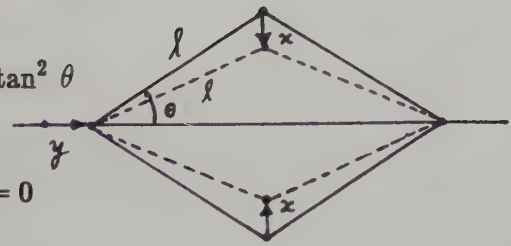
$$k_{eq} = (k_1 + k_2) \tan^2 \theta$$

Equation of motion:

$$m \ddot{x} + k_{eq} x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{(k_1 + k_2) g}{W}} \tan \theta$$



- 2.23 (a) Neglect masses of rigid links. Let x = displacement of W . Springs are in series.

$$k_{eq} = \frac{k}{2}$$

Equation of motion:

$$m \ddot{x} + k_{eq} x = 0$$

Natural frequency:

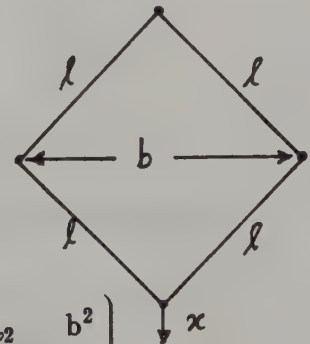
$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{2m}}$$

- (b) Under a displacement of x of mass, each spring will be compressed by an amount:

$$x_s = x \frac{2}{b} \sqrt{\ell^2 - \frac{b^2}{4}}$$

Equivalent spring constant:

$$\begin{aligned} \frac{1}{2} k_{eq} x^2 &= 2 \left(\frac{1}{2} k x_s^2 \right) \\ \text{or } k_{eq} &= 2 k \left(\frac{x_s}{x} \right)^2 = 2 k \left(\frac{2}{b} \sqrt{\ell^2 - \frac{b^2}{4}} \right)^2 = \frac{8 k}{b^2} \left(\ell^2 - \frac{b^2}{4} \right) \end{aligned}$$



Equation of motion:

$$m \ddot{x} + k_{eq} x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{8 k}{b^2 m} \left(\ell^2 - \frac{b^2}{4} \right)}$$

2.24

$$F_1 = F_3 = k_1 \times \cos 45^\circ$$

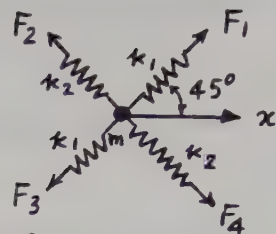
$$F_2 = F_4 = k_2 \times \cos 135^\circ$$

$$F = \text{force along } x = F_1 \cos 45^\circ + F_2 \cos 135^\circ + F_3 \cos 45^\circ + F_4 \cos 135^\circ$$

$$= 2x (k_1 \cos^2 45^\circ + k_2 \cos^2 135^\circ)$$

$$k_{eq} = \frac{F}{x} = 2 \left(\frac{k_1}{2} + \frac{k_2}{2} \right) = k_1 + k_2$$

$$\text{Equation of motion: } m \ddot{x} + (k_1 + k_2) x = 0$$



2.25

Let α_i denote the angle made by i^{th} spring with respect to X axis.

Let x = displacement of mass along the direction defined by θ .

If k_{eq} = equivalent spring constant, the equivalence of potential energies gives

$$\frac{1}{2} k_{eq} x^2 = \frac{1}{2} \sum_{i=1}^6 k_i \{x \cos(\theta - \alpha_i)\}^2$$

$$k_{eq} = \sum_{i=1}^6 k_i \cos^2(\theta - \alpha_i) = \sum_{i=1}^6 k_i (\cos \theta \cos \alpha_i + \sin \theta \sin \alpha_i)^2$$

$$= \sum_{i=1}^6 k_i (\cos^2 \alpha_i \cos^2 \theta + \sin^2 \alpha_i \sin^2 \theta)$$

$$+ 2 \sum_{i=1}^6 (\cos \alpha_i \sin \alpha_i \cos \theta \sin \theta)$$

$$\text{Natural frequency} = \omega_n = \sqrt{\frac{k_{eq}}{m}}$$

$$\text{For } \omega_n \text{ to be independent of } \theta, \quad \sum_{i=1}^6 k_i \cos^2 \alpha_i = \sum_{i=1}^6 k_i \sin^2 \alpha_i \quad (E_1)$$

$$\text{and} \quad \sum_{i=1}^6 k_i \cos \alpha_i \sin \alpha_i = 0 \quad (E_2)$$

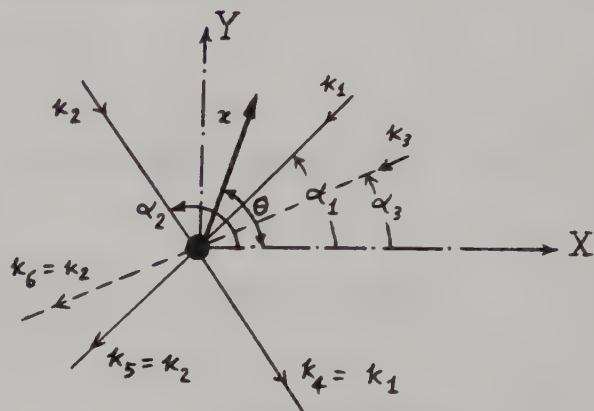
(E₁) and (E₂) can be rewritten as

$$\sum_{i=1}^6 k_i \left(\frac{1}{2} + \frac{1}{2} \cos 2\alpha_i \right) = \sum_{i=1}^6 k_i \left(\frac{1}{2} - \frac{1}{2} \cos 2\alpha_i \right)$$

$$\text{and} \quad \frac{1}{2} \sum_{i=1}^6 k_i \sin 2\alpha_i = 0$$

$$\text{i.e.} \quad \sum_{i=1}^6 k_i \cos 2\alpha_i = 0 \quad \dots (E_3)$$

$$\text{and} \quad \sum_{i=1}^6 k_i \sin 2\alpha_i = 0 \quad \dots (E_4)$$



In the present example, (E_3) and (E_4) become

$$k_1 \cos 60^\circ + k_2 \cos 240^\circ + k_3 \cos 2\alpha_3 + k_1 \cos 420^\circ + k_2 \cos 600^\circ + k_3 \cos (360^\circ + 2\alpha_3) = 0$$

$$k_1 \sin 60^\circ + k_2 \sin 240^\circ + k_3 \sin 2\alpha_3 + k_1 \sin 420^\circ + k_2 \sin 600^\circ + k_3 \sin (360^\circ + 2\alpha_3) = 0$$

$$\text{i.e., } \left. \begin{aligned} k_1 - k_2 + 2k_3 \cos 2\alpha_3 &= 0 \\ \sqrt{3} k_1 - \sqrt{3} k_2 + 2k_3 \sin 2\alpha_3 &= 0 \end{aligned} \right\}; \quad \begin{aligned} 2k_3 \cos 2\alpha_3 &= k_2 - k_1 \dots (E_5) \\ 2k_3 \sin 2\alpha_3 &= \sqrt{3}(k_2 - k_1) \dots (E_6) \end{aligned}$$

Squaring (E_5) and (E_6) and adding,

$$4k_3^2 = (k_2 - k_1)^2 (1 + 3)$$

$$\therefore k_3 = \pm (k_2 - k_1) \Rightarrow k_3 = |k_2 - k_1|$$

Dividing (E_6) by (E_5) ,

$$\tan 2\alpha_3 = \sqrt{3}$$

$$\therefore \alpha_3 = \frac{1}{2} \tan^{-1}(\sqrt{3}) = 30^\circ$$

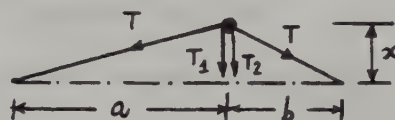
2.26

$$T_1 = \frac{x}{a} T, \quad T_2 = \frac{x}{b} T$$

$$(a) \quad m \ddot{x} + (T_1 + T_2) = 0$$

$$m \ddot{x} + \left(\frac{T}{a} + \frac{T}{b} \right) x = 0$$

$$(b) \quad \omega_n = \sqrt{\frac{\frac{T}{a} + \frac{T}{b}}{m}} = \sqrt{\frac{T}{mab} (a+b)}$$



2.27

$$m = \frac{160}{386.4} \frac{\text{lb-sec}^2}{\text{inch}}, \quad k = 10 \text{ lb/inch.}$$

Velocity of jumper as he falls through 200 ft:

$$mgh = \frac{1}{2} m v^2 \quad \text{or} \quad v = \sqrt{2gh} = \sqrt{2(386.4)(200(12))} = 1,361.8811 \text{ in/sec}$$

About static equilibrium position:

$$x_0 = x(t=0) = 0, \quad \dot{x}_0 = \dot{x}(t=0) = 1,361.8811 \text{ in/sec}$$

Response of jumper:

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = \frac{\dot{x}_0}{\omega_n} = \frac{\dot{x}_0 \sqrt{m}}{\sqrt{k}} = \frac{1361.8811}{\sqrt{10}} \sqrt{\frac{160}{386.4}} = 277.1281 \text{ in}$$

and

$$\phi_0 = \tan^{-1} \left(\frac{x_0 \omega_n}{\dot{x}_0} \right) = 0$$

2.28

The natural frequency of a vibrating rope is given by (see Problem 2.26):

$$\omega_n = \sqrt{\frac{T(a+b)}{mab}}$$

where T = tension in rope, m = mass, and a and b are lengths of the rope on both sides of the mass. For the given data:

$$10 = \left\{ \frac{T(80+160)}{\left(\frac{120}{386.4}\right)(80)(160)} \right\}^{\frac{1}{2}} = \sqrt{T(0.060375)}$$

which yields

$$T = \frac{100}{0.060375} = 1,656.3147 \text{ lb}$$

2.29

When $\omega = 0$, total
vertical height $= 2l + h$

When $\omega \neq 0$, total
vertical height $= (2l \cos \theta + h)$

$$\begin{aligned} \text{spring force} &= k[2l + h - (2l \cos \theta + h)] \\ &= 2kl(1 - \cos \theta) \end{aligned}$$

For vertical equilibrium of mass m ,

$$mg + T_2 \cos \theta = T_1 \cos \theta \quad \dots (E_1)$$

For horizontal equilibrium, $F_c = (T_1 + T_2) \sin \theta$

$$T_2 = (F_c - T_1 \sin \theta) / \sin \theta \quad \dots (E_2)$$

From (E_2) , (E_1) can be expressed as

$$mg + \left(\frac{F_c - T_1 \sin \theta}{\sin \theta} \right) \cos \theta = T_1 \cos \theta$$

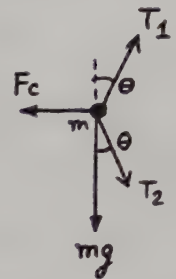
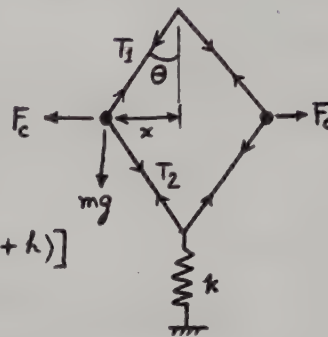
$$\text{i.e. } T_1 = \frac{mg + F_c \cot \theta}{2 \cos \theta} = \frac{mg + m\omega^2 l \cos \theta}{2 \cos \theta}$$

$$T_2 = \frac{F_c - T_1 \sin \theta}{\sin \theta} = \frac{m\omega^2 l - \frac{mg \tan \theta}{2} - \frac{m\omega^2 l}{2} \sin \theta}{\sin \theta}$$

$$= \frac{m l \omega^2}{2} - \frac{mg}{2 \cos \theta}$$

$$\begin{aligned} \text{spring force} &= 2kl(1 - \cos \theta) = 2T_2 \cos \theta \\ &= m l \omega^2 \cos \theta - mg \end{aligned}$$

$$\cos \theta = \left(\frac{2kl + mg}{2kl + m l \omega^2} \right) \quad \dots (E_3)$$



$$F_c = m \omega^2 x$$

$$x = l \sin \theta$$

This equation defines the equilibrium position of mass m .
For small oscillations about the equilibrium position,
Newton's second law gives

$$2m \ddot{y} + k y = 0, \quad \omega_n = \sqrt{\frac{2k}{m}}$$

2.30

- (a) Let P = total spring force, F = centrifugal force acting on each ball. Equilibrium of moments about the pivot of bell crank lever (O) gives:

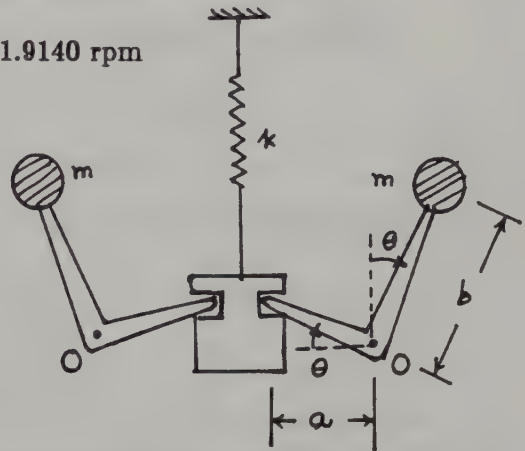
$$F \left(\frac{20}{100} \right) = \frac{P}{2} \left(\frac{12}{100} \right) \quad (1)$$

When $P = 10^4 \left(\frac{1}{100} \right) = 100 \text{ N}$, and

$$F = m r \omega^2 = m r \left(\frac{2 \pi N}{60} \right)^2 = \frac{25}{9.81} \left(\frac{16}{100} \right) \left(\frac{2 \pi N}{60} \right)^2 = 0.004471 N^2$$

where N = speed of the governor in rpm. Equation (1) gives:

$$0.004471 N^2 (0.2) = \frac{100}{2} (0.12) \quad \text{or} \quad N = 81.9140 \text{ rpm}$$



- (b) Consider a small displacement of the ball arm about the vertical position. Equilibrium about point O gives:

$$(m b^2) \ddot{\theta} + (k a \sin \theta) a \cos \theta = 0 \quad (2)$$

For small values of θ , $\sin \theta \approx \theta$ and $\cos \theta \approx 1$, and hence Eq. (2) gives

$$m b^2 \ddot{\theta} + k a^2 \theta = 0$$

from which the natural frequency can be determined as

$$\omega_n = \left(\frac{k a^2}{m b^2} \right)^{\frac{1}{2}} = \left((10)^4 \left(\frac{0.12}{0.20} \right)^2 \frac{9.81}{25} \right)^{\frac{1}{2}} = 37.5851 \text{ rad/sec}$$

2.31

$$SO' = \frac{a}{\sqrt{2}}, \quad OO' = h, \quad OS = \sqrt{h^2 + \frac{a^2}{2}}$$

When each wire stretches by x_s , let the resulting vertical displacement of the platform be x .

$$OS + x_s = \sqrt{(h+x)^2 + \frac{a^2}{2}}$$

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left\{ \frac{\sqrt{(h+x)^2 + \frac{a^2}{2}}}{\sqrt{h^2 + \frac{a^2}{2}}} - 1 \right\}$$

$$= \sqrt{h^2 + \frac{a^2}{2}} \left[\sqrt{1 + \left\{ \frac{2hx + x^2}{(h^2 + \frac{a^2}{2})} \right\}} - 1 \right]$$

For small x , x^2 is negligible compared to $2hx$ and $\sqrt{1+\theta} \approx 1 + \frac{\theta}{2}$ and hence

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left[1 + \frac{hx}{(h^2 + \frac{a^2}{2})} - 1 \right] = \frac{h}{\sqrt{h^2 + \frac{a^2}{2}}} x$$

Potential energy equivalence gives

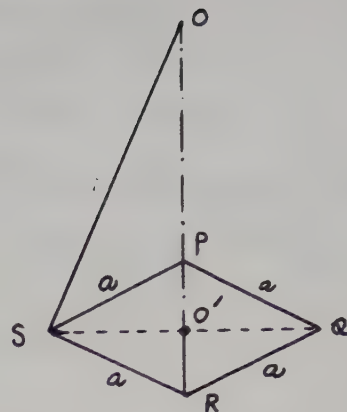
$$\frac{1}{2} k_{eq} x^2 = 4 \left(\frac{1}{2} k x_s^2 \right)$$

$$k_{eq} = 4k \left(\frac{x_s}{x} \right)^2 = \frac{4k h^2}{(h^2 + \frac{a^2}{2})}$$

Equation of motion of M :

$$M \ddot{x} + k_{eq} x = 0$$

$$\gamma_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{(k_{eq}/M)^{1/2}} = \frac{\pi \sqrt{M}}{h} \left(\frac{2h^2 + a^2}{2k} \right)^{1/2}$$



2.32

Equation of motion:

$$m \ddot{x} = \sum F_x$$

$$\text{i.e., } (L A \rho) \ddot{x} = -2 (A x \rho g)$$

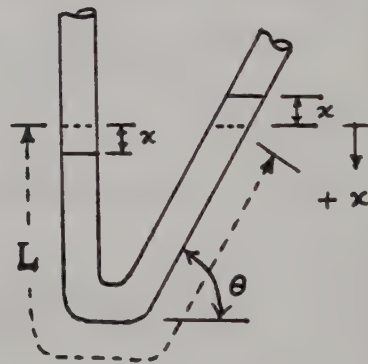
$$\text{i.e., } \ddot{x} + \frac{2g}{L} x = 0$$

where A = cross-sectional area of the tube and

ρ = density of mercury. Thus the

natural frequency is given by:

$$\omega_n = \sqrt{\frac{2g}{L}}$$



2.33

Assume same area of cross section for all segments of the cable. Speed of blades = 300 rpm = 5 Hz = 31.416 rad/sec.

$$\omega_n^2 = \frac{k_{eq}}{m} = (2 (31.416))^2 = (62.832)^2$$

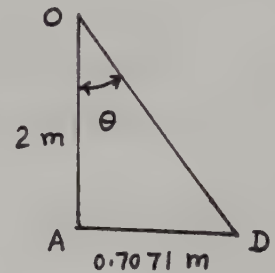
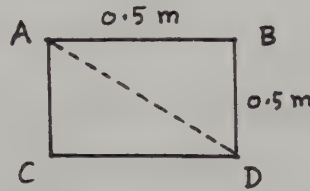
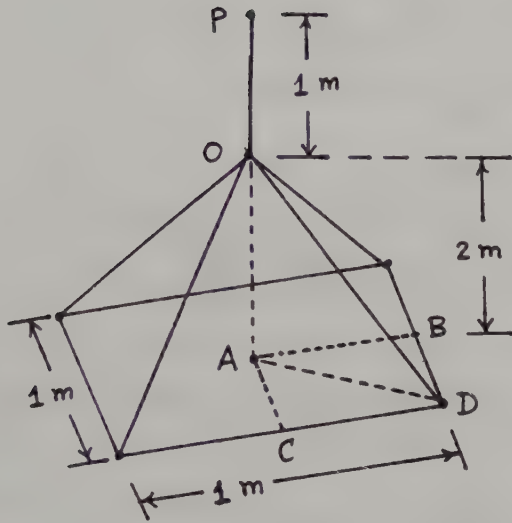
$$k_{eq} = m \omega_n^2 = 250 (62.832)^2 = 98.6965 (10^4) \text{ N/m} \quad (1)$$

$$AD = \sqrt{0.5^2 + 0.5^2} = 0.7071 \text{ m}, \quad OD = \sqrt{2^2 + 0.7071^2} = 2.1213 \text{ m}$$

Stiffness of cable segments:

$$k_{PO} = \frac{AE}{\ell_{PO}} = \frac{A (207) (10^9)}{1} = 207 (10^9) \text{ A N/m}$$

$$K_{OD} = \frac{AE}{\ell_{OD}} = \frac{A (207) (10^9)}{2.1213} = 97.5817 (10^9) \text{ A N/m}$$



$$\theta = \tan^{-1} \left(\frac{0.7071}{2} \right) = 19.4710^\circ$$

The total stiffness of the four inclined cables (k_{ic}) is given by:

$$k_{ic} = 4 k_{OD} \cos^2 \theta = 4 (97.5817) (10^9) \text{ A} \cos^2 19.4710^\circ = 346.9581 (10^9) \text{ A N/m}$$

Equivalent stiffness of vertical and inclined cables is given by:

$$\begin{aligned} \frac{1}{k_{eq}} &= \frac{1}{k_{PO}} + \frac{1}{k_{ic}} \\ \text{i.e., } k_{eq} &= \frac{k_{PO} k_{ic}}{k_{PO} + k_{ic}} \\ &= \frac{(207 (10^9) \text{ A}) (346.9581 (10^9) \text{ A})}{(207 (10^9) \text{ A}) + (346.9581 (10^9) \text{ A})} = 129.6494 (10^9) \text{ A N/m} \end{aligned} \quad (2)$$

Equating k_{eq} given by Eqs. (1) and (2), we obtain the area of cross section of cables as:

$$A = \frac{98.6965 (10^4)}{129.6494 (10^9)} = 7.6126 (10^{-6}) \text{ m}^2$$

2.34

$$\frac{1}{2\pi} \left\{ \frac{k_1}{m} \right\}^{\frac{1}{2}} = 5 ; \quad \frac{k_1}{m} = 4 (\pi)^2 (25) = 986.9651$$

$$\frac{1}{2\pi} \left\{ \frac{k_1}{m + 5000} \right\}^{\frac{1}{2}} = 4.0825 ; \quad \frac{k_1}{m + 5000} = 4 (\pi)^2 (16.6668) = 657.9822$$

Using $k_1 = \frac{A E}{\ell_1}$ we obtain

$$\begin{aligned} \frac{k_1}{m} &= \frac{A E}{\ell_1 m} = \frac{A (207) (10^9)}{2 m} = 986.9651 \\ \text{i.e., } A &= 9.5359 (10^{-9}) m \end{aligned} \quad (1)$$

Also

$$\begin{aligned} \frac{k_1}{m + 5000} &= \frac{A E}{\ell_1 (m + 5000)} = 657.9822 \\ \text{i.e., } \frac{A}{m + 5000} &= 6.3573 (10^{-9}) \end{aligned} \quad (2)$$

Using Eqs. (1) and (2), we obtain

$$\begin{aligned} A &= 9.5359 (10^{-9}) m = 6.3573 (10^{-9}) m + 31.7865 (10^{-6}) \\ \text{i.e., } 3.1786 (10^{-9}) m &= 31.7865 (10^{-6}) \\ \text{i.e., } m &= 10000.1573 \text{ kg} \end{aligned} \quad (3)$$

Equations (1) and (3) yield

$$A = 9.5359 (10^{-9}) m = 9.5359 (10^{-9}) (10000.1573) = 0.9536 (10^{-4}) \text{ m}^2$$

2.35

Longitudinal Vibration:

Let w_1 = part of weight w carried by length a of shaft

$w_2 = W - w_1$ = weight carried by length b

$$\begin{aligned} x &= \text{Elongation of length } a = \frac{w_1 a}{A E} \\ y &= \text{shortening of length } b = \frac{(W - w_1)(l - a)}{A E} \end{aligned} \quad \left. \begin{array}{l} E = \text{Young's modulus} \\ A = \text{area of cross-section} \\ = \pi d^2/4 \end{array} \right\}$$

$$\text{Since } x = y, \quad w_1 = \frac{W(l - a)}{l}$$

$$x = \text{elongation or static deflection of length } a = \frac{W a (l - a)}{A E l}$$

Considering the shaft of length a with end mass w_1/g as a spring-mass system,

$$\omega_n = \sqrt{\frac{g}{x}} = \left(\frac{g l A E}{W a (l - a)} \right)^{1/2}$$

Transverse vibration:

spring constant of a fixed-fixed beam with off-center load

$$= k = \frac{3EI l^3}{a^3 b^3} = \frac{3EI l^3}{a^3 (l-a)^3}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \left\{ \frac{3EI l^3 g}{W a^3 (l-a)^3} \right\}^{1/2} \quad \text{with } I = \left(\frac{\pi d^4}{64} \right) = \text{moment of inertia}$$

Torsional vibration:

If flywheel is given an angular deflection θ , resisting torques offered by lengths a and b are $\frac{GJ\theta}{a}$ and $\frac{GJ\theta}{b}$.

Total resisting torque = $M_t = GJ \left(\frac{1}{a} + \frac{1}{b} \right) \theta$

$$k_t = \frac{M_t}{\theta} = GJ \left(\frac{1}{a} + \frac{1}{b} \right) \quad \text{where } J = \frac{\pi d^4}{32} = \text{polar moment of inertia}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left\{ \frac{GJ}{J_0} \left(\frac{1}{a} + \frac{1}{b} \right) \right\}^{1/2}$$

where J_0 = mass polar moment of inertia of the flywheel.

2.36

$m_{eq_{end}}$ = equivalent mass of a uniform beam at the free end (see Problem 2.38) =

$$\frac{33}{140} m = \frac{33}{140} \left\{ 1 (1) (150 \times 12) \frac{0.283}{386.4} \right\} = 0.3107$$

Stiffness of tower (beam) at free end:

$$k_b = \frac{3EI}{L^3} = \frac{3(30 \times 10^6) \left(\frac{1}{12} (1) (1^3) \right)}{(150 \times 12)^3} = 0.001286 \text{ lb/in}$$

Length of each cable:

$$OA = \sqrt{2} = 1.4142 \text{ ft}, \quad OB = \sqrt{2} \cdot 15 = 21.2132 \text{ ft}, \quad AB = OB - OA = 19.7990 \text{ ft}$$

$$TB = \sqrt{TA^2 + AB^2} = \sqrt{100^2 + 19.7990^2} = 101.9412 \text{ ft}$$

$$\tan \theta = \frac{AT}{AB} = \frac{100}{19.7990} = 5.0508, \quad \theta = 78.8008^\circ$$

Axial stiffness of each cable:

$$k = \frac{AE}{\ell} = \frac{(0.5) (30 \times 10^6)}{(101.9412 \times 12)} = 12261.971 \text{ lb/in}$$

Axial extension of each cable (y_c) due to a horizontal displacement of x of tower:

$$\ell_1^2 = \ell^2 + x^2 - 2 \ell x \cos (180^\circ - \theta) = \ell^2 + x^2 + 2 \ell x \cos \theta$$

$$\text{or } \ell_1 = \ell \left\{ 1 + \left(\frac{x}{\ell} \right)^2 + 2 \frac{x}{\ell} \cos \theta \right\}^{\frac{1}{2}}$$

$$y_c = \ell_1 - \ell \approx \ell \left\{ 1 + \frac{1}{2} \frac{x^2}{\ell^2} + \frac{1}{2} (2) \frac{x}{\ell} \cos \theta \right\} - \ell$$

$$= \ell + x \cos \theta - \ell = x \cos \theta$$

Equivalent stiffness of each cable in horizontal direction:

$$\frac{1}{2} k y_c^2 = \frac{1}{2} k_{eqc} x^2 \quad \text{or} \quad k_{eqc} = k \left(\frac{y_c}{x} \right)^2 = k \cos^2 \theta$$

This gives

$$k_{eqc} = (12261.971) \cos^2 78.8008^\circ = 462.5419 \text{ lb/in}$$

In order to use the relation

$$k_{eqend} = k_b + 4 k_{eqc} \left(\frac{y_{L1}}{y_L} \right)^2$$

we find

$$\frac{y_{L1}}{y_L} = \left(\frac{F L_1^2 (3L - L_1)}{6EI} \frac{3EI}{F L^3} \right) = \frac{L_1^2 (3L - L_1)}{2L^3}$$

$$= \frac{100^2 (3(150) - 100)}{2(150)^3} = 0.5185$$

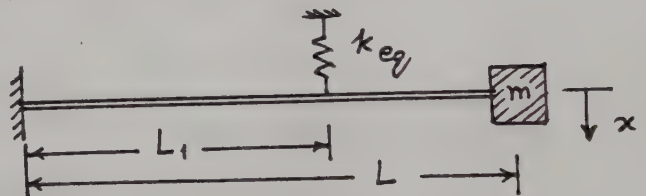
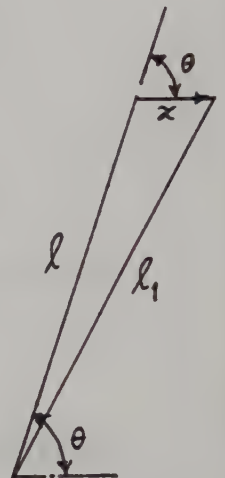
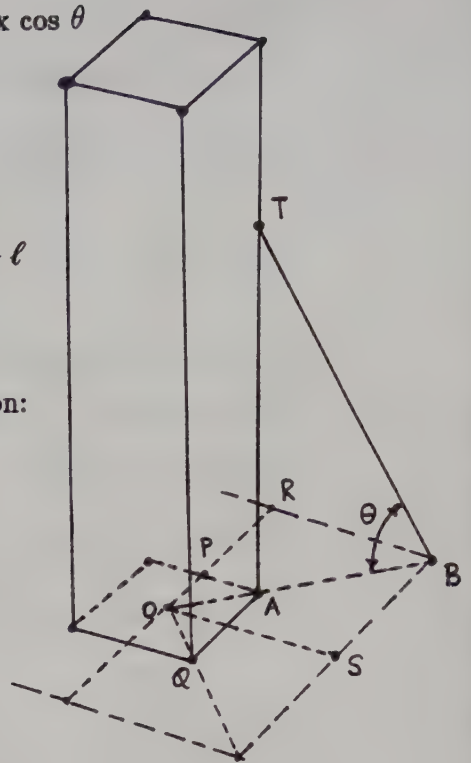
Thus

$$k_{eqend} = k_b + 4 k_{eqc} (0.5185)^2 = 0.001286 + 4 (462.5419) (0.5185)^2$$

$$= 497.4045 \text{ lb/in}$$

Natural frequency:

$$\omega_n = \left\{ \frac{k_{eqend}}{m_{eqend}} \right\}^{\frac{1}{2}} = \left\{ \frac{497.4045}{0.3107} \right\}^{\frac{1}{2}} = 40.0114 \text{ rad/sec}$$



2.37

Sides of the sign:

$$AB = \sqrt{8.8^2 + 8.8^2} = 12.44 \text{ in} ; BC = 30 - 8.8 - 8.8 = 12.4 \text{ in}$$

$$\text{Area} = 30(30) - 4\left(\frac{1}{2}(8.8)(8.8)\right) = 745.12 \text{ in}^2$$

$$\text{Thickness} = \frac{1}{8} \text{ in} ; \text{Weight density of steel} = 0.283 \text{ lb/in}^3$$

$$\text{Weight of sign} = (0.283)\left(\frac{1}{8}\right)(745.12) = 26.64 \text{ lb}$$

$$\text{Weight of sign post} = (72)(2)\left(\frac{1}{4}\right)(0.283) = 10.19 \text{ lb}$$

Stiffness of sign post (cantilever beam):

$$k = \frac{3EI}{\ell^3}$$

Area moments of inertia of the cross section of the sign post:

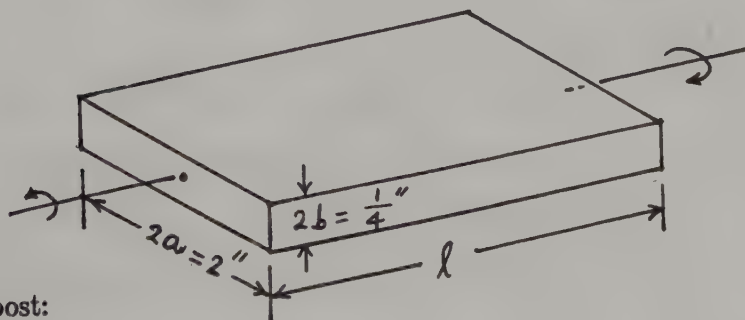
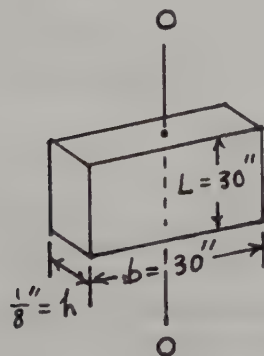
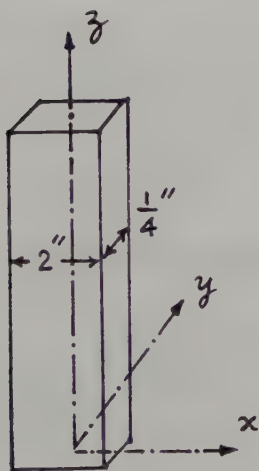
$$I_{xx} = \frac{1}{12}(2)\left(\frac{1}{4}\right)^3 = \frac{1}{384} \text{ in}^4$$

$$I_{yy} = \frac{1}{12}\left(\frac{1}{4}\right)(2)^3 = \frac{1}{6} \text{ in}^4$$

Bending stiffnesses of the sign post:

$$k_{xx} = \frac{3EI_{yy}}{\ell^3} = \frac{3(30(10^6))\left(\frac{1}{6}\right)}{72^3} = 40.1877 \text{ lb/in}$$

$$k_{yy} = \frac{3EI_{xx}}{\ell^3} = \frac{3(30(10^6))\left(\frac{1}{384}\right)}{72^3} = 0.6279 \text{ lb/in}$$



Torsional stiffness of the sign post:

$$k_t = 5.33 \frac{a b^3}{\ell} G \left\{ 1 - 0.63 \frac{b}{a} \left(1 - \frac{b^4}{12 a^4} \right) \right\}$$

(See Ref: N. H. Cook, *Mechanics of Materials for Design*, McGraw-Hill, New York, 1984, p. 342).

Thus

$$k_t = 5.33 \left\{ \frac{(1) \left(\frac{1}{8}\right)^3}{72} \right\} (11.5 (10^6)) \left\{ 1 - (0.63) \left(\frac{1}{8}\right) \left(1 - \frac{\left(\frac{1}{8}\right)^4}{12 (1)^4} \right) \right\}$$

$$= 1531.7938 \text{ lb-in/rad}$$

Natural frequency for bending in xz plane:

$$\omega_{xz} = \left\{ \frac{k_{xz}}{m} \right\}^{\frac{1}{2}} = \left\{ \frac{\frac{40.1877}{26.64}}{386.4} \right\}^{\frac{1}{2}} = 24.1434 \text{ rad/sec}$$

Natural frequency for bending in yz plane:

$$\omega_{yz} = \left\{ \frac{k_{yz}}{m} \right\} = \left\{ \frac{\frac{0.6279}{26.64}}{386.4} \right\}^{\frac{1}{2}} = 3.0178 \text{ rad/sec}$$

By approximating the shape of the sign as a square of side 30 in (instead of an octagon), we can find its mass moment of inertia as:

$$I_{oo} = \frac{\gamma L}{3} (b^3 h + h^3 b) = \left(\frac{0.283}{386.4} \right) \left(\frac{30}{3} \right) \left(30^3 \left(\frac{1}{8} \right) + \left(\frac{1}{8} \right)^3 (30) \right) = 24.7189$$

Natural torsional frequency:

$$\omega_t = \left\{ \frac{k_t}{I_{oo}} \right\}^{\frac{1}{2}} = \left\{ \frac{1531.7938}{24.7189} \right\}^{\frac{1}{2}} = 7.8720 \text{ rad/sec}$$

Thus the mode of vibration (close to resonance) is torsion in xy plane.

(2.38) (a) Pivoted;

Let $l = h$.

$$k_{eq} = 4 \quad k_{column} = 4 \left(\frac{3EI}{l^3} \right) = \frac{12EI}{l^3}$$

Let m_{eff1} = effective mass due to self weight of columns

Equation of motion: $\left(\frac{W}{g} + m_{eff1} \right) \ddot{x} + k_{eq} x = 0$

Natural frequency of horizontal vibration $= \omega_n = \sqrt{\frac{12EI}{l^3 \left(\frac{W}{g} + m_{eff1} \right)}}$

(b) Fixed:

since the joint between column and floor does not permit rotation, each column will bend with inflection point at middle.

When force F is applied at ends,

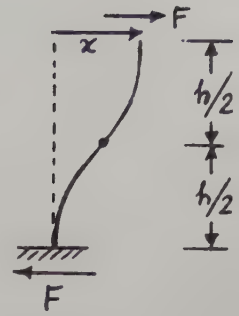
$$x = 2 \frac{F \left(\frac{l}{2}\right)^3}{3EI} = \frac{Fl^3}{12EI}$$

$$K_{\text{column}} = \frac{12EI}{l^3} ; K_{eq} = 4 K_{\text{column}} = \frac{48EI}{l^3}$$

Let m_{eff2} = effective mass of each column at top end

$$\text{Equation of motion: } \left(\frac{W}{g} + m_{eff2}\right) \ddot{x} + K_{eq} x = 0$$

$$\text{Natural frequency of horizontal vibration} = \omega_n = \sqrt{\frac{48EI}{l^3 \left(\frac{W}{g} + m_{eff2}\right)}}$$



Effective mass (due to self weight):

(a) Let m_{eff1} = effective part of mass of beam (m) at end.

Thus vibrating inertia force at end is due to $(M + m_{eff1})$.

Assume deflection shape during vibration same as the static deflection shape with a tip load:

$$y(x,t) = Y(x) \cos(\omega_n t - \phi) \quad \text{where} \quad Y(x) = \frac{F x^2 (3l - x)}{6EI}$$

$$Y(x) = \frac{Y_0}{2l^3} x^2 (3l - x) \quad \text{where} \quad Y_0 = \frac{Fl^3}{3EI} = \text{max. tip deflection}$$

$$y(x,t) = \frac{Y_0}{2l^3} (3x^2 l - x^3) \cos(\omega_n t - \phi) \quad (E_1)$$

Max. strain energy of beam = Max. work by force F

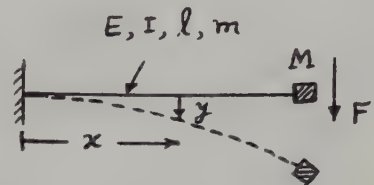
$$= \frac{1}{2} F Y_0 = \frac{3}{2} \frac{EI}{l^3} Y_0^2 \quad (E_2)$$

Max. kinetic energy due to distributed mass of beam

$$= \frac{1}{2} \frac{m}{l} \int_0^l \dot{y}^2(x,t) \Big|_{\max} dx + \frac{1}{2} (\dot{y}_{\max})^2 M$$

$$= \frac{1}{2} \omega_n^2 Y_0^2 \left(\frac{33}{140} m\right) + \frac{1}{2} \omega_n^2 Y_0^2 M \quad (E_3)$$

$$\therefore m_{eff1} = \frac{33}{140} m = 0.2357 m$$



(b) Let $Y(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$
 $Y(0) = 0, \frac{dY}{dx}(0) = 0, Y(l) = Y_0, \frac{dY}{dx}(l) = 0$

This leads to $Y(x) = \frac{3 Y_0}{l^2} x^2 - \frac{2 Y_0}{l^3} x^3$

$y(x, t) = Y_0 \left(3 \frac{x^2}{l^2} - 2 \frac{x^3}{l^3} \right) \cos(\omega_n t - \phi)$ (E4)

Maximum strain energy $= \frac{1}{2} EI \int_0^l \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx \Big|_{\max}$
 $= \frac{6 EI Y_0^2}{l^3}$ (E5)

Max. kinetic energy $= \frac{1}{2} M \omega_n^2 Y_0^2 + \frac{1}{2} \left(\frac{m}{l} \right) Y_0^2 \omega_n^2 \int_0^l \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3} \right)^2 dx$
 $= \frac{1}{2} \omega_n^2 Y_0^2 \left(M + \frac{13}{35} m \right)$ (E6)

$\therefore m_{eff 2} = \frac{13}{35} m = 0.3714 m$

2.39 Stiffnesses of segments:

$A_1 = \frac{\pi}{4} (D_1^2 - d_1^2) = \frac{\pi}{4} (2^2 - 1.75^2) = 0.7363 \text{ in}^2$

$k_1 = \frac{A_1 E_1}{L_1} = \frac{(0.7363) (10^7)}{12} = 61.3583 (10^4) \text{ lb/in}$

$A_2 = \frac{\pi}{4} (D_2^2 - d_2^2) = \frac{\pi}{4} (1.5^2 - 1.25^2) = 0.5400 \text{ in}^2$

$k_2 = \frac{A_2 E_2}{L_2} = \frac{(0.5400) (10^7)}{10} = 54.0 (10^4) \text{ lb/in}$

$A_3 = \frac{\pi}{4} (D_3^2 - d_3^2) = \frac{\pi}{4} (1^2 - 0.75^2) = 0.3436 \text{ in}^2$

$k_3 = \frac{A_3 E_3}{L_3} = \frac{(0.3436) (10^7)}{8} = 42.9516 (10^4) \text{ lb/in}$

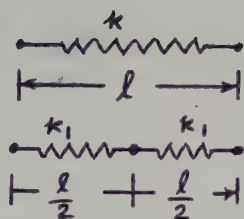
Equivalent stiffness (springs in series):

$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$
 $= 0.0162977 (10^{-4}) + 0.0185185 (10^{-4}) + 0.0232820 (10^{-4}) = 0.0580982 (10^{-4})$
 or $k_{eq} = 17.2122 (10^4) \text{ lb/in}$

Natural frequency:

$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_{eq} g}{W}} = \sqrt{\frac{17.2122 (10^4) (386.4)}{10}} = 2578.9157 \text{ rad/sec}$

2.40



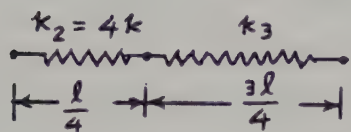
$$\frac{1}{k_{\text{total}}} = \frac{1}{k_1} + \frac{1}{k_1}$$

$$k_{\text{total}} = \frac{k_1}{2} \equiv k; \quad k_1 = 2k$$

$$\tau_n = 2\pi \sqrt{\frac{m}{k_{\text{eq}}}}$$

$$0.5 = 2\pi \sqrt{\frac{m}{4k}}$$

$$\sqrt{\frac{m}{k}} = \frac{1}{2\pi}$$



$$\frac{1}{k_{\text{total}}} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{4k} + \frac{1}{k_3} = \frac{1}{k}$$

$$k_3 = \frac{4}{3}k$$

$$\tau_n = 2\pi \sqrt{\frac{m}{k_{\text{eq}}}} \quad \text{where } k_{\text{eq}} = 4k + \frac{4}{3}k = \frac{16}{3}k$$

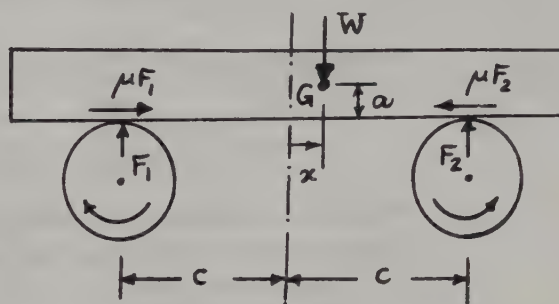
$$\therefore \tau_n = 2\pi \sqrt{\frac{3m}{16k}} = \frac{2\pi\sqrt{3}}{4} \sqrt{\frac{m}{k}} = \frac{2\pi\sqrt{3}}{4} \left(\frac{1}{2\pi}\right) = 0.4330 \text{ sec}$$

2.41

Let μ = coefficient of friction

x = displacement of c.g. of block

F_1, F_2 = net reactions between roller and block on left and right sides.



Reactions at left and right due to static load W are $W(c-x)/2c$ and $W(c+x)/2c$, respectively.

M = moment about G due to motion of block = $(\mu F_2 - \mu F_1)a$

Reactions at left and right to balance $M = \frac{M}{2c} = \frac{\mu a}{2c}(F_2 - F_1)$

$$F_1 = \frac{W(c-x)}{2c} - \frac{\mu a}{2c}(F_2 - F_1); \quad F_2 = \frac{W(c+x)}{2c} + \frac{\mu a}{2c}(F_2 - F_1)$$

subtraction gives $F_2 - F_1 = \frac{Wx}{c} + \frac{\mu a}{c}(F_2 - F_1)$

i.e., $F_2 - F_1 = \frac{Wx}{c} \left(\frac{c}{c - \mu a} \right) = \frac{Wx}{c - \mu a}$

Restoring force = $\mu(F_2 - F_1) = \left(\frac{\mu W x}{c - \mu a} \right)$

Equation of motion : $\frac{W}{g} \ddot{x} + \frac{\mu W}{(c - \mu a)} x = 0$

$$\omega_n = \omega = \sqrt{\frac{\mu W g}{W(c - \mu a)}} = \sqrt{\frac{\mu g}{c - \mu a}}$$

Solving this, we get $\omega = [c\omega^2/(g + a\omega^2)]$

2.42

From problem 2.41,

$$\text{Restoring force without springs} = \mu (F_2 - F_1) = \frac{\mu W x}{c - \mu a}$$

$$\text{spring restoring force} = 2 k x$$

$$\text{Total restoring force} = \frac{\mu W x}{c - \mu a} + 2 k x$$

$$\text{Equation of motion: } \frac{W}{g} \ddot{x} + \left(\frac{\mu W}{c - \mu a} + 2 k \right) x = 0$$

$$\omega_n = \omega = \left\{ \frac{[\mu W + 2 k (c - \mu a)] g}{(c - \mu a) W} \right\}^{1/2}$$

Solution of this equation gives

$$\mu = \left(\frac{\omega^2 W c - 2 k g c}{W g + W \omega^2 a - 2 k g a} \right)$$

2.43

(a) Natural frequency of vibration of electromagnet (without the automobile):

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10000.0 (386.4)}{3000.0}} = 35.8887 \text{ rad/sec}$$

(b) When the automobile is dropped, the electromagnet moves up by a distance (x_0) from its static equilibrium position. x_0 = static deflection (elongation of cable) under the weight of automobile

$$= \frac{W_{\text{auto}}}{k} = \frac{2000}{10000} = 0.2 \text{ in}$$

$$\dot{x}_0 = 0$$

Resultant motion of electromagnet (+x upwards):

$$x(t) = A_0 \sin (\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{1/2} = x_0 = 0.2$$

$$\text{and } \phi_0 = \tan^{-1} \left(\frac{x_0 \omega_n}{\dot{x}_0} \right) = \tan^{-1} (\infty) = 90^\circ$$

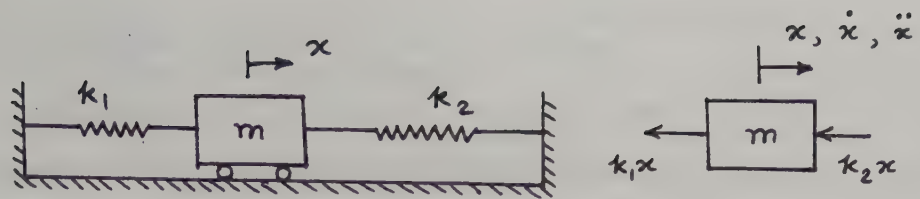
$$\text{Hence } x(t) = 0.2 \sin (35.8887 t + 90^\circ) = 0.2 \cos 35.8887 t$$

(c) Maximum $x(t)$:

$$x(t) |_{\max} = A_0 = 0.2 \text{ in}$$

$$\begin{aligned} \text{Maximum tension in cable during motion} &= k x(t) |_{\max} + \text{Weight of electromagnet} \\ &= 10000 (0.2) + 3000 = 5,000 \text{ lb.} \end{aligned}$$

2.44



- (a) Newton's second law of motion:

$$F(t) = -k_1 x - k_2 x = m \ddot{x} \text{ or } m \ddot{x} + (k_1 + k_2) x = 0$$

- (b) D'Alembert's principle:

$$F(t) - m \ddot{x} = 0 \text{ or } -k_1 x - k_2 x - m \ddot{x} = 0$$

$$\text{Thus } m \ddot{x} + (k_1 + k_2) x = 0$$

- (c) Principle of virtual work:

When mass m is given a virtual displacement δx ,

Virtual work done by the spring forces $= -(k_1 + k_2) x \delta x$

Virtual work done by the inertia force $= -(m \ddot{x}) \delta x$

According to the principle of virtual work, the total virtual work done by all forces must be equal to zero:

$$-m \ddot{x} \delta x - (k_1 + k_2) x \delta x = 0 \text{ or } m \ddot{x} + (k_1 + k_2) x = 0$$

- (d) Principle of conservation of energy:

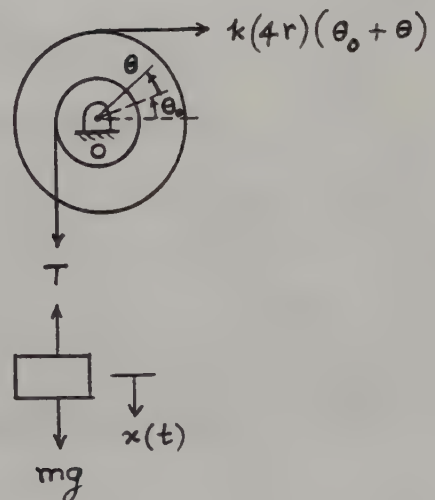
$$T = \text{kinetic energy} = \frac{1}{2} m \dot{x}^2$$

$$U = \text{strain energy} = \text{potential energy} = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 x^2$$

$$T + U = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} (k_1 + k_2) x^2 = c = \text{constant}$$

$$\frac{d}{dt} (T + U) = 0 \text{ or } m \ddot{x} + (k_1 + k_2) x = 0$$

2.45



Equation of motion:

$$\text{Mass } m: m g - T = m \ddot{x} \quad (1)$$

$$\text{Pulley } J_0: J_0 \ddot{\theta} = T r - k 4 r (\theta + \theta_0) 4 r \quad (2)$$

where $\theta_0 =$ angular deflection of the pulley under the weight, mg , given by:

$$m g r = k (4 r \theta_0) 4 r \quad \text{or} \quad \theta_0 = \frac{m g}{16 r k} \quad (3)$$

Substituting Eqs. (1) and (3) into (2), we obtain

$$J_0 \ddot{\theta} = (m g - m \ddot{x}) r - k 16 r^2 (\theta + \frac{m g}{16 r k}) \quad (4)$$

Using $x = r \theta$ and $\ddot{x} = r \ddot{\theta}$, Eq. (4) becomes

$$(J_0 + m r^2) \ddot{\theta} + (16 r^2 k) \theta = 0$$

2.46

Consider the springs connected to the pulleys (by rope) to be in series. Then

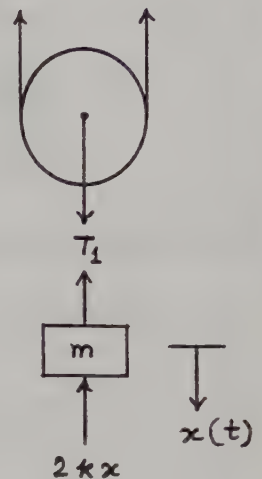
$$\frac{1}{k_{eq}} = \frac{1}{k} + \frac{1}{5k} \quad \text{or} \quad k_{eq} = \frac{5}{6} k$$

Let the displacement of mass m be x .

Then the extension of the rope (springs connected to the pulleys) $= 2 x$. From the free body diagram, the equation of motion of mass m :

$$m \ddot{x} + 2 k x + k_{eq} (2 x) = 0$$

$$\text{or} \quad m \ddot{x} + \frac{11}{3} k x = 0$$



2.47

$T = \text{kinetic energy} = T_{\text{mass}} + T_{\text{pulley}}$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} (m r^2 + J_0) \dot{\theta}^2$$

$$U = \text{potential energy} = \frac{1}{2} k x_s^2 = \frac{1}{2} k (4 r \theta)^2 = \frac{1}{2} k (16 r^2) \theta^2$$

Using $\frac{d}{dt} (T + U) = 0$ gives

$$(m r^2 + J_0) \ddot{\theta} + (16 r^2 k) \theta = 0$$

2.48

$$T = \text{kinetic energy} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

$$U = \text{potential energy} = \frac{1}{2} k x_s^2$$

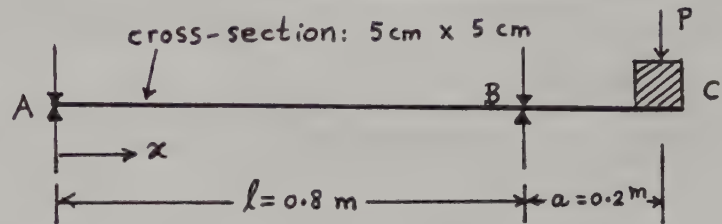
where $\theta = \frac{x}{r}$, $x_s = \text{extension of spring} = 4 r \theta = 4 x$. Hence

$$T = \frac{1}{2} \left(m + \frac{J_0}{r^2} \right) \dot{x}^2 ; U = \frac{1}{2} (16 k) x^2$$

Using the relation $\frac{d}{dt} (T + U) = 0$, we obtain the equation of motion of the system as:

$$\left(m + \frac{J_0}{r^2} \right) \ddot{x} + 16 k x = 0$$

2.49



Due to a load P at C , deflection at point C is given by (from Appendix B):

$$y(x) = \frac{P (x - \ell)}{6 E I \ell} \left[a (3 x - \ell) - (x - \ell)^2 \right] ; \ell \leq x \leq \ell + a$$

$$y_C = y(x = \ell + a) = \frac{P a^2}{3 E I \ell} (\ell + a)$$

Moment of inertia of cross section of beam:

$$I = \frac{1}{12} (0.05) (0.05)^3 = 52.0833 (10^{-8}) \text{ m}^4$$

Equivalent stiffness:

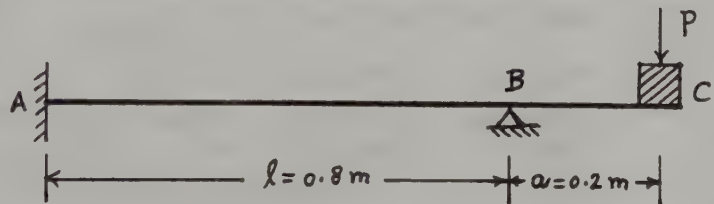
$$k_{eq} = \frac{P}{y_C} = \frac{3 E I \ell}{a^2 (\ell + a)} = \frac{3 (207 (10^9)) (52.0833 (10^{-8})) (0.8)}{(0.2)^2 (0.8 + 0.2)}$$

$$= 6.4687 (10^6) \text{ N/m}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{6.4687 (10^6)}{50}} = 359.6872 \text{ rad/sec}$$

2.50



From Appendix B, the deflection of fixed-pinned beam with an overhang, due to load P at the free end, is given by:

$$y(x) = \frac{P a}{4 E I \ell} \left[x^2 - \ell x^2 - \left(\frac{2 \ell}{3 a} + 1 \right) (x - \ell)^3 \right]; \ell \leq x \leq \ell + a$$

Using $a = 0.2$, $\ell = 0.8$, $x = a + \ell = 1.0$, and

$$I = \frac{1}{12} (0.05) (0.05)^3 = 52.0833 (10^{-8}) \text{ m}^4$$

we obtain

$$y_C = \frac{P (0.2)}{4 (207 (10^9)) (52.0833 (10^{-8})) (0.8)} \left[1^2 - 0.8 (1)^2 - \left(\frac{1.6}{0.6} + 1 \right) (0.2)^3 \right]$$

$$= P (9.895652 (10^{-8}))$$

$$k_{eq} = \frac{P}{y_C} = 1010.5448 (10^4) \text{ N/m}$$

$$\omega_n = \left\{ \frac{k_{eq}}{m} \right\}^{\frac{1}{2}} = \left\{ \frac{10.1054 (10^6)}{50} \right\}^{\frac{1}{2}} = 449.5642 \text{ rad/sec}$$

2.51

The system of Fig. (A)

can be drawn in equivalent form as shown in Fig. (B) where both pulleys have the same radius r_1 . We notice in

Fig. (B) that vibration can take place in only one way with one pulley moving clockwise and the other moving counter clockwise.

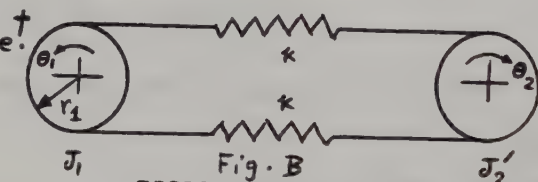
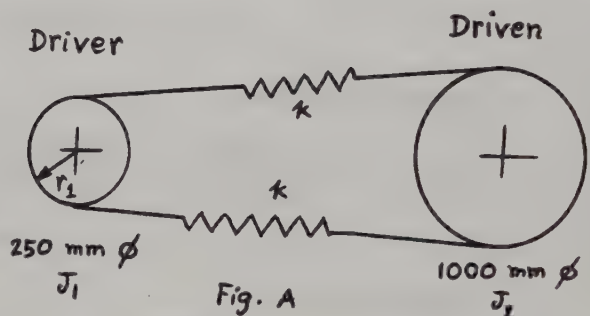
When pulleys rotate in opposite directions, $\frac{\theta_1}{\theta_2} = \frac{J_2}{J_1}$.

The spring force, which has the

same value on either pulley is $-k_t(\theta_1 + \theta_2)$ where k_t = torsional spring constant of the system. Equation of motion is

$$J_1 \ddot{\theta}_1 + k_t(\theta_1 + \theta_2) = 0 \quad \& \quad J_2 \ddot{\theta}_2 + k_t(\theta_1 + \theta_2) = 0$$

$$\text{i.e. } J_1 \ddot{\theta}_1 + k_t \left(1 + \frac{J_1}{J_2} \right) \theta_1 = 0 \quad \& \quad J_2 \ddot{\theta}_2 + k_t \left(\frac{J_2}{J_1} + 1 \right) \theta_2 = 0$$



$$\begin{aligned} k_t &= \frac{\Delta M_t}{\Delta \theta} = \left(\text{force in springs due to } \Delta \theta \right) \frac{r_1}{\Delta \theta} \\ &= (2k r_1 \Delta \theta) \frac{r_1}{\Delta \theta} = 2k r_1^2 \\ &= 2k \left(\frac{125}{1000} \right)^2 = k/32 \text{ N-m/rad} \\ \therefore E_8 \cdot (E_1) \text{ gives, for } \omega = 12\pi \text{ rad,} \\ k &= 454.7935 \text{ N/m.} \end{aligned}$$

Either of these equations gives

$$\omega = \left\{ k_t \left(\frac{J_1 + J_2}{J_1 J_2} \right) \right\}^{1/2} \dots (E_1)$$

Here $J_1 = 0.2/4 = 0.05 \text{ kg-m}^2$,

$$J_2' = J_2 (\text{speed ratio})^2 = 0.2 \left(\frac{1}{4} \right)^2 = 0.0125 \text{ kg-m}^2$$

† The other possible motion is rotation of the two pulleys as a whole (as rigid body) in same direction. This will have a natural frequency of zero. See section 5.7.

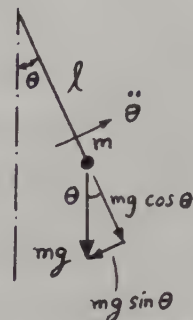
2.52

$$ml \ddot{\theta} + mg \sin \theta = 0$$

$$\text{For small } \theta, \quad ml \ddot{\theta} + mg \theta = 0$$

$$\omega_n = \sqrt{\frac{g}{l}}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{9.81}{0.5}}} = 1.4185 \text{ sec}$$



2.53

$$(a) \quad \omega_n = \sqrt{\frac{g}{l}}$$

$$(b) \quad ml^2 \ddot{\theta} + ka^2 \sin \theta + mgl \sin \theta = 0; \quad ml^2 \ddot{\theta} + (ka^2 + mgl) \theta = 0$$

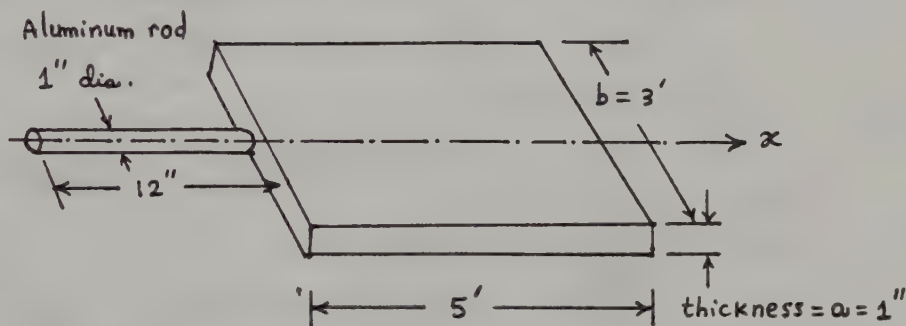
$$\omega_n = \sqrt{\frac{ka^2 + mgl}{ml^2}}$$

$$(c) \quad ml^2 \ddot{\theta} + ka^2 \sin \theta - mgl \sin \theta = 0; \quad ml^2 \ddot{\theta} + (ka^2 - mgl) \theta = 0$$

$$\omega_n = \sqrt{\frac{ka^2 - mgl}{ml^2}}$$

configuration (b) has the highest natural frequency.

2.54



$$m = \text{mass of a panel} = (5 \times 12) (3 \times 12) (1) \left(\frac{0.283}{386.4} \right) = 1.5820$$

$$\begin{aligned} J_0 &= \text{mass moment of inertia of panel about x-axis} = \frac{m}{12} (a^2 + b^2) \\ &= \frac{1.5820}{12} (1^2 + 36^2) = 170.9878 \end{aligned}$$

$$I_0 = \text{polar moment of inertia of rod} = \frac{\pi}{32} d^4 = \frac{\pi}{32} (1)^4 = 0.098175 \text{ in}^4$$

$$k_t = \frac{G I_0}{\ell} = \frac{(3.8 (10^6)) (0.098175)}{12} = 3.1089 (10^4) \text{ lb-in/rad}$$

$$\omega_n = \left\{ \frac{k_t}{J_0} \right\}^{\frac{1}{2}} = \left\{ \frac{3.1089 (10^4)}{170.9878} \right\}^{\frac{1}{2}} = 13.4841 \text{ rad/sec}$$

2.55

I_0 = polar moment of inertia of cross section of shaft AB

$$= \frac{\pi}{32} d^4 = \frac{\pi}{32} (1)^4 = 0.098175 \text{ in}^4$$

k_t = torsional stiffness of shaft AB = $\frac{G I_0}{\ell}$

$$= \frac{(12 (10^5)) (0.098175)}{6} = 19.635 (10^4) \text{ lb-in/rad}$$

J_0 = mass moment of inertia of the three blades about y-axis

$$= 3 J_0 |_{PQ} = 3 \left[\frac{1}{3} m \ell^2 \right] = m \ell^2 = \left[\frac{2}{386.4} \right] (12)^2 = 0.7453$$

Torsional natural frequency:

$$\omega_n = \left\{ \frac{k_t}{J_0} \right\}^{\frac{1}{2}} = \left\{ \frac{19.635 (10^4)}{0.7453} \right\}^{\frac{1}{2}} = 513.2747 \text{ rad/sec}$$

2.56

J_0 = mass moment of inertia of the ring = 1.0 kg-m^2 .

I_{os} = polar moment of inertia of the cross section of steel shaft

$$= \frac{\pi}{32} (d_{os}^4 - d_{is}^4) = \frac{\pi}{4} (0.05^4 - 0.04^4) = 36.2266 (10^{-8}) \text{ m}^4$$

I_{ob} = polar moment of inertia of cross section of brass shaft

$$= \frac{\pi}{32} (d_{ob}^4 - d_{ib}^4) = \frac{\pi}{32} (0.04^4 - 0.03^4) = 17.1806 (10^{-8}) \text{ m}^4$$

k_{ts} = torsional stiffness of steel shaft

$$= \frac{G_s I_{os}}{\ell} = \frac{(80 (10^9)) (36.2266 (10^{-8}))}{2} = 14490.64 \text{ N-m/rad}$$

k_{tb} = torsional stiffness of brass shaft

$$= \frac{G_b I_{ob}}{\ell} = \frac{(40 (10^9)) (17.1806 (10^{-8}))}{2} = 3436.12 \text{ N-m/rad}$$

$$k_{teq} = k_{ts} + k_{tb} = 17,926.76 \text{ N-m/rad}$$

Torsional natural frequency:

$$\omega_n = \sqrt{\frac{k_{teq}}{J_0}} = \sqrt{\frac{17926.76}{1}} = 133.8908 \text{ rad/sec}$$

Natural time period:

$$\tau_n = \frac{2 \pi}{\omega_n} = \frac{2 \pi}{133.8908} = 0.04693 \text{ sec}$$

Kinetic energy of system is

A diagram of a simple pendulum. A mass m is suspended by a string of length l from a fixed point. The string makes an angle θ with the vertical dashed line. The mass is shown at two positions, with dashed lines indicating its path.

Potential energy of system is

(since mass of the rod acts through its center)

Equation of motion:

$$\frac{d}{dt}(T+U) = 0$$

i.e. $(M + \frac{m}{3})l^2 \ddot{\theta} + (M + \frac{m}{2})gl \sin \theta = 0$

For small angles,

$$\therefore \ddot{\theta} + \frac{(M + \frac{m}{2})}{(M + \frac{m}{3})} \frac{g}{l} \cdot \theta = 0$$

$$\omega_n = \sqrt{\frac{(M + \frac{m}{2})g}{(M + \frac{m}{3})l}}$$

For the shaft,

$$J = \frac{\pi d^4}{32} = \frac{\pi (0.05)^4}{32} = 61.3594 \times 10^{-8} \text{ m}^4$$

$$k_t = \frac{GJ}{l} = \frac{(0.793 \times 10^{11}) (61.3594 \times 10^{-8})}{2} = 24329.002 \text{ N-m/rad}$$

For the disc,

$$J_0 = \frac{MD^2}{8} = \left(\rho \frac{\pi D^2}{4} h \right) \frac{D^2}{8} = \frac{\rho \pi D^4 h}{32}$$

$$= \frac{(7.83 \times 10^3) \pi (1)^4 (0.1)}{32} = 76.8710 \text{ kg-m}^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_o}} = \left(\frac{24329.002}{76.8710} \right)^{1/2} = 17.7902 \text{ rad/sec}$$

Equation of motion

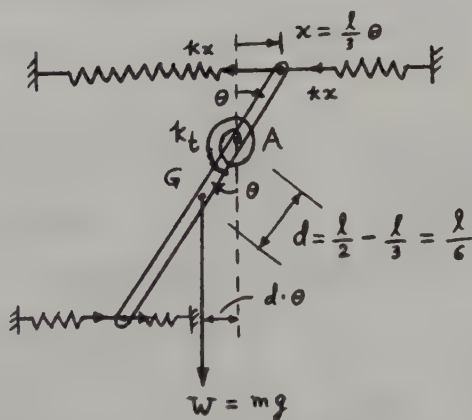
$$\begin{aligned} J_A \ddot{\theta} &= -W d \theta - 2k \left(\frac{l}{3} \theta \right) \frac{l}{3} \\ &\quad - 2k \left(\frac{2l}{3} \theta \right) \frac{2l}{3} - k_t \theta \end{aligned}$$

where

$$\begin{aligned} J_A &= J_G + m d^2 = \frac{1}{12} m l^2 + m \frac{l^2}{36} \\ &= \frac{1}{9} m l^2 \end{aligned}$$

$$\therefore \frac{m l^2}{9} \ddot{\theta} + (m g d + 2 \kappa \frac{l^2}{9} + \frac{8 \kappa l^2}{9} + \kappa_t) \theta = 0$$

$$\omega_n = \sqrt{\frac{(mgd + \frac{2}{9}kl^2 + \frac{8}{9}kl^2 + k_t)g}{ml^2}} = \sqrt{\frac{9mgd + 10kl^2 + 9k_t}{ml^2}}$$



For given data,

$$\omega_n = \sqrt{\frac{9(10)(9.81)(5/6) + 10(2000)(5)^2 + 9(1000)}{10(5)^2}} = 45.1547 \frac{\text{rad}}{\text{sec}}$$

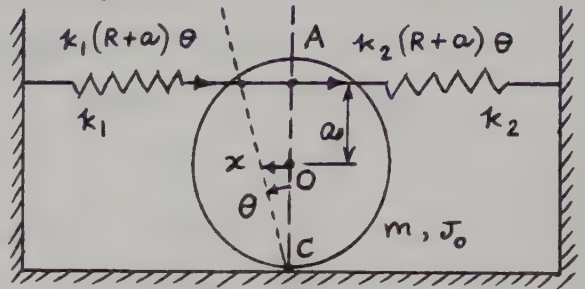
2.60 $J_O = \frac{1}{2} m R^2$, $J_C = \frac{1}{2} m R^2 + m R^2$

Let angular displacement = θ

Equation of motion:

$$J_C \ddot{\theta} + k_1(R+a)^2 \theta + k_2(R+a)^2 \theta = 0$$

$$\omega_n = \sqrt{\frac{(k_1 + k_2)(R+a)^2}{J_C}} = \sqrt{\frac{(k_1 + k_2)(R+a)^2}{1.5 m R^2}} \quad (E_1)$$



Equation (E1) shows that ω_n increases with the value of a .

$\therefore \omega_n$ will be maximum when $a = R$.

2.61 Net g acting on the pendulum = $9.81 - 5 = 4.81 \text{ m/sec}^2 = g_n$

$$\omega_n = \sqrt{\frac{g_n}{l}} = \sqrt{\frac{4.81}{5}} = 3.1016 \text{ rad/sec}$$

$$T_n = 2\pi/\omega_n = 2.0258 \text{ sec}$$

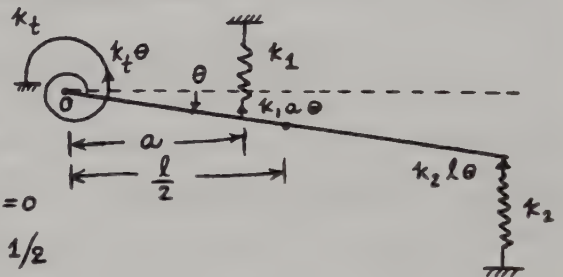
2.62 Equation of motion:

$$J_O \ddot{\theta} = -k_t \theta - (k_1 a \theta) a - (k_2 l \theta) l$$

$$\text{Where } J_O = \frac{1}{12} m l^2 + m \left(\frac{l}{2}\right)^2 = \frac{1}{3} m l^2$$

$$\therefore \frac{1}{3} m l^2 \ddot{\theta} + (k_t + k_1 a^2 + k_2 l^2) \theta = 0$$

$$\omega_n = \left\{ \frac{3(k_t + k_1 a^2 + k_2 l^2)}{m l^2} \right\}^{1/2}$$



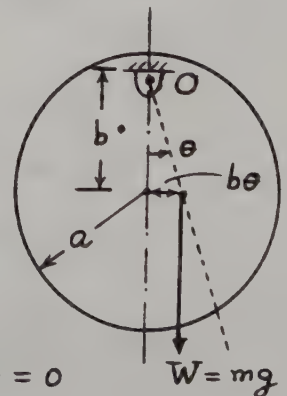
2.63 $J_O = J_G + m b^2 = \frac{1}{2} m a^2 + m b^2$

Equation of motion:

$$J_O \ddot{\theta} + m g b \theta = 0$$

$$\omega_n = \sqrt{\frac{m g b}{J_O}} = \sqrt{\frac{2 g b}{a^2 + 2 b^2}}$$

$$\frac{\partial \omega_n}{\partial b} = \frac{1}{2} \left(\frac{2 g b}{a^2 + 2 b^2} \right)^{-1/2} \left\{ \frac{(a^2 + 2 b^2)(2 g) - 2 g b(4 b)}{(a^2 + 2 b^2)^2} \right\} = 0$$



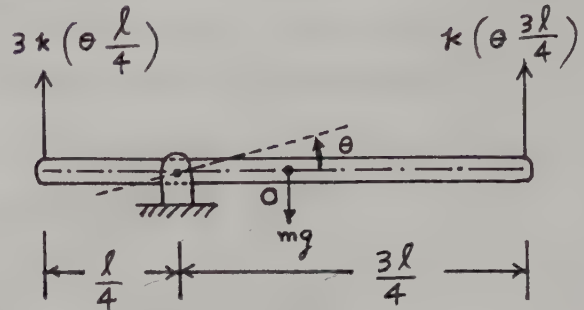
i.e., $b = \pm \frac{a}{\sqrt{2}}$

$$\omega_n \Big|_{b = + a/\sqrt{2}} = \sqrt{\frac{2g \frac{a}{\sqrt{2}}}{a^2 + 2(a^2/2)}} = \sqrt{\frac{g}{\sqrt{2} a}}$$

$b = -a/\sqrt{2}$ gives imaginary value for ω_n .

Since $\omega_n = 0$ when $b = 0$, we have $\omega_n|_{\max}$ at $b = \frac{a}{\sqrt{2}}$.

2.64



Let θ be measured from static equilibrium position so that gravity force need not be considered.

(a) Newton's second law of motion:

$$J_0 \ddot{\theta} = -3k \left(\theta \frac{\ell}{4} \right) \frac{\ell}{4} - k \left(\theta \frac{3\ell}{4} \right) \left(\frac{3\ell}{4} \right) \quad \text{or} \quad J_0 \ddot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

(b) D'Alembert's principle:

$$M(t) - J_0 \ddot{\theta} = 0 \quad \text{or} \quad -3k \left(\theta \frac{\ell}{4} \right) \left(\frac{\ell}{4} \right) - k \left(\theta \frac{3\ell}{4} \right) \left(\frac{3\ell}{4} \right) - J_0 \ddot{\theta} = 0$$

$$\text{or} \quad J_0 \ddot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

(c) Principle of virtual work:

Virtual work done by spring force:

$$\delta W_s = -3k \left(\theta \frac{\ell}{4} \right) \left(\frac{\ell}{4} \delta\theta \right) - k \left(\theta \frac{3\ell}{4} \right) \left(\frac{3\ell}{4} \delta\theta \right)$$

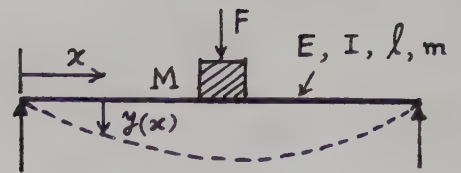
Virtual work done by inertia moment = $-(J_0 \ddot{\theta}) \delta\theta$

Setting total virtual work done by all forces/moments equal to zero, we obtain

$$J_0 \ddot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

2.65

Let m_{eff} = effective part of mass of beam (m) at middle. Thus vibratory inertia force at middle is due to $(M + m_{\text{eff}})$. Assume a deflection shape: $y(x, t) = Y(x) \cos(\omega_n t - \phi)$ where $Y(x)$ = static deflection shape due to load at middle given by:



$$Y(x) = Y_0 \left(3 \frac{x}{\ell} - 4 \frac{x^3}{\ell^3} \right) ; 0 \leq x \leq \frac{\ell}{2}$$

where Y_0 = maximum deflection of the beam at middle = $\frac{F \ell^3}{48 E I}$

Maximum strain energy of beam = maximum work done by force $F = \frac{1}{2} F Y_0$.

Maximum kinetic energy due to distributed mass of beam:

$$\begin{aligned} &= 2 \left\{ \frac{1}{2} \frac{m}{\ell} \int_0^{\frac{\ell}{2}} \dot{y}^2(x, t) |_{\max} dx \right\} + \frac{1}{2} (\dot{y}_{\max})^2 M \\ &= \frac{m \omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y^2(x) dx + \frac{1}{2} \omega_n^2 Y_{\max}^2 M \\ &= \frac{m \omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y_0^2 \left(\frac{9x^2}{\ell^2} + 16 \frac{x^6}{\ell^6} - 24 \frac{x^4}{\ell^4} \right) dx + \frac{1}{2} Y_0^2 M \omega_n^2 \\ &= \frac{m \omega_n^2 Y_0^2}{\ell} \left[\frac{9}{\ell^2} \frac{x^3}{3} + \frac{16}{\ell^6} \frac{x^7}{7} - \frac{24}{\ell^4} \frac{x^5}{5} \right] \Big|_0^{\frac{\ell}{2}} + \frac{1}{2} Y_0^2 M \omega_n^2 \\ &= \frac{1}{2} Y_0^2 \omega_n^2 \left(\frac{17}{35} m + M \right) \end{aligned}$$

This shows that $m_{\text{eff}} = \frac{17}{35} m = 0.4857 m$

2.66 For small angular rotation of bar PQ about P,

$$\frac{1}{2} (k_{12})_{eq} (\theta l_3)^2 = \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_2)^2$$

$$(k_{12})_{eq} = \frac{k_1 l_1^2 + k_2 l_2^2}{l_3^2}$$

Since $(k_{12})_{eq}$ and k_3 are in series,

$$k_{eq} = \frac{(k_{12})_{eq} k_3}{(k_{12})_{eq} + k_3} = \frac{k_1 k_3 l_1^2 + k_2 k_3 l_2^2}{k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2}$$

T = kinetic energy = $\frac{1}{2} m \dot{x}^2$, U = potential energy = $\frac{1}{2} k_{eq} x^2$

If $x = X \cos \omega_n t$,

$$T_{\max} = \frac{1}{2} m \omega_n^2 X^2, \quad U_{\max} = \frac{1}{2} k_{eq} X^2$$

$$T_{\max} = U_{\max} \text{ gives } \omega_n = \sqrt{\frac{k_1 k_3 l_1^2 + k_2 k_3 l_2^2}{m(k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$$

- 2.67 When mass m moves by x , spring k_1 deflects by $x/4$.

$$T = \text{kinetic energy} = \frac{1}{2} m (\dot{x})^2$$

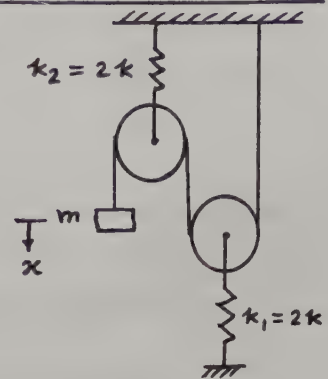
$$U = \text{potential energy} = 2 \left\{ \frac{1}{2} (2k) \left(\frac{x}{4} \right)^2 \right\} = \frac{1}{8} k x^2$$

For harmonic motion,

$$T_{\max} = \frac{1}{2} m \omega_n^2 X^2, \quad U_{\max} = \frac{1}{8} k X^2$$

$T_{\max} = U_{\max}$ gives

$$\omega_n = \sqrt{\frac{k}{4m}}$$



- 2.68 Refer to the figure of solution of problem 2.24.

$$T = \frac{1}{2} m \dot{x}^2, \quad U = \frac{1}{2} [2k_1 (x \cos 45^\circ)^2 + 2k_2 (x \cos 135^\circ)^2] = \frac{1}{2} (k_1 + k_2) x^2$$

For harmonic motion,

$$T_{\max} = \frac{1}{2} m \omega_n^2 X^2, \quad U_{\max} = \frac{1}{2} (k_1 + k_2) X^2$$

$T_{\max} = U_{\max}$ gives

$$\omega_n = \sqrt{\frac{k_1 + k_2}{m}}$$

- 2.69 Kinetic energy (K.E.) = $\frac{1}{2} m \dot{x}^2$

Potential energy (P.E.) = $\frac{1}{2} T_1 x + \frac{1}{2} T_2 x$ = work done in displacing mass m by distance x against the total force (tension) of $T_1 + T_2$.

$$T_1 = \frac{x}{a} T, \quad T_2 = \frac{x}{b} T \quad \text{from solution of problem 2.18}$$

$$\text{Max. K.E.} = \frac{1}{2} m \omega_n^2 X^2, \quad \text{Max. P.E.} = \frac{1}{2} T \left(\frac{1}{a} + \frac{1}{b} \right) X^2$$

Max. K.E. = Max. P.E. gives

$$\omega_n = \sqrt{\frac{T(a+b)}{mab}} = \sqrt{\frac{Tl}{ma(l-a)}}$$

- 2.70 $T = \text{K.E.} = \frac{1}{2} \mathcal{I}_A \dot{\theta}^2 = \frac{1}{2} (\mathcal{I}_G + md^2) \dot{\theta}^2 = \frac{1}{2} \left(\frac{1}{12} ml^2 + m \frac{l^2}{36} \right) \dot{\theta}^2 = \frac{1}{2} \left(\frac{m l^2}{9} \right) \dot{\theta}^2$

$$U = \text{P.E.} = mgd(1 - \cos \theta) + 2 \left(\frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 \right) + \frac{1}{2} k_t \theta^2$$

$$\text{with } \cos \theta \approx 1 - \frac{1}{2} \theta^2, \quad x_1 = \frac{l}{3} \theta \quad \text{and} \quad x_2 = \frac{2l}{3} \theta$$

$$U = mg \frac{l}{6} \frac{\theta^2}{2} + k \frac{l^2}{9} \theta^2 + k \frac{4l^2}{9} \theta^2 + \frac{1}{2} k_t \theta^2$$

$$T_{\max} = \frac{1}{2} \left(\frac{m l^2}{9} \right) \dot{\theta}^2, \quad U_{\max} = \frac{1}{2} \frac{mg l}{6} \theta^2 + \frac{1}{2} \left(\frac{10 k l^2}{9} \right) \theta^2 + \frac{1}{2} k_t \theta^2$$

$$T_{\max} = U_{\max} \text{ gives}$$

$$\omega_n = \sqrt{\frac{mg \frac{l}{6} + \frac{10 k l^2}{9} + k_t}{\frac{m l^2}{9}}} = 45.1547 \frac{\text{rad}}{\text{sec}} \quad \text{for given data}$$

2.71

Refer to the figure in the solution of problem 2.62.

$$T = \frac{1}{2} J_0 \dot{\theta}^2$$

$$U = \frac{1}{2} k_t \theta^2 + \frac{1}{2} k_1 (\theta a)^2 + \frac{1}{2} k_2 (\theta l)^2$$

$$\text{For } \theta(t) = \Theta \cos \omega_n t,$$

$$T_{\max} = \frac{1}{2} J_0 \omega_n^2 \Theta^2, \quad U_{\max} = \frac{1}{2} (k_t + k_1 a^2 + k_2 l^2) \Theta^2$$

$$T_{\max} = U_{\max} \text{ gives}$$

$$\omega_n = \sqrt{\frac{k_t + k_1 a^2 + k_2 l^2}{J_0}} = \sqrt{\frac{3(k_t + k_1 a^2 + k_2 l^2)}{m l^2}}$$

$$\text{since } J_0 = m l^2 / 3.$$

2.72

When prism is displaced by x from equilibrium position, the weight of oil displaced

$$= \rho_o g a b x = \text{restoring force.}$$

$$\text{Mass of prism} = m = \rho_w a b h$$

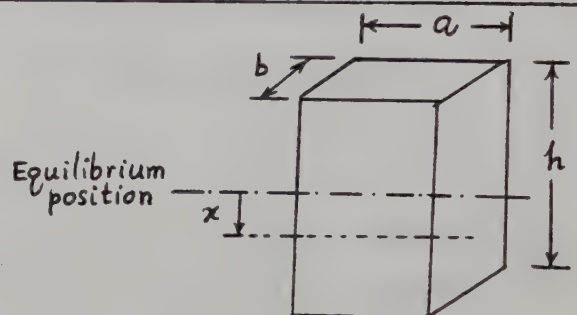
Equation of motion:

$$m \ddot{x} + \text{restoring force} = 0$$

$$\rho_w a b h \ddot{x} + \rho_o g a b x = 0$$

$$\omega_n = \sqrt{\frac{\rho_o g a b}{\rho_w a b h}} = \sqrt{\frac{\rho_o g}{\rho_w h}} \quad (E1)$$

Since ω_n is independent of cross-section of the prism, ω_n remains same even for a circular wooden prism.



2.73

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} \left(m R^2 + \frac{1}{2} m R^2 \right) \dot{\theta}^2$$

$$\text{since } x = R \theta \text{ and } J_0 = \frac{1}{2} m R^2.$$

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_1^2 = \frac{1}{2} (k_1 + k_2) (R + a)^2 \theta^2$$

where $x_1 = (R + a) \theta$. Using $\frac{d}{dt} (T + U) = 0$, we obtain

$$\left(\frac{3}{2} m R^2\right) \ddot{\theta} + (k_1 + k_2) (R + a)^2 \theta = 0$$

2.74 Let $x(t)$ be measured from static equilibrium position of mass. T = kinetic energy of the system:

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} \left(m + \frac{J_0}{r^2} \right) \dot{x}^2$$

since $\dot{\theta} = \frac{\dot{x}}{r}$ = angular velocity of pulley. U = potential energy of the system:

$$U = \frac{1}{2} k y^2 = \frac{1}{2} k (16 x^2)$$

since $y = \theta (4 r) = 4 x$ = deflection of spring. $\frac{d}{dt} (T + U) = 0$ leads to:

$$m \ddot{x} + \frac{J_0}{r^2} \ddot{x} + 16 k x = 0$$

This gives the natural frequency:

$$\omega_n = \sqrt{\frac{16 k r^2}{m r^2 + J_0}}$$

2.75 For pendulum, $\omega_n = \sqrt{\frac{g}{l}}$ in vacuum = 0.5 Hz = π rad/sec

$$l = \frac{g}{\pi^2} = \frac{9.81}{\pi^2} = 0.9940 \text{ m}$$

$\omega_d = \omega_n \sqrt{1 - \zeta^2}$ in viscous medium = 0.45 Hz = 0.9π rad/sec

$$\zeta^2 = \frac{\omega_n^2 - \omega_d^2}{\omega_n^2} = \pi^2 \left(\frac{1 - 0.81}{1} \right) = 1.8752$$

$$\zeta = 1.3694$$

Equation of motion: $m l^2 \ddot{\theta} + c_t \dot{\theta} + m g l \theta = 0$

$$c_{ct} = 2(m l^2) \omega_n = 2(1)(0.994)^2(\pi) = 6.2080$$

Since $\zeta = \frac{c_t}{c_{ct}} = 1.3694$, $c_t = 8.5013 \text{ N-m-sec/rad}$.

2.76 From Eq. (2.85),

$$\ln \left(\frac{x_j}{x_{j+1}} \right) = \ln(18) \Rightarrow \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 2.8904$$

$$\zeta = \left\{ \frac{(2.8904)^2}{(2.8904)^2 + 4\pi^2} \right\}^{\frac{1}{2}} = 0.4179$$

(a) If damping is doubled, $\zeta_{\text{new}} = 0.8358$

$$\ln \left(\frac{x_j}{x_{j+1}} \right) = \frac{2\pi \zeta_{\text{new}}}{\sqrt{1 - \zeta_{\text{new}}^2}} = \frac{2\pi (0.8358)}{\sqrt{1 - (0.8358)^2}} = 9.5656$$

$$\therefore \frac{x_j}{x_{j+1}} = 14265.362$$

(b) If damping is halved, $\zeta = 0.2090$

$$\ln \left(\frac{x_j}{x_{j+1}} \right) = \frac{2\pi \zeta_{\text{new}}}{\sqrt{1 - \zeta_{\text{new}}^2}} = \frac{2\pi (0.2090)}{\sqrt{1 - (0.2090)^2}} = 1.3428$$

$$\therefore \frac{x_j}{x_{j+1}} = 3.8296$$

2.77 $x(t) = X e^{-\zeta \omega_n t} \sin \omega_d t$ where $\omega_d = \sqrt{1 - \zeta^2} \omega_n$

For maximum or minimum of $x(t)$,

$$\frac{dx}{dt} = X e^{-\zeta \omega_n t} (-\zeta \omega_n \sin \omega_d t + \omega_d \cos \omega_d t) = 0$$

As $e^{-\zeta \omega_n t} \neq 0$ for finite t ,

$$-\zeta \omega_n \sin \omega_d t + \omega_d \cos \omega_d t = 0$$

$$\text{i.e. } \tan \omega_d t = \frac{\sqrt{1 - \zeta^2}}{\zeta}$$

Using the relation

$$\sin \omega_d t = \pm \frac{\tan \omega_d t}{\sqrt{1 + \tan^2 \omega_d t}} = \pm \frac{(\sqrt{1 - \zeta^2}/\zeta)}{\sqrt{1 + \left(\frac{\sqrt{1 - \zeta^2}}{\zeta}\right)^2}} = \pm \sqrt{1 - \zeta^2}$$

we obtain

$$\sin \omega_d t = \sqrt{1 - \zeta^2}, \quad \cos \omega_d t = \zeta$$

and

$$\sin \omega_d t = -\sqrt{1 - \zeta^2}, \quad \cos \omega_d t = -\zeta$$

$$\frac{d^2 x}{dt^2} = X e^{-\zeta \omega_n t} [\zeta^2 \omega_n^2 \sin \omega_d t - 2\zeta \omega_n \omega_d \cos \omega_d t - \omega_d^2 \sin \omega_d t]$$

When $\sin \omega_d t = \sqrt{1 - \zeta^2}$ and $\cos \omega_d t = \zeta$,

$$\frac{d^2 x}{dt^2} = -X e^{-\zeta \omega_n t} \omega_n^2 \sqrt{1 - \zeta^2} < 0$$

$\therefore \sin \omega_d t = \sqrt{1 - \zeta^2}$ corresponds to maximum of $x(t)$.

When $\sin \omega_d t = -\sqrt{1 - \zeta^2}$ and $\cos \omega_d t = -\zeta$,

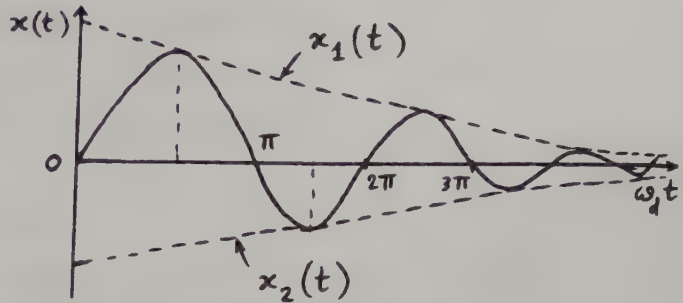
$$\frac{d^2 x}{dt^2} = X e^{-\gamma \omega_n t} \omega_n^2 \sqrt{1-\gamma^2} > 0$$

$\therefore \sin \omega_d t = -\sqrt{1-\gamma^2}$ corresponds to minimum of $x(t)$.

Enveloping curves:

Let the curve passing through the maximum (or minimum) points be

$$x(t) = C e^{-\gamma \omega_n t}$$



For maximum points, $t_{\max} = \frac{\sin^{-1}(\sqrt{1-\gamma^2})}{\omega_d}$

and

$$C e^{-\gamma \omega_n t_{\max}} = X e^{-\gamma \omega_n t_{\max}} \sin \omega_d t_{\max}$$

i.e. $C = X \sqrt{1-\gamma^2}$

$$\therefore x_1(t) = X \sqrt{1-\gamma^2} e^{-\gamma \omega_n t}$$

Similarly for minimum points, $t_{\min} = \frac{\sin^{-1}(-\sqrt{1-\gamma^2})}{\omega_d}$

and

$$C e^{-\gamma \omega_n t_{\min}} = X e^{-\gamma \omega_n t_{\min}} \sin \omega_d t_{\min}$$

i.e. $C = -X \sqrt{1-\gamma^2}$

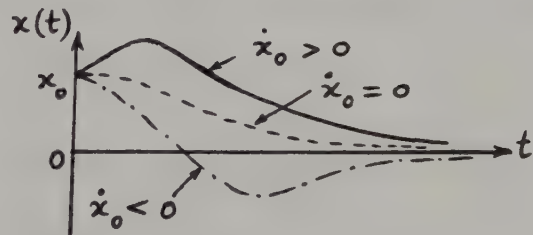
$$\therefore x_2(t) = -X \sqrt{1-\gamma^2} e^{-\gamma \omega_n t}$$

(2.78) $x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t] e^{-\omega_n t}$ ----- (E1)

For $x_0 > 0$, graph of $E_p(E_1)$

is shown for different $\dot{x}_0$.

We assume $\dot{x}_0 > 0$ as it is the only case that gives a maximum.



For maximum of $x(t)$,

$$\frac{dx}{dt} = e^{-\omega_n t} \{ -(\dot{x}_0 + \omega_n x_0) \omega_n t + \dot{x}_0 \} = 0$$

$$t_m = \frac{\dot{x}_0}{\omega_n (\dot{x}_0 + \omega_n x_0)} \text{ ----- (E2)}$$

$$\frac{d^2x}{dt^2} = -e^{-\omega_n t} \left\{ 2\omega_n \dot{x}_0 + \omega_n^2 x_0 - \omega_n^2 (\dot{x}_0 + \omega_n x_0) t \right\} \dots (E_3)$$

(E₂) and (E₃) give

$$\begin{aligned} \left. \frac{d^2x}{dt^2} \right|_{t=t_m} &= -e^{-\omega_n t_m} \left\{ 2\omega_n \dot{x}_0 + \omega_n^2 x_0 - \omega_n^2 (\dot{x}_0 + \omega_n x_0) t_m \right\} \\ &= -e^{-\omega_n \left(\frac{\dot{x}_0}{\omega_n (\dot{x}_0 + \omega_n x_0)} \right)} \left\{ \omega_n \dot{x}_0 + \omega_n^2 x_0 \right\} \dots (E_4) \end{aligned}$$

For $x_0 > 0$ and $\dot{x}_0 > 0$, $\left. \frac{d^2x}{dt^2} \right|_{t_m} < 0$

Hence t_m given by Eq. (E₂) corresponds to a maximum of $x(t)$.

$$\begin{aligned} x|_{t=t_m} &= \left\{ x_0 + (\dot{x}_0 + \omega_n x_0) \frac{\dot{x}_0}{\omega_n (\dot{x}_0 + \omega_n x_0)} \right\} e^{-\omega_n t_m} \\ &= \left(x_0 + \frac{\dot{x}_0}{\omega_n} \right) e^{-\left(\frac{\dot{x}_0}{\dot{x}_0 + \omega_n x_0} \right)} \dots (E_5) \end{aligned}$$

(2.79) Equation (2.92) can be expressed as $\delta = \frac{1}{m} \ln \left(\frac{x_0}{x_m} \right)$

For half cycle, $m = \frac{1}{2}$ and hence $\delta = 2 \ln \left(\frac{x_0}{x_{\frac{1}{2}}} \right) = 2 \ln \left(\frac{1}{0.15} \right)$

Necessary damping ratio ζ_0 is $= 3.7942$

$$\begin{aligned} \zeta_0 &= \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{3.7942^2}{\sqrt{4\pi^2 + 3.7942^2}} \\ &= 0.5169 \end{aligned}$$

(a)

If $\zeta = \frac{3}{4} \zeta_0 = 0.3877$, the overshoot can be determined by finding δ from Eq. (2.85):

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.3877)}{\sqrt{1-0.3877^2}} = 2.6427 = 2 \ln \left(\frac{x_0}{x_{\frac{1}{2}}} \right)$$

$$\ln \left(\frac{x_0}{x_{\frac{1}{2}}} \right) = 1.32135$$

$$x_{\frac{1}{2}} = x_0 / e^{1.32135} = 0.266775 x_0$$

$\therefore$ Overshoot is 26.6775%

(b)

If $\zeta = \frac{5}{4} \zeta_0 = 0.6461$, δ is given by

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.6461)}{\sqrt{1-(0.6461)^2}} = 5.3189 = 2 \ln \left(\frac{x_0}{x_{\frac{1}{2}}} \right)$$

$$\frac{x_0}{x_{\frac{1}{2}}} = 14.2888, \quad x_{\frac{1}{2}} = 0.0700 x_0$$

$$\therefore \text{overshoot} = 7\%$$

2.80

(i) (a) Viscous damping, (b) Coulomb damping.

- (iii) (a) $\tau_d = 0.2 \text{ sec}$, $f_d = 5 \text{ Hz}$, $\omega_d = 31.416 \text{ rad/sec}$.
 (b) $\tau_n = 0.2 \text{ sec}$, $f_n = 5 \text{ Hz}$, $\omega_n = 31.416 \text{ rad/sec}$.

(ii) (a) $\frac{x_i}{x_{i+1}} = e^{\zeta \omega_n \tau_d}$

$$\ln \left(\frac{x_i}{x_{i+1}} \right) = \ln 2 = 0.6931 = \frac{2 \pi \zeta}{\sqrt{1 - \zeta^2}} \cdot \text{sp}$$

$$\text{or } 39.2590 \zeta^2 = 0.4804 \quad \text{or } \zeta = 0.1096$$

Since $\omega_d = \omega_n \sqrt{1 - \zeta^2}$, we find

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} = \frac{31.416}{\sqrt{0.98798}} = 31.6065 \text{ rad/sec}$$

$$k = m \omega_n^2 = \left(\frac{500}{9.81} \right) (31.6065)^2 = 5.0916 (10^4) \text{ N/m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2 m \omega_n}$$

$$\text{Hence } c = 2 m \omega_n \zeta = 2 \left(\frac{500}{9.81} \right) (31.6065) (0.1096) = 353.1164 \text{ N-s/m}$$

(b) From Eq. (2.116):

$$k = m \omega_n^2 = \frac{500}{9.81} (31.416)^2 = 5.0304 (10^4) \text{ N/m}$$

Using $N = W = 500 \text{ N}$,

$$\mu = \frac{0.002 k}{4 W} = \frac{(0.002) (5.0304 (10^4))}{4 (500)} = 0.0503$$

2.81

(a) $c_c = 2 \sqrt{k m} = 2 \sqrt{5000 \times 50} = 1000 \text{ N-s/m}$

(b) $c = c_c/2 = 500 \text{ N-s/m}$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = \sqrt{\frac{k}{m}} \sqrt{1 - \left(\frac{c}{c_c} \right)^2} = \sqrt{\frac{5000}{50}} \sqrt{1 - \left(\frac{1}{2} \right)^2}$$

$$= 8.6603 \text{ rad/sec}$$

(c) From Eq. (2.85), $\delta = \frac{2\pi}{\omega_d} \left(\frac{c}{2m} \right) = \frac{2\pi}{8.6603} \left(\frac{500}{2 \times 50} \right)$

$$= 3.6276$$

2.82

$m = 2000 \text{ kg}$, $v = \dot{x}_0 = 10 \text{ m/sec}$, $k = 40,000 \text{ N/m}$
 $c = 20,000 \text{ N-sec/m}$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40000}{2000}} = 4.4721 \text{ rad/sec}$$

$$c_c = 2\sqrt{km} = 25,298.221 \text{ N-sec/m}$$

$$\zeta = c/c_c = 0.7906$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 4.4721 \sqrt{1 - (0.7906)^2} = 2.7384 \text{ rad/sec}$$

$$\tau_d = 2\pi/\omega_d = 2.2945 \text{ sec}$$

(a) For $x_0 = 0$ and $\dot{x}_0 = 10 \text{ m/sec}$, Eq. (2.72) gives

$$x(t) = e^{-\zeta\omega_n t} \frac{\dot{x}_0}{\omega_n \sqrt{1 - \zeta^2}} \sin \omega_n \sqrt{1 - \zeta^2} t$$

$$\text{At } x_{\max}, \omega_n t \simeq \frac{\pi}{2} \text{ and } \sin \omega_n \sqrt{1 - \zeta^2} t \simeq 1$$

$$\therefore x_{\max} \simeq e^{-0.7906(\pi/2)} \cdot \left(\frac{10}{2.7384}\right) \cdot (1) = 1.0548 \text{ m}$$

$$(b) t = \tau_d/4 = 2.2945/4 = 0.5736 \text{ sec.}$$

2.83

$$\omega_n = 200 \text{ cycles/min} = 20.944 \text{ rad/sec}, \quad \omega_d = 180 \text{ cycles/min} = 18.8496 \frac{\text{rad}}{\text{sec}}$$

$$J_0 = 0.2 \text{ kg-m}^2$$

$$\text{Since } \omega_d = \sqrt{1 - \zeta^2} \omega_n, \quad \zeta = \sqrt{1 - \left(\frac{\omega_d}{\omega_n}\right)^2} = \sqrt{1 - \left(\frac{18.8496}{20.944}\right)^2} = 0.4359$$

$$= \frac{c_t}{(c_t)_{\text{cri}}} = \frac{c_t}{2 J_0 \omega_n}$$

$$c_t = 2 J_0 \omega_n \zeta = 2(0.2)(20.944)(0.4359)$$

$$= 3.6518 \text{ N-m-s/rad}$$

Eq. (2.72) can be used to obtain $\theta(t)$ for $\dot{\theta}_0 = 0$, $\theta_0 = 2^\circ = 0.03491 \text{ rad}$ and $t = \tau_d = \frac{2\pi}{\omega_d} = 0.3333 \text{ sec}$,

$$\begin{aligned} \theta(t) &= e^{-\zeta\omega_n t} \theta_0 \left\{ \cos \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t \right\} \\ &= e^{-(0.4359)(20.944)(0.3333)} (0.03491) \left\{ \cos 18.8496 \times 0.3333 \right. \\ &\quad \left. + \frac{0.4359 \times 20.944}{18.8496} \sin 18.8496 \times 0.3333 \right\} \end{aligned}$$

$$= 0.001665 \text{ rad} = 0.09541^\circ$$

2.84

Assume that the bicycle and the boy fall as a rigid body by 5 cm at point A. Thus the mass (m_{eq}) will be subjected to an initial downward displacement of 5 cm ($t = 0$ assumed at point A):

$$\omega_n = \sqrt{\frac{k_{\text{eq}}}{m_{\text{eq}}}} = \sqrt{\frac{(50000)(9.81)}{800}} = 24.7614 \text{ rad/sec}$$

$$c_c = 2 m \omega_n = 2 \left(\frac{800}{9.81} \right) (24.7614) = 4038.5566 \text{ N-s/m}$$

$$\zeta = \frac{c}{c_c} = \frac{1000.0}{4038.5566} = 0.2476 \text{ (underdamping)}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 24.7614 \sqrt{1 - 0.2476^2} = 23.9905 \text{ rad/sec}$$

Response of the system:

$$x(t) = X e^{-\zeta \omega_n t} \sin(\omega_d t + \phi)$$

$$\text{where } X = \left\{ x_0^2 + \left(\frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d} \right)^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ (0.05)^2 + \left(\frac{(0.2476)(24.7614)(0.05)}{23.9905} \right)^2 \right\}^{\frac{1}{2}} = 0.051607 \text{ m}$$

$$\text{and } \phi = \tan^{-1} \left(\frac{x_0 \omega_d}{\dot{x}_0 + \zeta \omega_n x_0} \right) = \tan^{-1} \left(\frac{0.05 (23.9905)}{0.2476 (24.7614) (0.05)} \right) = 75.6645^\circ$$

Thus the displacement of the boy (positive downward) in vertical direction is given by

$$x(t) = 0.051607 e^{-6.1309 t} \sin(23.9905 t + 75.6645^\circ) \text{ m}$$

2.85

Reduction in amplitude of viscously damped free vibration in one cycle = 0.5 in.

$$\frac{x_1}{x_2} = \frac{6.0}{5.5} = 1.0909; \quad \ln \frac{x_1}{x_2} = 0.08701 = \frac{2 \pi \zeta}{\sqrt{1 - \zeta^2}}$$

i.e., $0.007571 (1 - \zeta^2) = 39.478602 \zeta^2$ or $\zeta = 0.013847$

2.86

$$\tau_d = 0.2 \text{ sec} = \frac{2\pi}{\omega_d}, \quad \omega_d = 31.416 \text{ rad/sec}$$

$$\text{From Eq. (2.92)} \quad \delta = \frac{1}{50} \ln 10 = 0.04605$$

$$\gamma = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.04605}{\sqrt{(2\pi)^2 + 0.04605^2}} = 0.007329$$

When damping is neglected,

$$\omega_n = \omega_d / \sqrt{1 - \gamma^2} = 31.417 \text{ rad/sec}; \quad \tau_n = \frac{2\pi}{\omega_n} = 0.19999 \text{ sec}$$

$$\text{Proportional decrease in period} = \left(\frac{0.2 - 0.19999}{0.2} \right) = 0.00005$$

2.87

For critically damped system, Eq. (2.80) gives

$$x(t) = \{ x_0 + (\dot{x}_0 + \omega_n x_0) t \} e^{-\omega_n t} \quad (E_1)$$

$$\dot{x}(t) = e^{-\omega_n t} \{ \dot{x}_0 - \dot{x}_0 \omega_n t - \omega_n^2 x_0 t \} \quad (E_2)$$

Let t_m = time at which $x = x_{\max}$ and $\dot{x} = 0$ occur.
 Here $x_0 = 0$ and $\dot{x}_0$ = initial recoil velocity. By setting $\dot{x}(t) = 0$, Eq. (E₂) gives

$$t_m = \frac{\dot{x}_0}{\omega_n (\dot{x}_0 + \omega_n x_0)} = \frac{\dot{x}_0}{\omega_n \dot{x}_0} = \frac{1}{\omega_n} \quad (E_3)$$

With Eq. (E₃) for t_m and $x_0 = 0$, (E₁) gives

$$x_{\max} = \dot{x}_0 t_m e^{-\omega_n t_m} = \frac{\dot{x}_0 e^{-1}}{\omega_n}$$

$$\text{i.e. } \dot{x}_0 = \omega_n x_{\max} e = \omega_n (0.5) (2.7183) \quad (E_4)$$

$$\text{Using } \dot{x}_0 = 10 \text{ m/sec, } \omega_n = 10 / (0.5 * 2.7183) = 7.3575 \frac{\text{rad}}{\text{sec}}$$

When mass of gun is 500 kg,
 the stiffness of the spring is

$$k = \omega_n^2 m = (7.3575)^2 (500) = 27,066.403 \text{ N/m}$$

2.88

$$k = 5000 \text{ N/m, } c_c = 0.2 \text{ N-s/mm} = 200 \text{ N-s/m}$$

$$= 2 \sqrt{k m} = 2 \sqrt{5000 m}$$

$$m = 2 \text{ kg}$$

$$\omega_n = \sqrt{k/m} = \sqrt{5000/2} = 50 \text{ rad/sec}$$

$$\text{Logarithmic decrement} = \delta = \frac{2\pi \gamma}{\sqrt{1-\gamma^2}} = 2.0$$

$$\text{i.e., } \gamma = \frac{c}{c_c} = 0.3033 \text{ and } c = 0.3033 (0.2) = 60.66 \text{ N-s/m}$$

Assuming $x_0 = 0$ and $\dot{x}_0 = 1 \text{ m/s}$,

$$x(t) = e^{-\gamma \omega_n t} \frac{\dot{x}_0}{\omega_n \sqrt{1-\gamma^2}} \sin \sqrt{1-\gamma^2} \omega_n t$$

For $x_{\max}$, $\omega_n t \approx \pi/2$ and $\sin \sqrt{1-\gamma^2} \omega_n t \approx 1$

$$\therefore x_{\max} \approx e^{-0.3033(\pi/2)} \frac{1}{50 \sqrt{1-0.3033^2}} (1) = 0.01303 \text{ m}$$

2.89

For an overdamped system, Eq. (2.81) gives

$$x(t) = e^{-\gamma \omega_n t} (C_1 e^{\omega_d t} + C_2 e^{-\omega_d t}) \quad (E_1)$$

$$\text{Using the relations } e^{\pm x} = \cosh x \pm \sinh x \quad (E_2)$$

Eq. (E₁) can be rewritten as

$$x(t) = e^{-\gamma \omega_n t} (C_3 \cosh \omega_d t + C_4 \sinh \omega_d t) \quad (E_3)$$

where $C_3 = C_1 + C_2$ and $C_4 = C_1 - C_2$.

Differentiating (E₃),

$$\dot{x}(t) = e^{-\gamma \omega_n t} [C_3 \omega_d \sinh \omega_d t + C_4 \omega_d \cosh \omega_d t] - \gamma \omega_n e^{-\gamma \omega_n t} [C_3 \cosh \omega_d t + C_4 \sinh \omega_d t] \quad (E_4)$$

Initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$ with (E₃) and (E₄) give

$$C_3 = x_0, \quad C_4 = (\dot{x}_0 + \gamma \omega_n x_0) / \omega_d \quad (E_5)$$

Thus (E₃) becomes

$$x(t) = x_0 e^{-\gamma \omega_n t} \left(\cosh \omega_d t + \frac{\gamma \omega_n}{\omega_d} \sinh \omega_d t \right) + \frac{\dot{x}_0}{\omega_d} e^{-\gamma \omega_n t} \sinh \omega_d t \quad (E_6)$$

(i) When $\dot{x}_0 = 0$, Eq. (E₆) gives

$$x(t) = x_0 e^{-\gamma \omega_n t} \left(\cosh \omega_d t + \frac{\gamma \omega_n}{\omega_d} \sinh \omega_d t \right) \quad (E_7)$$

since $e^{-\gamma \omega_n t}$, $\cosh \omega_d t$, $\frac{\gamma \omega_n}{\omega_d}$ and $\sinh \omega_d t$ do not change sign (always positive) and $e^{-\gamma \omega_n t}$ approaches zero with increasing t , $x(t)$ will not change sign.

(ii) When $x_0 = 0$, Eq. (E₆) gives

$$x(t) = \frac{\dot{x}_0}{\omega_d} e^{-\gamma \omega_n t} \sinh \omega_d t \quad (E_8)$$

Here also, ω_d , $e^{-\gamma \omega_n t}$ and $\sinh \omega_d t$ do not change sign (always positive) and $e^{-\gamma \omega_n t}$ approaches zero with increasing t , $x(t)$ will not change sign.

2.90

Newton's second law of motion:

$$\sum F = m \ddot{x} = -kx - c\dot{x} + F_f \quad (1)$$

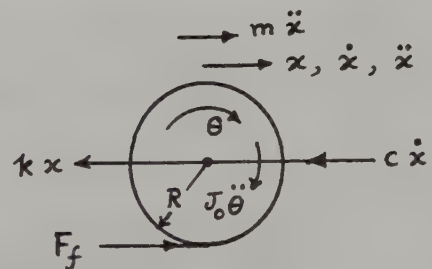
$$\sum M = J_0 \ddot{\theta} = -F_f R \quad (2)$$

where F_f = friction force.

Using $J_0 = \frac{m R^2}{2}$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

$$F_f = -\frac{1}{2R} (m R^2) \frac{\ddot{x}}{R} = -\frac{1}{2} m \ddot{x} \quad (3)$$

Substitution of Eq. (3) into (1) yields:



$$\frac{3}{2} m \ddot{x} + c \dot{x} + k x = 0 \quad (4)$$

The undamped natural frequency is: $\omega_n = \sqrt{\frac{2k}{3m}}$ (5)

2.91 Newton's second law of motion: (measuring x from static equilibrium position of cylinder)

$$\sum F = m \ddot{x} = -kx - c\dot{x} - kx + F_f \quad (1)$$

$$\sum M = J_0 \ddot{\theta} = -F_f R \quad (2)$$

where F_f = friction force. Using $J_0 = \frac{1}{2} m R^2$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

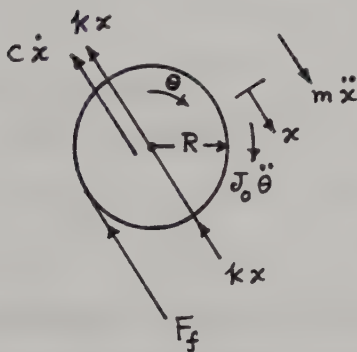
$$F_f = -\frac{1}{2} m \ddot{x} \quad (3)$$

Substitution of Eq. (3) into (1) gives

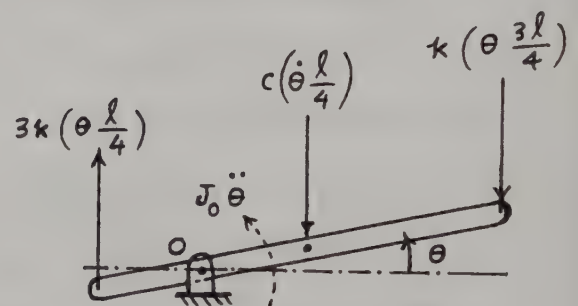
$$\frac{3}{2} m \ddot{x} + c \dot{x} + 2kx = 0 \quad (4)$$

Undamped natural frequency of the system:

$$\omega_n = \sqrt{\frac{4k}{3m}} \quad (4)$$



2.92



Consider a small angular displacement of the bar θ about its static equilibrium position. Newton's second law gives:

$$\begin{aligned} \sum M = J_0 \ddot{\theta} &= -k \left(\theta \frac{3\ell}{4} \right) \left(\frac{3\ell}{4} \right) - c \left(\dot{\theta} \frac{\ell}{4} \right) \left(\frac{\ell}{4} \right) - 3k \left(\theta \frac{\ell}{4} \right) \left(\frac{\ell}{4} \right) \\ \text{i.e., } J_0 \ddot{\theta} + \frac{c\ell^2}{16} \dot{\theta} + \frac{3}{4} k \ell^2 \theta &= 0 \end{aligned}$$

where $J_0 = \frac{7}{48} m \ell^2$. The undamped natural frequency of torsional vibration is given by:

$$\omega_n = \sqrt{\frac{3 k \ell^2}{4 J_0}} = \sqrt{\frac{36 k}{7 m}}$$

2.93

Let δx = virtual displacement given to cylinder. Virtual work done by various forces:

Inertia forces: $\delta W_i = -(J_0 \ddot{\theta}) (\delta\theta) - (m \ddot{x}) \delta x = -(J_0 \ddot{\theta}) \left(\frac{\delta x}{R}\right) - (m \ddot{x}) \delta x$

Spring force: $\delta W_s = -(k x) \delta x$

Damping force: $\delta W_d = -(c \dot{x}) \delta x$

By setting the sum of virtual works equal to zero, we obtain:

$$-\frac{J_0}{R} \left(\frac{\ddot{x}}{R}\right) - m \ddot{x} - k x - c \dot{x} = 0 \quad \text{or} \quad \frac{3}{2} m \ddot{x} + c \dot{x} + k x = 0$$

2.94

Let δx = virtual displacement given to cylinder from its static equilibrium position. Virtual works done by various forces:

Inertia forces: $\delta W_i = -(J_0 \ddot{\theta}) \delta\theta - (m \ddot{x}) \delta x = -(J_0 \frac{\ddot{x}}{R}) \left(\frac{\delta x}{R}\right) - (m \ddot{x}) \delta x$

Spring force: $\delta W_s = -(k x) \delta x - (k x) \delta x = -2 k x \delta x$

Damping force: $\delta W_d = -(c \dot{x}) \delta x$

By setting the sum of virtual works equal to zero, we find

$$-\frac{J_0}{R} \frac{\ddot{x}}{R} - m \ddot{x} - 2 k x - c \dot{x} = 0 \quad (1)$$

Using $J_0 = \frac{1}{2} m R^2$, Eq. (1) can be rewritten as

$$\frac{3}{2} m \ddot{x} + c \dot{x} + 2 k x = 0 \quad (2)$$

2.95

See figure given in the solution of Problem 2.92. Let $\delta\theta$ be virtual angular displacement given to the bar about its static equilibrium position. Virtual works done by various forces:

Inertia force: $\delta W_i = -(J_0 \ddot{\theta}) \delta\theta$

Spring forces:

$$\delta W_s = -(k \theta \frac{3 \ell}{4}) \left(\frac{3 \ell}{4} \delta\theta\right) - (3 k \theta \frac{\ell}{4}) \left(\frac{\ell}{4} \delta\theta\right) = -\left(\frac{3}{4} k \ell^2 \theta\right) \delta\theta$$

Damping force: $\delta W_d = -(c \dot{\theta} \frac{\ell}{4}) \left(\frac{\ell}{4} \delta\theta\right)$

By setting the sum of virtual works equal to zero, we get the equation of motion as:

$$J_0 \ddot{\theta} + c \frac{\ell^2}{16} \dot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

2.96

See solution of Problem 2.72. When wooden prism is given a displacement x , equation of motion becomes: $m \ddot{x} + \text{restoring force} = 0$
 where $m = \text{mass of prism} = 40 \text{ kg}$ and $\text{restoring force} = \text{weight of fluid displaced}$
 $= \rho_0 g a b x = \rho_0 (9.81) (0.4) (0.6) x = 2.3544 \rho_0 x$ where ρ_0 is the density of the fluid. Thus the equation of motion becomes:

$$40 \ddot{x} + 2.3544 \rho_0 x = 0$$

$$\text{Natural frequency} = \omega_n = \sqrt{\frac{2.3544 \rho_0}{40}}$$

$$\text{Since } \tau_n = \frac{2\pi}{\omega_n} = 0.5, \text{ we find}$$

$$\omega_n = \frac{2\pi}{0.5} = 4\pi = \sqrt{\frac{2.3544 \rho_0}{40}}$$

$$\text{Hence } \rho_0 = 2682.8816 \text{ kg/m}^3.$$

2.97

Let $x = \text{displacement of mass}$ and $P = \text{tension in rope on the left of mass}$.
 Equations of motion:

$$\sum F = m \ddot{x} = -kx - P \text{ or } P = -m \ddot{x} - kx \quad (1)$$

$$\sum M = J_0 \ddot{\theta} = P r_2 - c (\dot{\theta} r_1) \quad (2)$$

Using Eq. (1) in (2), we obtain

$$-(m \ddot{x} + kx) r_2 - c \dot{\theta} r_1 = J_0 \ddot{\theta} \quad (3)$$

With $x = \theta r_2$, Eq. (3) can be written as:

$$(J_0 + m r_2^2) \ddot{\theta} + c r_1 \dot{\theta} + k r_2^2 \theta = 0 \quad (4)$$

For given data, Eq. (4) becomes

$$[5 + 10 (0.25)^2] \ddot{\theta} + c (0.1) \dot{\theta} + k (0.25)^2 \theta = 0$$

$$\text{or } 5.625 \ddot{\theta} + 0.1 c \dot{\theta} + 0.0625 k \theta = 0 \quad (5)$$

Since amplitude is reduced by 80% in 10 cycles,

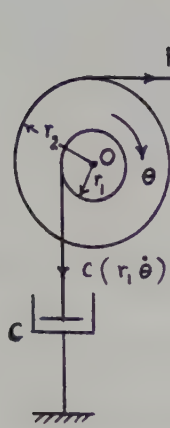
$$\frac{x_1}{x_{11}} = \frac{1.0}{0.2} = 5 = e^{10 \zeta \omega_n \tau_d}$$

$$\ln \frac{x_1}{x_{11}} = \ln 5 = 1.6094 = 10 \zeta \omega_n \tau_d \quad (6)$$

Since the natural frequency (assumed to be undamped torsional vibration frequency) is 5 Hz, $\omega_n = 2\pi (5) = 31.416 \text{ rad/sec}$. Also

$$\tau_d = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{0.2}{\sqrt{1 - \zeta^2}} \quad (7)$$

Eq. (6) gives



$$1.6094 = 10 \zeta (31.416) \left(\frac{0.2}{\sqrt{1-\zeta^2}} \right) = \frac{62.832 \zeta}{\sqrt{1-\zeta^2}}$$

$$\text{i.e., } \sqrt{1-\zeta^2} = \frac{62.832}{1.6094} \zeta = 39.0406 \zeta$$

$$\text{i.e., } \zeta = 0.02561$$

Thus we obtain:

$$\omega_n = \sqrt{\frac{0.0625 k}{5.625}} = 31.416 \text{ or } k = 8.8827 (10^4) \text{ N/m}$$

$$\zeta = 0.02561 = \frac{c}{c_c} = \frac{c}{2 m_{eq} \omega_n} = \frac{0.10 c}{2 (5.625) (31.416)}$$

$$\text{or } c = 90.5134 \text{ N-s/m}$$

2.98 $m = 20 \text{ kg}, \quad k = 4000 \text{ N/m}$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{20}} = 14.1421 \text{ rad/sec}$$

Amplitudes of successive cycles : 50, 45, 40, 35 mm

Amplitudes of successive cycles diminish by $5 \text{ mm} = 5 \times 10^{-3} \text{ m}$

System has Coulomb damping.

$$\frac{4 \mu N}{k} = 5 \times 10^{-3} \Rightarrow \mu N = \left\{ \frac{(5 \times 10^{-3})(4000)}{4} \right\} = 5 \text{ N}$$

= damping force

Frequency of damped vibration = 14.1421 rad/sec .

2.99 $m = 20 \text{ kg}, \quad k = 10000 \text{ N/m}, \quad \frac{4 \mu N}{k} = \frac{150 - 100}{4} \text{ mm} = 12.5 \times 10^{-3} \text{ m}$

$$\mu = \frac{(12.5 \times 10^{-3})(10000)}{4(20 \times 9.81)} = 0.1593$$

$$\text{Time elapsed} = 4 \tau_n = 4 \times \frac{2\pi}{\omega_n} = 8\pi \sqrt{\frac{m}{k}} = 1.124 \text{ sec}$$

2.100 $m = 10 \text{ kg}, \quad k = 3000 \text{ N/m}, \quad \mu = 0.12, \quad X = 100 \text{ mm}$

$$\frac{4 \mu N}{k} = \frac{4(0.12)(10 \times 9.81)}{3000} = 0.0157 \text{ m} = 15.7 \text{ mm}$$

$$\text{As } 6 \left(\frac{4 \mu N}{k} \right) = 94.2 \text{ mm, mass comes to rest at } (100 - 94.2) = 5.8 \text{ mm}$$

2.101 $mg = 25 \text{ N}, \quad k = 1000 \text{ N/m}, \quad \text{damping force} = \text{constant}$
 mass released with $x_0 = 10 \text{ cm}$ and $\dot{x}_0 = 0$.

$$\text{Static deflection of spring due to self weight of mass} = \frac{25}{1000}$$

$$= 0.025 \text{ m}$$

$$\text{at } t = 0: \quad x = 0.1 \text{ m}, \quad \dot{x} = 0$$

$$x_0 = 0.1$$

$$x_1 = x_0 - 2 \frac{\mu N}{k}, \quad x_2 = x_0 - \frac{4 \mu N}{k}$$

$$x_3 = x_0 - \frac{6 \mu N}{k}, \quad x_4 = x_0 - \frac{8 \mu N}{k} = 0$$

$$\text{i.e., } x_0 = \frac{8 \mu N}{k} = 0.1$$

$$\text{Magnitude of damping force} = \mu N = \frac{x_0 k}{8} = \frac{(0.1)(1000)}{8} = 12.5 \text{ N}$$

2.102 $m = 20 \text{ kg}, \quad k = 10,000 \text{ N/m}, \quad \mu N = 50 \text{ N}, \quad x_0 = 0.05 \text{ m}$

(a) Number of half cycles elapsed before mass comes to rest (r) is given by:

$$r \geq \left\{ \frac{x_0 - \frac{\mu N}{k}}{2 \frac{\mu N}{k}} \right\} = \frac{0.05 - \left(\frac{50}{10000} \right)}{2 \left(\frac{50}{10000} \right)} = 4.5$$

$$\therefore r = 5$$

(b) Time elapsed before mass comes to rest:

$$t_p = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{20}{10000}} = 0.2810 \text{ sec}$$

$$\text{Time taken} = (2.5 \text{ cycles}) t_p = 0.7025 \text{ sec}$$

(c) Final extension of spring after 5 half-cycles:

$$x_5 = 0.05 - 5 \left(\frac{2 \mu N}{k} \right) = 0.05 - 5 \left(2 * \frac{50}{10000} \right) = 0$$

(displacement from static equilibrium position = 0)

$$\text{But static deflection} = \frac{mg}{k} = \frac{20 * 9.81}{10000} = 0.01962 \text{ m}$$

$$\therefore \text{Final extension of spring} = 1.9620 \text{ cm.}$$

2.103 (a) Equation of motion for angular oscillations of pendulum:

$$J_0 \ddot{\theta} + mgl \sin \theta \pm mg \mu \frac{d}{2} \cos \theta = 0$$

$$\text{For small angles, } \ddot{\theta} + \frac{mgl}{J_0} \left(\theta \pm \frac{\mu d}{2l} \right) = 0$$

This shows that the angle of swing decreases by $\left(\frac{\mu d}{2l} \right)$ in each quarter cycle.

(b) For motion from right to left:

$$\theta(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t + \frac{\mu d}{2l}$$

$$\text{where } \omega_n = \sqrt{\frac{mgl}{J_0}}$$

$$\text{Let } \theta(t=0) = \theta_0 \text{ and } \dot{\theta}(t=0) = 0. \text{ Then } A_1 = \theta_0 - \frac{\mu d}{2l}, \quad A_2 = 0$$

$$\theta(t) = \left(\theta_0 - \frac{\mu d}{2l} \right) \cos \omega_n t + \frac{\mu d}{2l}$$

For motion from left to right:

$$\theta(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t - \frac{\mu d}{2l}$$

At $\omega_n t = \pi$, $\theta = -\theta_0 + \frac{2\mu d}{2l}$, $\dot{\theta} = 0$ from previous solution.

$$A_3 = \theta_0 - \frac{3\mu d}{2l}, \quad A_4 = 0$$

$$\theta(t) = \left(\theta_0 - \frac{3\mu d}{2l} \right) \cos \omega_n t - \frac{\mu d}{2l}$$

(c) The motion ceases when $\left(\theta_0 - n \frac{4\mu d}{2l} \right) < \frac{\mu d}{2l}$
where n denotes the number of cycles.

2.104 $x(t) = X \sin \omega t$ (under sinusoidal force $F_0 \sin \omega t$)

Damping force = μN

Total displacement per cycle = $4X$

Energy dissipated per cycle = $\Delta W = 4\mu N X$ (E₁)

If c_{eq} = equivalent viscous damping constant, energy dissipated per cycle is given by E_d (2.98):

$$\Delta W = \pi c_{eq} \omega X^2$$
 (E₂)

Equating (E₁) and (E₂) gives

$$c_{eq} = \frac{4\mu N X}{\pi \omega X^2} = \frac{4\mu N}{\pi \omega X}$$
 (E₃)

2.105 Due to viscous damping:

$$\delta = \ln \left(\frac{X_m}{X_{m+1}} \right) \approx 2\pi \zeta$$

ζ_1 = percent decrease in amplitude per cycle at X_m

$$= 100 \left(\frac{X_m - X_{m+1}}{X_m} \right) = 100 \left(1 - \frac{X_{m+1}}{X_m} \right) = 100 (1 - e^{-2\pi \zeta})$$

Due to Coulomb damping:

ζ_2 = percent decrease in amplitude per cycle at X_m

$$= 100 \left(\frac{X_m - X_{m+1}}{X_m} \right) = 100 \left(\frac{4\mu N}{\pi X_m} \right)$$

When both types of damping are present:

$$\zeta_1 + \zeta_2 \Big|_{X_m = 20 \text{ mm}} = 2 ; \quad \zeta_1 + \zeta_2 \Big|_{X_m = 10 \text{ mm}} = 3$$

i.e.,

$$100 (1 - e^{-2\pi\gamma}) + \frac{400}{0.02} \left(\frac{\mu N}{k} \right) = 2$$

$$100 (1 - e^{-2\pi\gamma}) + \frac{400}{0.01} \left(\frac{\mu N}{k} \right) = 3$$

The solution of these equations gives

$$50 (1 - e^{-2\pi\gamma}) = 0.5 \quad \text{and} \quad \frac{\mu N}{k} = 0.5 \times 10^{-6} \text{ m}$$

1.106

Coulomb damping.

- (a) Natural frequency $= \omega_n = \frac{2\pi}{\tau_n} = \frac{2\pi}{1} = 6.2832 \text{ rad/sec}$. Reduction in amplitude in each cycle:

$$\begin{aligned} \frac{4\mu N}{k} &= 4\mu g \frac{m}{k} = \frac{4\mu g}{\omega_n^2} = 4\mu \left(\frac{9.81}{6.2832^2} \right) \\ &= 0.9940 \mu = \frac{0.5}{100} = 0.005 \text{ m} \end{aligned}$$

Kinetic coefficient of friction $= \mu = 0.00503$

- (b) Number of half-cycles executed (r) is:

$$\begin{aligned} r &\geq \frac{(x_0 - \frac{\mu N}{k})}{(\frac{2\mu N}{k})} = \frac{(x_0 - \frac{\mu g}{\omega_n^2})}{(\frac{2\mu g}{\omega_n^2})} \\ &\geq \frac{\left(0.1 - \frac{0.00503(9.81)}{6.2832^2} \right)}{\left(\frac{2(0.00503)(9.81)}{6.2832^2} \right)} \\ &\geq 39.5032 \\ &\geq 40 \end{aligned}$$

Thus the block stops oscillating after 20 cycles.

2.107

Friction force $= \mu N = 0.2(5) = 1 \text{ N}$. $k = \frac{25}{0.10} = 250 \text{ N/m}$. Reduction in amplitude in each cycle $= \frac{4\mu N}{k} = \frac{4(1)}{250} = 0.016 \text{ m}$. Number of half-cycles executed before the motion ceases (r):

$$r \geq \left(\frac{x_0 - \frac{\mu N}{k}}{\frac{2\mu N}{k}} \right) = \frac{0.1 - 0.004}{0.008} \geq 12$$

Thus after 6 cycles, the mass stops at a distance of $0.1 - 6(0.016) = 0.004$ m from the unstressed position of the spring.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{250(9.81)}{5}} = 22.1472 \text{ rad/sec}$$

$$\tau_n = \frac{2\pi}{\omega_n} = 0.2837 \text{ sec}$$

Thus total time of vibration $= 6 \tau_n = 1.7022$ sec.

2.108

Energy dissipated in each full load cycle is given by the area enclosed by the hysteresis loop.

The area can be found by counting the squares enclosed by the hysteresis loop. In Fig. 2.99, the number of squares is ≈ 33 . Since each square $= \frac{100 \times 1}{1000} = 0.1$ N-m, the energy dissipated in a cycle is

$$\Delta W = 33 \times 0.1 = 3.3 \text{ N-m} = \pi k \beta X^2$$

Since the maximum deflection $= X = 4.3$ mm, and the slope of the force-deflection curve is

$$k \approx \frac{1800 \text{ N}}{11 \text{ mm}} = 1.6364 \times 10^5 \text{ N/m},$$

the hysteresis damping constant β is given by

$$\beta = \frac{\Delta W}{\pi k X^2} = \frac{3.3}{\pi (1.6364 \times 10^5) (0.0043)^2} = 0.3472$$

$$\delta = \pi \beta = \text{logarithmic decrement} = \pi (0.3472) = 1.0908$$

$$\text{Equivalent viscous damping ratio} = \zeta_{eq} = \beta/2 = 0.1736.$$

2.109

$$\frac{X_j}{X_{j+1}} = \frac{2 + \pi \beta}{2 - \pi \beta} = 1.1, \quad \beta = 0.03032$$

$$c_{eq} = \beta \sqrt{mk} = 0.03032 \sqrt{1 \times 2} = 0.04288 \text{ N-s/m}$$

$$\Delta W = \pi k \beta X^2 = \pi (2) (0.03032) \left(\frac{10}{1000}\right)^2 = 19.05 \times 10^{-6} \text{ N-m}$$

2.110

$$\text{Logarithmic decrement} = \delta = \ln \left(\frac{X_j}{X_{j+1}} \right) \approx \pi \beta$$

$$\text{For } n \text{ cycles, } \delta = \frac{1}{n} \ln \left(\frac{X_0}{X_n} \right) \approx \pi \beta$$

$$\frac{1}{100} \ln \left(\frac{30}{20} \right) = 0.004055 = \pi \beta$$

$$\beta = 0.001291$$

2.111

$$\text{Torque} = 2 \times 10^{-3} \text{ N-m}$$

$$\text{angle} = 50^\circ = 80 \text{ divisions}$$

For a torsional system, Eq. (2.84) gives

$$\frac{\theta_1}{\theta_2} = e^{\gamma \omega_n \tau_d} \quad (E_1)$$

(b) For one cycle, $\tau_d = 2 \text{ sec}$ and (E₁) gives

$$\frac{80}{5} = e^{2\gamma \omega_n} \quad \text{or} \quad \gamma \omega_n = \frac{1}{2} \ln(16) = 1.3863 \quad (E_2)$$

$$\text{Since} \quad \tau_d = \frac{2\pi}{\sqrt{\omega_n^2 - \gamma^2 \omega_n^2}},$$

$$\omega_n^2 = \frac{(2\pi)^2}{\tau_d^2} + \gamma^2 \omega_n^2 = \frac{4\pi^2}{4} + 1.3863^2 = 11.7915$$

$$\text{i.e., } \omega_n = 3.4339 \text{ rad/sec} \quad (E_3)$$

(d) Since angular displacement of rotor under applied torque
 $= 50^\circ = 0.8727 \text{ rad},$

$$\begin{aligned} k_t &= \text{torque/angular displacement} = 2 \times 10^{-3} / 0.8727 \\ &= 2.2917 \times 10^{-3} \text{ N-m/rad} \end{aligned} \quad (E_4)$$

(a) Mass moment of inertia of rotor is

$$J_o = \frac{k_t}{\omega_n^2} = 2.2917 \times 10^{-3} / 11.7915 = 1.9436 \times 10^{-4} \text{ N-m-s}^2 \quad (E_5)$$

$$(c) \quad c_t = 2 J_o \gamma \omega_n \quad (E_6)$$

$$\text{Eqs. (E}_2\text{) and (E}_3\text{) give } \gamma = \frac{\gamma \omega_n}{\omega_n} = \frac{1.3863}{3.4339} = 0.4037$$

$$\text{Eq. (E}_6\text{) gives } c_t = 5.3887 \times 10^{-4} \text{ N-m-s/rad.}$$

2.112 The input data for Program 2 and results are given below.

```
C      THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(50),XD(50),XDD(50),T(50)
      DATA M,K,C,X0,XD0,N,DELT/
      2 4.0,2500.0,0.0,0.1,-10.0,50,0.01/
C      END OF PROBLEM-DEPENDENT DATA
```

FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

```
M      = 0.40000000E+01
K      = 0.25000000E+04
C      = 0.00000000E+00
X0     = 0.10000000E+00
XD0    = -0.10000000E+02
N      = 50
DELT   = 0.99999998E-02
```

SYSTEM IS UNDAMPED

RESULTS:

I	TIME(I)	X(I)	XD(I)	XDD(I)
1	0.100000E-01	-0.207036E-02	-0.103076E+02	0.129397E+01
2	0.200000E-01	-0.104012E+00	-0.997438E+01	0.650075E+02
3	0.300000E-01	-0.199487E+00	-0.902097E+01	0.124679E+03
4	0.400000E-01	-0.282558E+00	-0.750667E+01	0.176599E+03
5	0.500000E-01	-0.348062E+00	-0.552565E+01	0.217539E+03
6	0.600000E-01	-0.391924E+00	-0.320107E+01	0.244953E+03
7	0.700000E-01	-0.411419E+00	-0.677464E+00	0.257137E+03
8	0.800000E-01	-0.405334E+00	0.188826E+01	0.253334E+03
9	0.900000E-01	-0.374047E+00	0.433659E+01	0.233779E+03
10	0.100000E+00	-0.319503E+00	0.651528E+01	0.199690E+03
:				
46	0.460000E+00	0.398512E+00	-0.264442E+01	-0.249070E+03
47	0.470000E+00	0.359954E+00	-0.502704E+01	-0.224971E+03
48	0.480000E+00	0.299016E+00	-0.709711E+01	-0.186885E+03
49	0.490000E+00	0.219486E+00	-0.872591E+01	-0.137179E+03
50	0.500000E+00	0.126310E+00	-0.981218E+01	-0.789439E+02

The problem- dependent data and results are given(given by Program 2).

2.113

```
C      THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(50),XD(50),XDD(50),T(50)
      DATA M,K,C,X0,XD0,N,DELT/
      2 4.0,2500.0,100.0,0.1,-10.0,50,0.01/
C      END OF PROBLEM-DEPENDENT DATA
```

FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

M = 0.40000000E+01
 K = 0.25000000E+04
 C = 0.10000000E+03
 X0 = 0.10000000E+00
 XD0 = -0.10000000E+02
 N = 50
 DELT = 0.99999998E-02

SYSTEM IS UNDER DAMPED

RESULTS:

I	TIME(I)	X(I)	XD(I)	XDD(I)
1	0.100000E-01	-0.957279E-02	0.807168E+01	-0.195809E+03
2	0.200000E-01	0.613786E-01	0.612586E+01	-0.191508E+03
3	0.300000E-01	0.113259E+00	0.427345E+01	-0.177623E+03
4	0.400000E-01	0.147433E+00	0.259570E+01	-0.157038E+03
5	0.500000E-01	0.165937E+00	0.114627E+01	-0.132367E+03
6	0.600000E-01	0.171219E+00	-0.456091E-01	-0.105872E+03
7	0.700000E-01	0.165914E+00	-0.971334E+00	-0.794127E+02
8	0.800000E-01	0.152654E+00	-0.163885E+01	-0.544377E+02
9	0.900000E-01	0.133930E+00	-0.206856E+01	-0.319924E+02
10	0.100000E+00	0.111980E+00	-0.228942E+01	-0.127520E+02
:				
45	0.450000E+00	-0.113169E-03	-0.310021E-01	0.845783E+00
46	0.460000E+00	-0.381386E-03	-0.227080E-01	0.806065E+00
47	0.470000E+00	-0.569294E-03	-0.149992E-01	0.730789E+00
48	0.480000E+00	-0.684317E-03	-0.817052E-02	0.631961E+00
49	0.490000E+00	-0.736251E-03	-0.240284E-02	0.520228E+00
50	0.500000E+00	-0.736195E-03	0.222121E-02	0.404592E+00

2.114

The problem - dependent data (to be used in Program 2) and output are given.

```

C THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION X(50),XD(50),XDD(50),T(50)
  DATA M,K,C,X0,XD0,N,DELT/
2 4.0,2500.0,200.0,0.1,-10.0,50,0.01/
C END OF PROBLEM-DEPENDENT DATA
  
```

 FREE VIBRATION ANALYSIS
 OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

~~M = 0.40000000E+01~~
~~K = 0.25000000E+04~~
~~C = 0.20000000E+03~~
~~X0 = 0.10000000E+00~~
~~XD0 = -0.10000000E+02~~
~~N = 50~~
~~DELT = 0.99999998E-02~~

~~SYSTEM IS CRITICALLY DAMPED~~

RESULTS:

I	TIME(I)	X(I)	XD(I)	XDD(I)
1	0.100000E-01	0.194700E-01	-0.632776E+01	0.304219E+03
2	0.200000E-01	-0.303265E-01	-0.379082E+01	0.208495E+03
3	0.300000E-01	-0.590458E-01	-0.206660E+01	0.140234E+03
4	0.400000E-01	-0.735759E-01	-0.919699E+00	0.919699E+02
5	0.500000E-01	-0.787888E-01	-0.179066E+00	0.581963E+02
6	0.600000E-01	-0.780956E-01	0.278913E+00	0.348641E+02
7	0.700000E-01	-0.738539E-01	0.543044E+00	0.190065E+02
8	0.800000E-01	-0.676676E-01	0.676676E+00	0.845846E+01
9	0.900000E-01	-0.606045E-01	0.724619E+00	0.164687E+01
10	0.100000E+00	-0.533553E-01	0.718244E+00	-0.256515E+01
:				
46	0.460000E+00	-0.339359E-04	0.772422E-03	-0.174112E-01
47	0.470000E+00	-0.270210E-04	0.616356E-03	-0.139296E-01
48	0.480000E+00	-0.215048E-04	0.491539E-03	-0.111364E-01
49	0.490000E+00	-0.171069E-04	0.391783E-03	-0.889736E-02
50	0.500000E+00	-0.136023E-04	0.312109E-03	-0.710396E-02

2.115

The problem-dependent data (to be used in Program 2) and results are given.

```
C THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION X(50),XD(50),XDD(50),I(50)
  DATA M,K,C,X0,XD0,N,DELT/
2 4.0,2500.0,400.0,0.1,-10.0,50,0.01/
C END OF PROBLEM-DEPENDENT DATA
```

FREE VIBRATION ANALYSIS
OF A SINGLE DEGREE OF FREEDOM SYSTEM

DATA

```
M = 0.400000000E+01
K = 0.250000000E+04
C = 0.400000000E+03
X0 = 0.100000000E+00
XD0 = -0.100000000E+02
N = 50
DELT = 0.99999998E-02
```

~~SYSTEM IS OVER DAMPED~~

RESULTS:

I	TIME(I)	X(I)	XD(I)	XDD(I)
---	---------	------	-------	--------

1	0.100000E-01	0.351455E-01	-0.390559E+01	0.368593E+03
2	0.200000E-01	0.990550E-02	-0.151007E+01	0.144816E+03
3	0.300000E-01	0.230870E-03	-0.569458E+00	0.568015E+02
4	0.400000E-01	-0.333730E-02	-0.201042E+00	0.221900E+02
5	0.500000E-01	-0.451879E-02	-0.576071E-01	0.858495E+01
6	0.600000E-01	-0.477582E-02	-0.257606E-02	0.324249E+01
7	0.700000E-01	-0.468266E-02	0.177700E-01	0.114966E+01
8	0.800000E-01	-0.446433E-02	0.245564E-01	0.334569E+00
9	0.900000E-01	-0.420854E-02	0.260878E-01	0.215607E-01
10	0.100000E+00	-0.394902E-02	0.256257E-01	-0.944359E-01
:				
46	0.460000E+00	-0.354993E-03	0.237800E-02	-0.159296E-01
47	0.470000E+00	-0.331992E-03	0.222393E-02	-0.148975E-01
48	0.480000E+00	-0.310481E-03	0.207983E-02	-0.139322E-01
49	0.490000E+00	-0.290364E-03	0.194507E-02	-0.130295E-01
50	0.500000E+00	-0.271551E-03	0.181905E-02	-0.121853E-01

2.116

The equations for the natural frequencies of vibration were derived in Problem 2.35.

Operating speed of turbine is :

$$\omega_0 = (2400) \frac{2\pi}{60} = 251.328 \text{ rad/sec}$$

Thus we need to satisfy :

$$\omega_n|_{\text{axial}} = \left\{ \frac{-g l A E}{W a (l-a)} \right\}^{1/2} \geq \omega_0 \quad (E_1)$$

$$\omega_n|_{\text{transverse}} = \left\{ \frac{3 E I l^3 g}{W a^3 (l-a)^3} \right\}^{1/2} \geq \omega_0 \quad (E_2)$$

$$\omega_n|_{\text{circumferential}} = \left\{ \frac{G J}{J_0} \left(\frac{1}{a} + \frac{1}{l-a} \right) \right\}^{1/2} \geq \omega_0 \quad (E_3)$$

where $A = \frac{\pi d^2}{4}$, $W = 1000 \times 9.81 = 9810 \text{ N}$,

$$I = \frac{\pi d^4}{64} , \quad J = \frac{\pi d^2}{32} , \quad J_0 = 500 \text{ kg-m}^2 ,$$

and $E = 207 \times 10^9 \text{ N/m}^2$, $G = 79.3 \times 10^9 \text{ N/m}^2$ (for steel).

The unknowns d , l and a can be determined to satisfy the inequalities (E_1) , (E_2) and (E_3) using a trial and error procedure.

2.117

From solution of problem 2.38, the requirements can be stated as:

$$\omega_n \Big|_{\text{pivot ends}} = \sqrt{\frac{12EI}{l^3 \left(\frac{W}{g} + m_{\text{eff}1} \right)}} \geq \omega_0 \quad (E_1)$$

Where $E = 30 \times 10^6 \text{ psi}$ and $I = \frac{\pi}{64} [d^4 - (d-2t)^4]$

$$\omega_n \Big|_{\text{fixed ends}} = \sqrt{\frac{48EI}{l^3 \left(\frac{W}{g} + m_{\text{eff}2} \right)}} \geq \omega_0 \quad (E_2)$$

with

$$m_{\text{eff}1} = (0.2357 \text{ m}), \quad m_{\text{eff}2} = (0.3714 \text{ m}),$$

$$m = \text{mass of each column} = \frac{\pi}{4} [d^2 - (d-2t)^2] \frac{l \rho}{g},$$

$$\rho = 0.283 \text{ lb/in}^3, \quad g = 386.4 \text{ in/sec}^2,$$

$$l = \text{length of column} = 96 \text{ in.},$$

$$W = \text{weight of floor} = 4000 \text{ lb.},$$

$$\underline{W} = \text{weight of columns} = 4 \left\{ \frac{\pi}{4} [d^2 - (d-2t)^2] l \rho \right\} \quad (E_3)$$

$$\text{Frequency limit} = \omega_0 = 50 \times 2\pi = 314.16 \text{ rad/sec.}$$

Problem: Find d and t such that $\underline{W}$ given by Eq. (E₃) is minimized while satisfying the inequalities (E₁) and (E₂).

This problem can be solved either by graphical optimization or by using a trial and error procedure.

2.118

$$J_0 = \frac{ml^2}{12} + \frac{ml^2}{4} + Ml^2 = \frac{1}{3}ml^2 + Ml^2 \quad \dots (E_1)$$

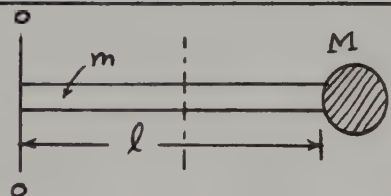
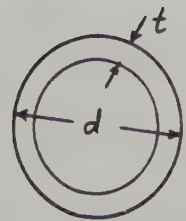
(i) Viscous damping:

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left(\frac{k_t}{\frac{1}{3}ml^2 + Ml^2} \right)^{\frac{1}{2}} \quad \dots (E_2)$$

$$(c_t)_{\text{cri}} = 2J_0\omega_n = 2\sqrt{J_0 k_t} \quad \dots (E_3)$$

For critical damping, Eq. (2.80) gives

$$\theta(t) = \left\{ \theta_0 + (\dot{\theta}_0 + \omega_n \theta_0) t \right\} e^{-\omega_n t} \quad \dots (E_4)$$



For $\theta_0 = 75^\circ = 1.309 \text{ rad}$ and $\dot{\theta}_0 = 0$,

$$\theta(t) = (1.309 + 1.309 \omega_n t) e^{-\omega_n t} \quad \text{--- (E5)}$$

For $\theta = 5^\circ = 0.08727 \text{ rad}$, Eq. (E5) becomes

$$0.08727 = 1.309 (1 + \omega_n t) e^{-\omega_n t} \quad \text{--- (E6)}$$

Let time to return = 2 sec. Then Eq. (E6) gives

$$0.08727 = 1.309 (1 + 2\omega_n) e^{-2\omega_n} \quad \text{--- (E7)}$$

Solve (E7) by trial and error to find ω_n . Then choose the values of m , M and k_t to get the desired value of ω_n . Find the damping constant $(c_t)_{\text{cri}}$ using Eq. (E3).

(ii) Coulomb damping:

(a) Follow the procedure of part(i) to find the value of ω_n .

(b) Derive expression for the equivalent torsional viscous damping constant $(c_t)_{\text{eq}}$ for Coulomb damping. This expression, for small amounts of damping, is

$$(c_t)_{\text{eq}} = \left\{ 4 T_d / \pi \omega_n \Theta \right\} \quad \text{--- (E8)}$$

where T_d = friction (damping) torque, and Θ = amplitude of angular oscillations.

(c) If $(c_t)_{\text{eq}}$ is to be equal to $(c_t)_{\text{cri}} = 2\sqrt{J_o k_t}$, we find

$$T_d = \frac{\pi \omega_n \Theta}{4} (2\sqrt{J_o k_t}) \quad \text{--- (E9)}$$

2.119

Let x = vertical displacement of the mass (lunar excursion module), x_s = resulting deflection of each inclined leg (spring). From equivalence of potential energy, we find:

$$k_{eq1} = \text{stiffness of each leg in vertical direction} = k \cos^2 \alpha$$

Hence for the four legs, the equivalent stiffness in vertical direction is:

$$k_{eq} = 4 k \cos^2 \alpha$$

Similarly, the equivalent damping coefficient of the four legs in vertical direction is:

$$c_{eq} = 4 c \cos^2 \alpha$$

where c = damping constant of each leg (in axial motion). Modeling the system as a single degree of freedom system, the equation of motion is:

$$m_{eq} \ddot{x} + c_{eq} \dot{x} + k_{eq} x = 0$$

and the damped period of vibration is:

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{2\pi}{\sqrt{\frac{k_{eq}}{m_{eq}}} \sqrt{1 - \left[\frac{c_{eq}^2}{4 k_{eq} m_{eq}} \right]}}$$

Using $m_{eq} = 2000$ kg, $k_{eq} = 4 k \cos^2 \alpha$, $c_{eq} = 4 c \cos^2 \alpha$, and $\alpha = 20^\circ$, the values of k and c can be determined (by trial and error) so as to achieve a value of τ_d between 1 s and 2 s. Once k and c are known, the spring (helical) and damper (viscous) can be designed suitably.

2.120

Assume no damping. Neglect masses of telescoping boom and strut. Find stiffness of telescoping boom in vertical direction (see Example 2.2). Find the equivalent stiffness of telescoping boom together with the strut in vertical direction. Model the system as a single degree of freedom system with natural time period:

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m_{eq}}{k_{eq}}}$$

Using $\tau_n = 1$ s and $m_{eq} = \left(\frac{W_c + W_f}{g} \right) = \frac{300}{386.4}$, determine the axial stiffness of the strut (k_s). Once k_s is known, the cross section of the strut (A_s) can be found from:

$$k_s = \frac{A_s E_s}{\ell_s}$$

with $E_s = 30 (10^6)$ psi and ℓ_s = length of strut (known).

Chapter 3

Harmonically Excited Vibration

$$(3.1) \quad (a) \quad \delta = \frac{W}{k} = \frac{50}{4000} = 0.0125 \text{ m}$$

$$(b) \quad \delta_{st} = \frac{F_0}{k} = \frac{60}{4000} = 0.015 \text{ m}$$

$$(c) \quad \omega_n = \sqrt{\frac{k}{m}} = \left(\frac{4000 \times 9.81}{50} \right)^{1/2} = 28.0143 \text{ rad/sec}$$

$$\omega = 6 \text{ Hz} = 37.6992 \text{ rad/sec}$$

$$X = \delta_{st} \left| \frac{1}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \right| = 0.015 \left| \frac{1}{1 - \left(\frac{37.6992}{28.0143} \right)^2} \right| = 0.0185 \text{ m}$$

$$(3.2) \quad \tau_b = \frac{2\pi}{\omega_n - \omega} = \frac{2\pi}{2\pi(40.0 - 39.8)} = 5 \text{ sec}$$

$$(3.3) \quad \delta_{st} = \frac{F_0}{k} = \frac{25}{2000} = 0.0125 \text{ m}$$

$$\text{steady state solution at resonance} = x(t) = \frac{\delta_{st} \cdot \omega_n t}{2} \sin \omega_n t$$

$$= 0.00625 \omega_n t \sin \omega_n t \text{ m}$$

$$(a) \text{ At end of } \frac{1}{4} \text{ cycle, } \omega_n t = \frac{\pi}{2} \text{ and } x(t) = 0.00625 \left(\frac{\pi}{2} \right) \sin \frac{\pi}{2} = 0.009817 \text{ m}$$

$$(b) \text{ At end of } 2\frac{1}{2} \text{ cycles, } \omega_n t = 5\pi \text{ and } x(t) = 0.00625(5\pi) \sin 5\pi = 0$$

$$(c) \text{ At end of } 5\frac{3}{4} \text{ cycles, } \omega_n t = 11\frac{1}{2}\pi \text{ and}$$

$$x(t) = 0.00625 \left(\frac{23}{2} \pi \right) \sin \frac{23}{2} \pi = -0.2258 \text{ m}$$

$$(3.4) \quad \delta_{st} = \frac{F_0}{k} = \frac{100}{4000} = 0.025 \text{ m}$$

$$X = \delta_{st} \cdot \frac{1}{\left| 1 - \left(\frac{\omega}{\omega_n} \right)^2 \right|}, \quad \left| 1 - \left(\frac{\omega}{\omega_n} \right)^2 \right| = \frac{\delta_{st}}{X} = \frac{0.025}{20 \times 10^{-3}} = 1.25$$

$$\frac{\omega}{\omega_n} = \sqrt{1.25 + 1} = 1.5$$

$$\omega_n = \omega / 1.5 = 5(2\pi) / 1.5 = 20.944 \text{ rad/sec}$$

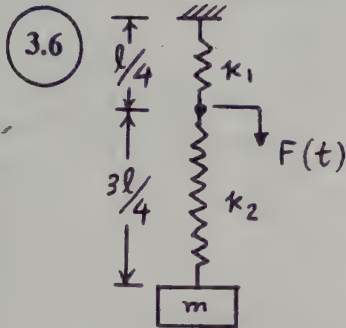
$$m = k / \omega_n^2 = 4000 / (20.944)^2 = 9.1189 \text{ kg}$$

$$(3.5) \quad \omega_n = \sqrt{k/m} = \sqrt{5000/10} = 22.3607 \text{ rad/sec}$$

$$\delta_{st} = F_0/k = 250/5000 = 0.05 \text{ m}$$

$$X = \delta_{st} \left\{ \frac{1}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \right\}$$

$$\text{i.e., } \omega = \omega_n \left(1 - \frac{\delta_{st}}{X} \right)^{\frac{1}{2}} = 22.3607 \left[1 - \frac{0.05}{0.10} \right]^{\frac{1}{2}} \\ = 15.8114 \text{ rad/sec}$$



$$k_1 = 4k ; \quad \frac{1}{4k} + \frac{1}{k_2} = \frac{1}{k} , \quad k_2 = \frac{4}{3}k$$

Force transmitted to the mass through k_2 :

$$\tilde{F}(t) = \frac{k_2}{k_1 + k_2} F(t) = \frac{k_1 k_2}{k_1 + k_2} \left(\frac{F_0}{k_1} \right) \cos \omega t \\ = k \delta_{st} \cos \omega t \quad \text{where } \delta_{st} = \frac{F_0}{k_1}$$

Steady state response of m :

$$x(t) = \frac{\tilde{F}_0}{k \left\{ 1 - \left(\frac{\omega}{\omega_n} \right)^2 \right\}} \cos \omega t \\ = \left\{ \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \right\} \cos \omega t \quad \text{with } \tilde{F}_0 = k \cdot \delta_{st}$$

3.7

Equivalent stiffness of wing (beam) at location of engine:

$$k = \frac{\text{force}}{\text{deflection}} = \frac{3EI}{\ell^3} = \frac{3E \left(\frac{1}{12} b a^3 \right)}{\ell^3} = \frac{E b a^3}{4 \ell^3}$$

$$\text{Magnitude of unbalanced force: } = m r \omega^2 = m r \left(\frac{2\pi N}{60} \right)^2 = \frac{m r \pi^2 N^2}{900}$$

$$\text{Equivalent mass of wing at location of engine: } M = \frac{33}{140} m_w = \frac{33}{140} (a b \ell \rho)$$

$$\text{Equation of motion: } M \ddot{x} + k x = m r \omega^2 \sin \omega t$$

Maximum steady state displacement of wing at location of engine:

$$X = \left| \frac{m r \omega^2}{k - M \omega^2} \right| = \left| \frac{\left(\frac{m r \pi^2 N^2}{900} \right)}{\left\{ \frac{E b a^3}{4 \ell^3} - \frac{33}{140} a b \ell \rho \left(\frac{2\pi N}{60} \right)^2 \right\}} \right| \\ = \left| \frac{m r \ell^3 N^2}{22.7973 E b a^3 - 0.2357 \rho a b \ell^4 N^2} \right|$$

3.8

Rotating unbalanced force, $m r \omega^2$, can be resolved into two components as:

$$F_y = m r \omega^2 \sin \omega t \quad (\text{parallel } y\text{-axis})$$

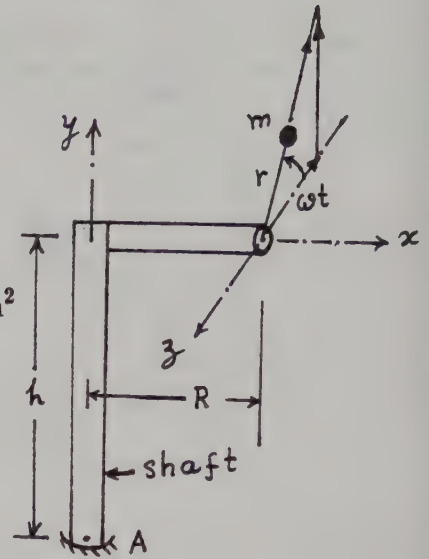
$$F_z = m r \omega^2 \cos \omega t \quad (\text{parallel } z\text{-axis})$$

Maximum bending stress at A:

$$\begin{aligned} \sigma_b &= \frac{1}{I_z} |M_z| \frac{d_o}{2} = \frac{m r \omega^2 R \left(\frac{d_o}{2} \right)}{\frac{\pi}{64} (d_o^4 - d_i^4)} \\ &= \frac{(0.1) (0.1) (31.416^2) (0.5) \left(\frac{0.1}{2} \right)}{\frac{\pi}{64} (0.1^4 - 0.08^4)} = 8.5124 (10^4) \text{ N/m}^2 \end{aligned}$$

Maximum torsional stress at A:

$$\begin{aligned} \sigma_t &= \frac{1}{J_y} |M_y| \left(\frac{d_o}{2} \right) = \frac{m r \omega^2 R \left(\frac{d_o}{2} \right)}{\frac{\pi}{32} (d_o^4 - d_i^4)} \\ &= 4.2562 (10^4) \text{ N/m}^2 \end{aligned}$$



3.9

Total stiffness with steel specimen:

$$k_{eq} = k_1 + k_2 = 10,217.0296 + 750,000.0 = 760,217.0296 \text{ lb/in}$$

Force in specimen due to magnets (static) due to elongation $X = k_2 X$.

Force in specimen due to a.c. current in magnets (dynamic) due to elongation $X = k_{eq} X - m \omega^2 X$.

$$\begin{aligned} \text{Ratio of stresses} &= \left| \frac{k_2 X}{k_{eq} X - m \omega^2 X} \right| = \frac{1}{2} \quad \text{i.e.,} \quad \left| \frac{k_2}{k_{eq} - m \omega^2} \right| = \frac{1}{2} \text{ sp} \\ \text{i.e.,} \quad &\left| \frac{750,000.0}{760,217.0296 - \left(\frac{40}{386.4} \right) \omega^2} \right| = \frac{1}{2} \end{aligned}$$

Squaring both sides of this equation and rearranging gives:

$$\begin{aligned} 107.1225 \omega^4 - 15.7365 (10^8) \omega^2 - 167.207 (10^{14}) &= 0 \\ \text{or } \omega^2 &= 0.218378 (10^8) \quad (\text{positive value}) \\ \omega &= 4673.0935 \text{ rad/sec} = 743.7442 \text{ Hz} \end{aligned}$$

3.10

Equation of motion: $m_{eq} \ddot{x} + k_{eq} x = F(t)$

where m_{eq} = mass of valve and valve rod plus mass of spring at end = $(20 + (15/3))/386.4 = 0.0647 \text{ lb-sec}^2/\text{in}$.

$k_{eq} = 400 \text{ lb/in}$, $F(t) = A p(t) = A p_0 \sin \omega t = 100 (10) \sin \omega t = 1000 \sin 8t \text{ lb}$.

Response of valve (steady state) = $x_p(t) = X \sin 8t$ in where

$$X = \frac{1000}{400 - 0.0647 (8)^2} = 2.5261 \text{ in}$$

3.11

(a) Equation of motion:

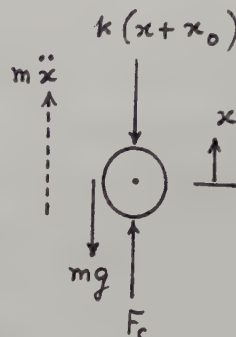
$$m_0 \ddot{x} + k(x + x_0) + m_0 g = F_c; \ddot{x} > 0 \quad (1)$$

where F_c = force exerted on the follower by the cam, m_0 = mass of follower plus one third the mass of the spring, and x_0 = initial displacement of the spring.

(b) Force exerted on the follower by the cam:

$$F_c = m_0 \ddot{x} + k(x + x_0) + m_0 g \quad (2)$$

with $x = e \cos \omega t$.



(c) Condition under which follower loses contact with the cam is when F_c is zero and $\ddot{x}$ is negative. Equation (1) can be used to state this condition as:

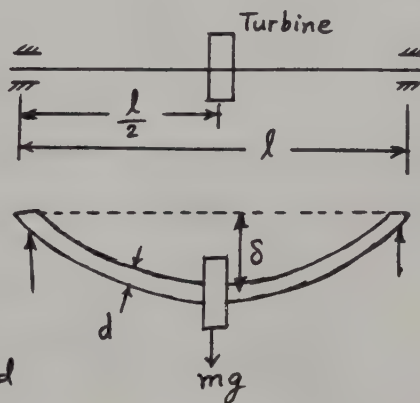
$$k(x + x_0) + m_0 g \geq |m_0 \ddot{x}| \quad (3)$$

3.12

δ_{st} = static radial displacement of shaft under weight of turbine

δ = radial deflection of shaft during rotation

$k = \frac{48EI}{l^3}$ = stiffness of centrally loaded simply supported beam



$$m \delta \omega^2 = k(\delta - \delta_{st}) \quad \text{or} \quad m \omega^2 = k - k \cdot \frac{\delta_{st}}{\delta}$$

$$\text{or} \quad \frac{\delta}{\delta_{st}} = \frac{k}{k - m \omega^2} \quad (E_1)$$

$$\text{Critical speed is } \omega_{cri} = \sqrt{\frac{k}{m}} \quad (E_2)$$

If critical speed = $\frac{1}{5}$ th of operating speed,

$$\sqrt{\frac{k}{m}} = \frac{1}{5} \omega \quad (E_3)$$

$$\text{Here } m = 500/386.4 = 1.2940 \text{ lb-s}^2/\text{in}$$

$$\text{and } \omega = 3000 \times 2\pi/60 = 314.16 \text{ rad/sec}$$

For solid shaft (steel) of diameter d and length l , $E_g \cdot (E_3)$ gives

$$\frac{48 E I}{m l^3} = \frac{\omega^2}{25} \quad \text{with } E = 30 \times 10^6 \text{ psi and } I = \frac{\pi d^4}{64}$$

$$\text{i.e., } \frac{l^3}{d^4} = 13836.8 \quad (E_4)$$

Let $l = 30 d$ in Eq. (E4):

$$d = \frac{27000}{13836.8} = 1.9513 \text{ inch and hence } l = 58.5395 \text{ inch.}$$

$$(3.13) \quad I = \frac{\pi}{64} (d_o^4 - d_i^4) = \frac{\pi}{64} (4^4 - 3.5^4) = 5.2002 \text{ in}^4$$

$$k = \frac{48 E I}{l^3} = \frac{48 (30 \times 10^6) (5.2002)}{(100)^3} = 7488.288 \text{ lb/in}$$

$$m = 500 / 386.4 = 1.2940 \text{ lb-s}^2/\text{in}$$

$$\omega_n = \sqrt{k/m} = \sqrt{7488.288 / 1.2940} = 76.0719 \text{ rad/sec}$$

$$\text{Eccentricity} = r = 2 \text{ in, eccentric mass} = m_o = \frac{0.5}{386.4} \frac{\text{lb-s}^2}{\text{in}}$$

Radial force due to eccentric mass at resonance

$$= F_o = m_o r \omega^2 = \left(\frac{5}{386.4}\right)(2)(76.0719)^2 = 149.7654 \text{ lb}$$

Let $x(t)$ = radial displacement of turbine.

At resonance, Eq. (3.15) gives, for $x_o = \dot{x}_o = 0$,

$$x(t) = \frac{1}{2} \delta_{st} \omega_n t \sin \omega_n t$$

$$\text{where } \delta_{st} = \frac{F_o}{k} = \frac{149.7654}{7488.288} = 0.02 \text{ in}$$

To activate the limit switch, $x(t) = 0.5 \text{ in.}$ and hence

$$0.5 = \frac{1}{2} (0.02) (76.0719) t \sin 76.0719 t$$

$$\text{i.e., } t \sin 76.0719 t = 0.6573 \quad (E_1)$$

Eq. (E1) is solved by trial and error (assuming values

of $t = 1.0, 0.9, 0.8, 0.7$, etc.) as

$$t \approx 0.6760 \text{ sec.}$$

$$(3.14) \quad \text{Tip load} = 0.1 \text{ lb, tip mass} = m_o = \frac{0.1}{386.4} = 2.588 \times 10^{-4} \text{ lb-s}^2/\text{in}$$

$$I = \frac{1}{12} (0.2) (0.05)^3 = 2.0833 \times 10^{-6} \text{ in}^4$$

$$k = \frac{3EI}{l^3} = \frac{3(30 \times 10^6)(2.0833 \times 10^{-6})}{(10)^3} = 0.1875 \text{ lb/in}$$

$$m = \text{mass of beam} = \frac{0.283}{386.4} (10 \times 0.2 \times 0.05) = 7.324 \times 10^{-5} \text{ lb-s}^2/\text{in}$$

$$\omega_n = \left(\frac{k}{m_0 + 0.23 m} \right)^{\frac{1}{2}} = \left[\frac{0.1875}{(2.588 + 0.7324) 10^{-4}} \right]^{\frac{1}{2}} = 5.6469 \frac{\text{rad}}{\text{sec}}$$

Eq. (3.68) gives

$$\frac{x}{Y} = \left\{ \frac{1 + (2\gamma r)^2}{(1-r^2)^2 + (2\gamma r)^2} \right\}^{\frac{1}{2}}$$

i.e.,

$$\frac{2.5}{0.05} = 50 = \left\{ \frac{1 + (2 \times 0.01 r)^2}{(1-r^2)^2 + (2 \times 0.01 r)^2} \right\}^{\frac{1}{2}}$$

i.e.

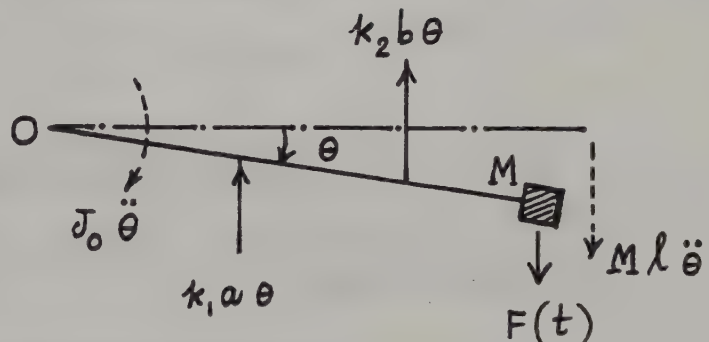
$$r^4 - 1.9996 r^2 + 0.9996 = 0$$

i.e.

$$r = \frac{\omega}{\omega_n} = 0.9999$$

$$\therefore \omega = 0.9999 \omega_n = 5.6463 \text{ rad/sec.}$$

3.15



Equation of motion for rotational motion about the hinge O:

$$(J_0 + M \ell^2) \ddot{\theta} + (k_1 a^2 + k_2 b^2) \theta = F(t) \ell = F_0 \ell \sin \omega t \quad (1)$$

Steady state response (using Eqs. (3.3) and (3.6)):

$$\theta_p(t) = \Theta \sin \omega t \quad (2)$$

$$\text{where } \Theta = \frac{F_0 \ell}{(k_1 a^2 + k_2 b^2) - (J_0 + M \ell^2) \omega^2} \quad (3)$$

$$\text{and } J_0 = \frac{m \ell^2}{12} + m \left(\frac{\ell}{2} \right)^2 = \frac{1}{3} m \ell^2 \quad (4)$$

For given data, $J_0 = \frac{1}{3} (10) (1^2) = 3.3333 \text{ kg-m}^2$, $\omega = \frac{1000 (2\pi)}{60} = 104.72 \text{ rad/sec}$,
and

$$\Theta = \frac{500 (1)}{5000 (0.25^2 + 0.5^2) - (3.3333 + 50 (1^2)) (104.72^2)} = -8.5718 (10^{-4}) \text{ rad}$$

3.16 Equation of motion for rotation about O:

$$J_0 \ddot{\theta} = -k \frac{\theta \ell}{4} \frac{\ell}{4} - k \frac{\theta 3 \ell}{4} \frac{3 \ell}{4} + M_0 \cos \omega t$$

$$\text{i.e., } J_0 \ddot{\theta} + \left(\frac{5}{8} k \ell^2 \right) \theta = M_0 \cos \omega t$$

$$\text{where } J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4} \right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10) (1^2) = 1.4583 \text{ kg-m}^2$$

and $\omega = 1000 \text{ rpm} = 104.72 \text{ rad/sec}$. Steady state solution is:

$$\theta_p(t) = \Theta \cos \omega t$$

where

$$\Theta = \frac{M_0}{\frac{5}{8} k \ell^2 - J_0 \omega^2} = \frac{100}{5000 \left(\frac{5}{8} \right) (1^2) - 1.4583 (104.72^2)} = -0.007772 \text{ rad}$$

3.17 $m = \frac{500}{386.4} \text{ lb-sec}^2/\text{in}$, $F(t) = 200 \sin 100 \pi t \text{ lb}$. Let $X_{\max} = 0.05 \text{ in} < 0.1 \text{ in}$ (maximum permissible value). From Eq. (3.33),

$$X_{\max} = \delta_{st} \frac{1}{2 \zeta \sqrt{1 - \zeta^2}} = 0.05 \quad (1)$$

$$\text{Let } \zeta = 0.01. \text{ Then } \delta_{st} = \frac{F_0}{k_{eq}} = \frac{200}{k_{eq}} \text{ and Eq. (1) gives}$$

$$k_{eq} = \frac{200}{2 (0.01) \sqrt{1 - 0.0001} (0.05)} = 20.0020 (10^4) \text{ lb/in}$$

Since shock mounts are in parallel, stiffness of each mount $= k = \frac{k_{eq}}{3} = 6.6673 (10^4) \text{ lb/in}$.

$$\zeta = \frac{c_{eq}}{c_c} = \frac{c_{eq}}{\sqrt{2 k_{eq} m}}$$

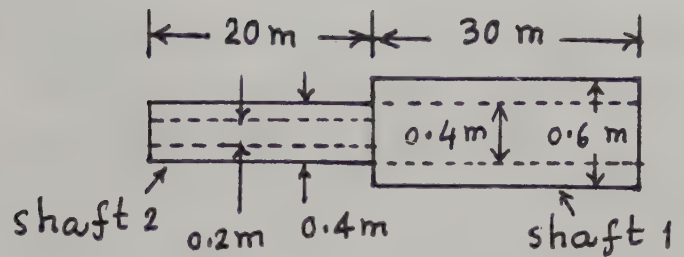
$$\text{or } c_{eq} = \zeta \sqrt{2 k_{eq} m} = 0.01 \sqrt{2 (20.0020 (10^4)) \left(\frac{500}{386.4} \right)} = 7.1948 \text{ lb-sec/in}$$

$$\text{and hence } c = \frac{c_{eq}}{3} = 2.3983 \text{ lb-sec/in}$$

3.18 Equation of motion for torsional system:

$$J_0 \ddot{\theta} + c_t (\dot{\theta} - \dot{\alpha}) + k_t (\theta - \alpha) = 0 \quad (1)$$

where θ = angular displacement of shaft and α = angular displacement of base of shaft $= \alpha_0 \sin \omega t$. Steady state response of propeller (Eq. (3.67)):



$$\theta_p(t) = \Theta \sin(\omega t - \phi) \quad (2)$$

$$\text{where } \Theta = \alpha_0 \left\{ \frac{k_t^2 + (c_t \omega)^2}{(k_t - J_0 \omega^2)^2 - (c_t \omega)^2} \right\}^{\frac{1}{2}} \quad (3)$$

$$\text{and } \phi = \tan^{-1} \left\{ \frac{J_0 c_t \omega^3}{k_t (k_t - J_0 \omega^2) + (c_t \omega)^2} \right\} \quad (4)$$

Here $J_0 = 10^4 \text{ kg-m}^2$, $\zeta_t = 0.1$, and $\omega = 314.16 \text{ rad/sec}$. Torsional stiffnesses of shafts:

$$(k_t)_1 = \frac{G_1 J_1}{\ell_1} = \frac{(80 (10^9)) \left(\frac{\pi}{32} (0.6^4 - 0.4^4) \right)}{30} = 27.2272 (10^6) \text{ N-m/rad}$$

$$(k_t)_2 = \frac{G_2 J_2}{\ell_2} = \frac{(80 (10^9)) \left(\frac{\pi}{32} (0.4^4 - 0.2^4) \right)}{20} = 9.4248 (10^6) \text{ N-m/rad}$$

Series springs give:

$$k_t = \frac{(k_t)_1 (k_t)_2}{(k_t)_1 + (k_t)_2} = \frac{(27.2272 (10^6)) (9.4248 (10^6))}{27.2272 (10^6) + 9.4248 (10^6)} = 7.0013 (10^6) \text{ N-m/rad}$$

$$c_t = \zeta (2 \sqrt{J_0 k_t}) = 0.1 (2) \sqrt{(10^4) (7.0013 (10^6))} = 52,919.8624 \text{ N-m-s/rad}$$

From Eq. (3),

$$\Theta = 0.05 \left[\frac{(7.0013 (10^6))^2 + \left\{ 5.2920 (10^4) (314.16^2) \right\}^2}{\left\{ 7.0013 (10^6) - (10^4) (314.16^2) \right\}^2 + \left\{ 5.2920 (10^4) (314.16) \right\}^2} \right]^{\frac{1}{2}} \\ = 9.2028 (10^{-4}) \text{ rad}$$

$$\phi = \tan^{-1} \left\{ \frac{(10^4) (5.2920 (10^4)) (314.16^3)}{7.0013 (10^6) \left[7.0013 (10^6) - (10^4) (314.16^2) \right] + (5.2920 (10^4) (314.16))^2} \right\} \\ = \tan^{-1} (59.3664) = 89.0350^\circ = 1.5540 \text{ rad}$$

$$(3.19) \quad X = \frac{\delta_{st}}{\{(1-r^2)^2 + (2\gamma r)^2\}^{1/2}}$$

$$\text{For maximum } X, \quad \frac{dX}{dr} = -\delta_{st} \cdot \frac{1}{2} \frac{1}{\{(1-r^2)^2 + (2\gamma r)^2\}^{3/2}} \cdot \{2(1-r^2)(-2r) + 2(2\gamma r)(2\gamma)\} = 0$$

$$\text{i.e.,} \quad -4r(1-r^2) + 8r\gamma^2 = 0$$

$$\text{i.e.,} \quad r = \sqrt{1-2\gamma^2}$$

$$X \Big|_{at \ r = \sqrt{1-2\gamma^2}} = \frac{\delta_{st}}{\left[\{1 - (1-2\gamma^2)\}^2 + (2\gamma \sqrt{1-2\gamma^2})^2 \right]^{1/2}} = \frac{\delta_{st}}{2\gamma \sqrt{1-\gamma^2}}$$

$$\therefore \left(\frac{X}{\delta_{st}} \right)_{max} = \frac{1}{2\gamma \sqrt{1-\gamma^2}}$$

(3.20) Under a d.c. current (I) through the coil, core rotates by angle θ . Torque developed due to I balances the restoring torque of spring: $aI = k_t \theta$ where a is a constant and k_t is the torsional spring constant. Under an a.c. current $I(t)$, torque developed is $T(t) = aI(t)$ and the equation of motion is:

$$J_0 \ddot{\theta} + c_t \dot{\theta} + k_t \theta = T(t) = aI(t) = aI_0 \cos \omega t \quad (1)$$

Steady state angular displacement of core:

$$\theta_p(t) = \Theta \cos(\omega t - \phi).sp \quad (2)$$

$$\text{where } \Theta = \frac{aI_0}{\left\{ (k_t - J_0 \omega^2)^2 + (c_t \omega)^2 \right\}^{1/2}} = \frac{\left(\frac{aI_0}{k_t} \right)}{\left[\left\{ 1 - \left(\frac{\omega}{\omega_n} \right)^2 \right\}^2 + \left\{ 2\zeta \frac{\omega}{\omega_n} \right\}^2 \right]^{1/2}} \quad (3)$$

When $\omega = 0$ (d.c. current) and $I_0 = 1$ ampere, Eq. (1) gives

$$\Theta_{dc} = \left(\frac{a}{k_t} \right) = 1 \text{ (reading corresponding } \Theta_{dc})$$

and hence $a = k_t = 62.5$.

When $\omega = 50 \text{ Hz} = 314.16 \text{ rad/sec}$ and $I_0 = 5$ amperes, Eq. (3) gives:

$$\Theta_{ac} = \frac{\left(\frac{a(5)}{k_t} \right)}{\left[\left\{ 1 - \left(\frac{314.16}{250} \right)^2 \right\}^2 + \left\{ 2(1) \left(\frac{314.16}{250} \right) \right\}^2 \right]^{1/2}} = 1.9386 \text{ amperes}$$

where $J_0 = 0.001 \text{ N-m}^2$, $k_t = 62.5 \text{ N-m/rad}$, $c_t = 0.5 \text{ N-m-s/rad}$, and $\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{62.5}{0.001}} = 250 \text{ rad/s}$. The steady state value of current indicated by ammeter = 1.9386 amperes (this shows that the ammeter is not accurate).

3.21 Eg. (3.34): $\frac{X_{res}}{\delta_{st}} = \frac{X}{\delta_{st}} \Big|_{\omega = \omega_n} = \frac{1}{2\gamma}$

i.e., $\delta_{st} = 2\gamma \left(\frac{20}{1000} \right) = 0.04 \gamma \quad (E_1)$

Eg. (3.30): $\frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\gamma r)^2}} ; r = 0.75 = \frac{\omega}{\omega_n}$

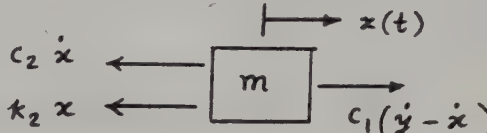
i.e., $\frac{0.01}{\delta_{st}} = \frac{1}{\sqrt{(1-0.75^2)^2 + (2\gamma \times 0.75)^2}} \quad (E_2)$

Eqs. (E₁) and (E₂) give

$$\frac{0.01}{0.04 \gamma} = \frac{1}{\sqrt{0.1914 + 2.25 \gamma^2}}$$

i.e., $0.1914 + 2.25 \gamma^2 = 16 \gamma^2$

i.e., $\gamma = 0.1180$

3.22 (a) Equation of motion of mass: 

$$m\ddot{x} = c_1(\dot{y} - \dot{x}) - c_2\dot{x} - k_2x$$

i.e., $m\ddot{x} + (c_1 + c_2)\dot{x} + k_2x = c_1\dot{y} = -c_1\omega\gamma \sin \omega t$

(b) $x_p(t) = \frac{-(c_1\omega\gamma/k_2)}{\sqrt{(1-r^2)^2 + (2\gamma r)^2}} \sin(\omega t - \phi)$

where $r = \omega/\omega_n$, $\gamma = (c_1 + c_2)\omega/(2rk)$ and $\phi = \tan^{-1}\left(\frac{2\gamma r}{1-r^2}\right)$.

(c) steady-state force transmitted to point P:

$$= k_2 x_p + c_2 \dot{x}_p$$

$$= \frac{-(c_1\omega\gamma)}{\sqrt{(1-r^2)^2 + (2\gamma r)^2}} \left\{ \sin(\omega t - \phi) + \frac{c_2\omega}{k_2} \cos(\omega t - \phi) \right\}$$

3.23 Eq. (3.34) gives $\left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta}$

If $X = \frac{1}{\sqrt{2}} X_{\max} = \frac{1}{\sqrt{2}} X \Big|_{\omega=\omega_n}$, Eq. (3.30) gives

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{2\zeta} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Squaring and rearranging

$$8\zeta^2 = (1-r^2)^2 + 4\zeta^2 r^2 = 1 - 2r^2 + r^4 + 4r^2\zeta^2$$

$$r^4 + r^2(4\zeta^2 - 2) + (1 - 8\zeta^2) = 0$$

$$r^2 = 1 - 2\zeta^2 \pm 2\zeta \sqrt{1 + \zeta^2}$$

Neglecting terms involving ζ^2 ,

$$r^2 = \frac{\omega^2}{\omega_n^2} = 1 \pm 2\zeta$$

Let $\omega = \omega_1$ when $r^2 = 1 - 2\zeta$ and $\omega = \omega_2$ when $r^2 = 1 + 2\zeta$

$$\frac{\omega_2^2 - \omega_1^2}{\omega_n^2} = \frac{(\omega_2 + \omega_1)(\omega_2 - \omega_1)}{\left(\frac{\omega_2 + \omega_1}{2}\right)^2} = 4\zeta$$

$$\therefore \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} = \zeta$$

3.24 $k_t = \frac{\pi G}{32 l} d^4 = \frac{\pi (79.3 \times 10^9)}{32 (1)} \left(\frac{4}{100}\right)^4 = 19930.31 \text{ N-m/rad}$

$$\omega_n = \sqrt{\frac{k_t}{J_o}} = \sqrt{\frac{19930.31}{10}} = 44.6434 \text{ rad/sec}$$

$$\theta_{st} = M_{to}/k_t = 1000/19930.31 = 0.0502 \text{ rad}$$

$$\zeta_t = \frac{c_t}{2 J_o \omega_n} = \frac{300}{2(10)(44.6434)} = 0.336$$

(a) Eq. (3.30), when written for a torsional system, gives

$$\frac{\theta}{\theta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\text{i.e., } \frac{(2/57.2956)}{0.0502} = \frac{1}{\sqrt{(1-r^2)^2 + (2 \times 0.336 r)^2}}$$

$$\text{i.e., } r^4 - 1.5484 r^2 - 1.0679 = 0$$

$$\text{i.e., } r^2 = 2.0655, -0.5171$$

$$\therefore \omega = r \omega_n = \sqrt{2.0655} (44.6434) = 64.16 \text{ rad/sec}$$

(b) Maximum torque transmitted to the support:

$$\begin{aligned}
 M_t(t) &= k_t \theta(t) + c_t \dot{\theta}(t) \\
 &= k_t \oplus \cos(\omega t - \phi) - c_t \oplus \omega \sin(\omega t - \phi) \\
 (M_t)_{\max} &= \sqrt{(k_t \oplus)^2 + (c_t \oplus \omega)^2} \\
 &= \sqrt{\left\{19930.31 \left(\frac{2}{57.2956}\right)\right\}^2 + \left\{300 \left(\frac{2}{57.2956}\right)(64.16)\right\}^2} \\
 &= 967.2 \text{ N-m}
 \end{aligned}$$

3.25

Complete solution is $x(t) = X_0 e^{-\gamma \omega_n t} \cos(\omega_d t + \phi_0) + X \cos(\omega t - \phi)$

$$\omega = 2\pi(3.5) = 21.9912 \text{ rad/sec}, \quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{10}} = 15.8114 \text{ rad/sec}$$

$$\delta_{st} = \frac{F_0}{k} = \frac{180}{2500} = 0.072 \text{ m}$$

$$\gamma = \frac{c}{2m\omega_n} = \frac{45}{2(10)(15.8114)} = 0.1423, \quad r = \frac{\omega}{\omega_n} = \frac{21.9912}{15.8114} = 1.3908$$

$$\gamma \omega_n = 2.25, \quad \omega_d = \sqrt{1 - \gamma^2} \omega_n = 15.6505$$

$$\begin{aligned}
 X &= \frac{\delta_{st}}{\sqrt{(1 - r^2)^2 + (2\gamma r)^2}} = \frac{0.072}{[(1 - 1.3908^2)^2 + (2 \times 0.1423 \times 1.3908)^2]^{1/2}} \\
 &= 0.07095 \text{ m}
 \end{aligned}$$

$$\phi = \tan^{-1} \left(\frac{2\gamma r}{1 - r^2} \right) = \tan^{-1} \left(\frac{0.3958}{-0.9343} \right) = -22.9591^\circ$$

$$x(t) = X_0 e^{-2.25t} \cos(15.6505t + \phi_0) + 0.07095 \cos(21.9912t + 22.9591^\circ)$$

$$\begin{aligned}
 \dot{x}(t) &= -2.25 X_0 e^{-2.25t} \cos(15.6505t + \phi_0) - 15.6505 X_0 e^{-2.25t} \sin(15.6505t + \phi_0) \\
 &\quad - 21.9912(0.07095) \sin(21.9912t + 22.9591^\circ)
 \end{aligned}$$

$$x(0) = 0.015 = X \cos \phi_0 + 0.07095 \cos 22.9591^\circ$$

$$X \cos \phi_0 = -0.05033 \quad \text{--- (E}_1\text{)}$$

$$\dot{x}(0) = 5 = -2.25 X_0 \cos \phi_0 - 15.6505 X_0 \sin \phi_0 - 1.5603 \sin 22.9591^\circ$$

$$X \sin \phi_0 = \frac{-0.6086 - 2.25 X_0 \cos \phi_0 - 5}{15.6505} = -0.3511 \quad \text{--- (E}_2\text{)}$$

Eqs. (E₁) and (E₂) give

$$X_0 = \{(-0.05033)^2 + (-0.3511)^2\}^{1/2} = 0.3547$$

$$\phi_0 = \tan^{-1} \left(\frac{0.3511}{0.05033} \right) = \tan^{-1}(6.9760) = 81.8423^\circ$$

3.26 (a) Eq. (3.38) gives $\frac{1}{25} \approx \left(\frac{X}{\delta_{st}} \right)_{\max} = \frac{0.2}{0.1} = 2$
 $\therefore 5 = 0.25$

(b) Eqs. (3.42) yield

$$\left(\frac{\omega_1}{\omega_n} \right)^2 \approx 1 - 25 = 0.5, \quad \omega_1 = \omega_n \sqrt{0.5} = (5 \times 2\pi) \sqrt{0.5}$$

$$= 22.2145 \text{ rad/sec}$$

$$\left(\frac{\omega_2}{\omega_n} \right)^2 \approx 1 + 25 = 1.5, \quad \omega_2 = \omega_n \sqrt{1.5} = (5 \times 2\pi) \sqrt{1.5}$$

$$= 38.4766 \text{ rad/sec}$$

3.27 Amplitude of vibration under base excitation:

$$X = Y \left\{ \frac{\sqrt{k^2 + (c\omega)^2}}{\left[\left(k - m\omega^2 \right)^2 + (c\omega)^2 \right]^{\frac{1}{2}}} \right\}$$

$$= \frac{(0.2) \sqrt{k^2 + c^2 (157.08)^2}}{\left[\left\{ k - 2000 (157.08)^2 \right\}^2 + c^2 (157.08)^2 \right]^{\frac{1}{2}}} = 0.1 \text{ m} \quad (1)$$

Let $k = 5 (10^6) \text{ N/m}$. Then Eq. (1) gives:

$$\frac{\sqrt{25 (10^{12}) + 2.4674 (10^4) c^2}}{\sqrt{1966.7717 (10^{12}) + 2.4674 (10^4) c^2}} = 0.5$$

i.e., $1.85055 (10^4) c^2 = 466.6929 (10^{12})$ i.e., $c = 158805.0 \text{ N-s/m}$

3.28

$$\frac{X}{Y} = \left[\frac{k^2 + c^2 \omega^2}{\left(k - m\omega^2 \right)^2 + c^2 \omega^2} \right]^{\frac{1}{2}} \text{ .sp}$$

$$\text{or } \frac{10^{-6}}{Y} = \left[\frac{(10^6) + (10^3 (200\pi))^2}{\left\{ 10^6 - \left(\frac{5000}{9.81} \right) (200\pi)^2 \right\}^2 + \left\{ (10^3) (200\pi) \right\}^2} \right]^{\frac{1}{2}}$$

or $Y = 169.5294 (10^{-6}) \text{ m}$

3.29 Equation of motion:

$$I_0 \ddot{\theta} + \left(k \frac{\ell}{4} \theta \right) \frac{\ell}{4} + \left(c \frac{\ell}{4} \dot{\theta} \right) \frac{\ell}{4} + \left(k \frac{3\ell}{4} \theta \right) \frac{3\ell}{4} = M_0 \cos \omega t$$

$$\text{or } I_0 \ddot{\theta} + c \frac{\ell^2}{16} \dot{\theta} + \frac{5}{8} k \ell^2 \theta = M_0 \cos \omega t$$

$$\begin{aligned}\text{where } I_0 &= \frac{m \ell^2}{12} + m \left(\frac{\ell}{4}\right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10) (1^2) = 1.4583 \text{ kg-m}^2 \\ \frac{c \ell^2}{16} &= \frac{(1000) (1^2)}{16} = 62.5 \text{ N-m-s/rad} \\ \frac{5}{8} k \ell^2 &= \frac{5}{8} (5000) (1^2) = 3125.0 \text{ N-m/rad} \\ \omega &= \frac{1000 (2 \pi)}{60} = 104.72 \text{ rad/sec}\end{aligned}$$

Equation of motion becomes:

$$1.4583 \ddot{\theta} + 62.5 \dot{\theta} + 3125.0 \theta = 100 \cos 104.72 t$$

Steady state response is given by Eq. (3.28):

$$\theta_p(t) = \Theta \cos(\omega t - \phi) = \Theta \cos(104.72 t - \phi).sp$$

$$\text{where } \Theta = \frac{100}{\left[\left\{ 3125.0 - 1.4583 (104.72^2) \right\}^2 + \left\{ 62.5 (104.72) \right\}^2 \right]^{\frac{1}{2}}} = 0.006927 \text{ rad}$$

$$\text{and } \phi = \tan^{-1} \left(\frac{62.5 (104.72)}{3125.0 - 1.4583 (104.72^2)} \right) = -0.4705 \text{ rad} = -26.9606^\circ$$

3.30

$m = 100 \text{ kg}$, $F_0 = 100 \text{ N}$, $X_{\max} = 0.005 \text{ m}$ at $\omega = 300 \text{ rpm} = 31.416 \text{ rad/sec}$. Equations (3.33) and (3.34) yield:

$$\begin{aligned}\omega &= \omega_n \sqrt{1 - 2 \zeta^2} = \sqrt{\frac{k}{m}} \sqrt{1 - 2 \zeta^2} = 31.416 \\ \text{or } k (1 - 2 \zeta^2) &= (100) (31.416^2) = 98,696.5056\end{aligned} \quad (1)$$

$$\begin{aligned}\text{and } X_{\max} &= \delta_{st} \frac{1}{2 \zeta \sqrt{1 - \zeta^2}} = \frac{F_0}{k} \frac{1}{2 \zeta \sqrt{1 - \zeta^2}} = 0.005 \\ \text{or } k \zeta \sqrt{1 - \zeta^2} &= \frac{F_0}{2 (0.005)} = 10,000.0\end{aligned} \quad (2)$$

Divide Eq. (1) by (2):

$$\frac{1 - 2 \zeta^2}{\zeta \sqrt{1 - \zeta^2}} = 9.8696 \quad (3)$$

Squaring Eq. (3) and rearranging leads to:

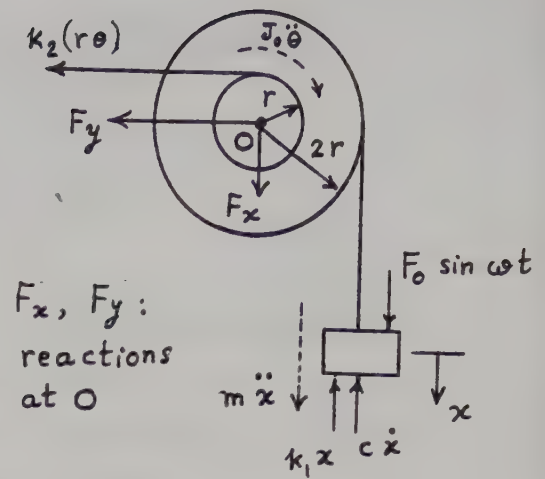
$$101.4090 \zeta^4 - 101.4090 \zeta^2 + 1 = 0 \quad \text{or } \zeta = 0.0998, 0.9950$$

Using $\zeta = 0.0998$ in Eq. (1), we obtain

$$k = \frac{98696.5056}{1 - 2 (0.0998^2)} = 100,702.4994 \text{ N/m}$$

Since $\zeta = \frac{c}{2 m \omega_n}$, we find

$$c = 2 m \omega_n \zeta = 2 (100) \sqrt{\frac{100702.4944}{1000}} (0.0998) = 633.4038 \text{ N-s/m}$$



Equation of motion for rotation of pulley about O:

$$-k_2 (\theta r) r - J_0 \ddot{\theta} - k_1 x (2r) - c \dot{x} (2r) + F_0 \sin \omega t (2r) - m \ddot{x} (2r) = 0 \quad (1)$$

where $\theta = x/(2r)$. Equation (1) can be rearranged as:

$$\left(\frac{J_0}{2r} + 2m \right) \ddot{x} + 2c r \dot{x} + \left(2k_1 r + \frac{1}{2} k_2 r \right) x = 2r F_0 \sin \omega t \quad (2)$$

For given data, Eq. (2) becomes

$$11 \ddot{x} + 50 \dot{x} + 112.5 x = 5 \sin 20 t \quad (3)$$

Steady state response is given by Eq. (3.25):

$$x_p(t) = X \cos (\omega t - \phi)$$

$$\text{where } X = \frac{5}{\left[\left\{ 112.5 - 11 (20^2) \right\}^2 + \left\{ 50 (20) \right\}^2 \right]^{\frac{1}{2}}} = 0.001136 \text{ m}$$

$$\text{and } \phi = \tan^{-1} \left(\frac{50 (20)}{112.5 - 11 (20^2)} \right) = -0.2291 \text{ rad} = -13.1287^\circ$$

(a)

$$\begin{aligned} \sum M_0 &= 0 \text{ (about hinge):} \\ I_0 \ddot{\theta} + \left(k \theta \frac{3\ell}{4} \right) \frac{3\ell}{4} + (c \ell \dot{\theta}) \ell &= \frac{\ell}{2} F_0 \sin \omega t \\ \text{or } I_0 \ddot{\theta} + c \ell^2 \dot{\theta} + \frac{9}{16} k \ell^2 \theta &= \frac{F_0 \ell}{2} \sin \omega t \end{aligned}$$

Magnitude of steady state response:

$$\Theta_a = \left(\frac{F_0 \ell}{2} \right) / \left[\left(\frac{9}{16} k \ell^2 - I_0 \omega^2 \right)^2 + (c \ell^2 \omega)^2 \right]^{\frac{1}{2}} \quad (1)$$

(b)

$$\sum M_0 = 0 \text{ (about hinge):}$$

$$I_0 \ddot{\theta} + (k \ell \theta) \ell + \left[c \frac{3 \ell}{4} \dot{\theta} \right] \frac{3 \ell}{4} = \frac{\ell}{2} F_0 \sin \omega t$$

$$\text{or } I_0 \ddot{\theta} + \frac{9}{16} c \ell^2 \dot{\theta} + k \ell^2 \theta = \frac{F_0 \ell}{2} \sin \omega t$$

Magnitude of steady state response:

$$\Theta_b = \left(\frac{F_0 \ell}{2} \right) / \left[\left\{ k \ell^2 - I_0 \omega^2 \right\}^2 + \left\{ \frac{9}{16} c \ell^2 \omega \right\}^2 \right]^{\frac{1}{2}} \quad (2)$$

Usually, c is small compared to k . If the term containing c is negligible, Θ_a will be smaller than Θ_b . Hence arrangement (a) is desirable.

3.33 $\ddot{y}(t) = \ddot{x}_g(t) = A \cos \omega t$; $\dot{y}(t) = \frac{A}{\omega} \sin \omega t + B_1$

$$y(t) = -\frac{A}{\omega^2} \cos \omega t + B_1 t + B_2$$

Assuming $y(0) = \dot{y}(0) = 0$, we get

$$y(t) = -\frac{A}{\omega^2} \cos \omega t$$

Equation of motion:

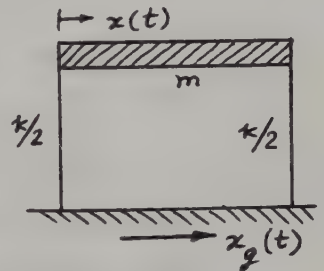
$$m \ddot{x} + k(x - y) = 0$$

i.e., $m \ddot{z} + k z = -m \ddot{y} = -m \ddot{x}_g(t) = -mA \cos \omega t$
 where $z = x - y$

Solution is:

$$z(t) = \frac{-mA \cos \omega t}{k - m \omega^2}$$

$$\therefore x(t) = z(t) + y(t) = -\left(\frac{m}{k - m \omega^2} + \frac{1}{\omega^2} \right) A \cos \omega t$$



3.34 From solution of problem 3.33,

$$x(t) = \left| \frac{-mA}{k - m \omega^2} \right| \sin \omega t - \frac{A}{\omega^2} \sin \omega t$$

$$= \left| \frac{-2000 \left(\frac{100}{1000} \right)}{0.1 \times 10^6 - 2000 (25)^2} \right| \sin 25t - \left(\frac{100}{1000} \right) \frac{1}{(25)^2} \sin 25t$$

For maximum $x(t)$,

$$x(t) = \left(\frac{-200}{1.15 \times 10^6} - \frac{1}{6250} \right) \sin 25t = -3.3391 \times 10^{-4} \sin 25t \text{ m}$$

$\therefore$ Maximum horizontal displacement of floor = 0.3339 mm

$$(3.35) \quad m(\ddot{x} - \ddot{y}) + k(x - y) = -m\ddot{y} = -m\ddot{x}_g \quad (E_1)$$

Here $y(t) = x_g(t) = X_g \cos \omega t$, and Eq. (E₁) becomes

$$m\ddot{z} + kz = m\omega^2 X_g \cos \omega t \quad \text{with } z = x - y$$

Solution is:

$$z(t) = \frac{m\omega^2 X_g \cos \omega t}{k - m\omega^2} = \frac{X_g r^2 \cos \omega t}{1 - r^2}$$

$$\text{with } \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.5 \times 10^6}{2000}} = 15.8114 \text{ rad/sec}$$

$$\text{and } r = \omega/\omega_n = 200/15.8114 = 12.6491$$

$$z(t) = \left(\frac{15}{1000}\right) \left\{ \frac{12.6491^2}{1 - 12.6491^2} \right\} \cos 200t = -0.01509 \cos 200t \text{ m}$$

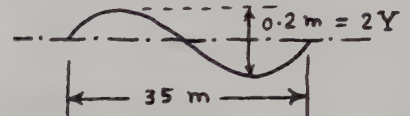
$$x(t) = y(t) + z(t) = \{0.015 \cos 200t - |0.01509| \cos 200t\} \text{ m}$$

$$\therefore \text{Amplitude of vibration of floor} = 0.03009 \text{ m} = 30.09 \text{ mm.}$$

(3.36) Time taken by car to travel one cycle

(35 m) is

$$\tau = \frac{35 \times 3600}{60 \times 1000} = 2.1 \text{ sec}$$



$$\text{Excitation frequency} = \omega = \frac{2\pi}{\tau} = 2.992 \text{ rad/sec}$$

$$\omega_n = 2\pi(2) = 12.5664 \text{ rad/sec}, \quad r = \frac{\omega}{\omega_n} = 0.2381, \quad \gamma = 0.15$$

Amplitude of vibration of car is given by Eq. (3.68):

$$\frac{X}{Y} = \left[\frac{1 + (2\gamma r)^2}{(1 - r^2)^2 + (2\gamma r)^2} \right]^{1/2} \quad (E_1)$$

$$X = 0.1 \left\{ \frac{1 + (2 \times 0.15 \times 0.2381)^2}{(1 - 0.2381^2)^2 + (2 \times 0.15 \times 0.2381)^2} \right\}^{1/2}$$

$$= 0.105977 \text{ m}$$

The most unfavorable speed corresponds to the maximum of $\frac{X}{Y}$ in Eq. (E₁). For maximum of $\frac{X}{Y}$ with respect to r ,

$$\frac{d}{dr} \left[\frac{1 + 4\gamma^2 r^2}{1 + r^4 - 2r^2 + 4\gamma^2 r^2} \right] = 0$$

$$\text{i.e., } \frac{(1 + r^4 - 2r^2 + 4\gamma^2 r^2)(8\gamma^2 r) - (1 + 4\gamma^2 r^2)(4r^3 - 4r + 8\gamma^2 r)}{(1 + r^4 - 2r^2 + 4\gamma^2 r^2)^2} = 0$$

$$\text{i.e., } -4r(2\gamma^2 r^4 + r^2 - 1) = 0$$

$$\text{i.e., } r = 0 \text{ or } r^2 = \frac{-1 \pm \sqrt{1 + 8\gamma^2}}{4\gamma^2}$$

Feasible value of $r^2 = \frac{-1 + \sqrt{1 + 8(0.15)^2}}{4(0.15)^2} = 0.9586$

$$r = \frac{\omega}{\omega_n} = 0.9791$$

$$\omega = 0.9791 (12.5664) = 12.3035 \text{ rad/sec} = \frac{2\pi}{\tau}$$

where $\tau = \frac{35 \times 3600}{s \times 1000}$ and $s = \text{speed of car in km/hr.}$

$$\therefore s = \frac{12.3035 \times 35 \times 3.6}{2\pi} = 246.7279 \text{ km/hr.}$$

3.37 Equations (3.73) and (3.68) give

$$F_T = m \omega^2 X = m \omega^2 Y \left[\frac{1 + (2\gamma r)^2}{(1-r^2)^2 + (2\gamma r)^2} \right]^{1/2}$$

$$\begin{aligned} \frac{F_T}{kY} &= \frac{m \omega^2}{k} \left[\frac{1 + (2\gamma r)^2}{(1-r^2)^2 + (2\gamma r)^2} \right]^{1/2} \\ &= r^2 \left[\frac{1 + (2\gamma r)^2}{(1-r^2)^2 + (2\gamma r)^2} \right]^{1/2} \end{aligned}$$

3.38 Eg. (3.75): $m \ddot{z} + c \dot{z} + k z = -m \ddot{y} = m \omega^2 Y \cos \omega t$

steady-state solution is:

$$z(t) = \frac{m \omega^2 Y \cos(\omega t - \phi_1)}{(k - m \omega^2)^2 + (c \omega)^2} = Z \cos(\omega t - \phi_1)$$

where $\phi_1 = \tan^{-1} \left(\frac{c \omega}{k - m \omega^2} \right)$

Damping force = $c \frac{dz}{dt} = -c \omega Z \sin(\omega t - \phi_1)$

Energy absorbed per cycle by the damper (E):

$$\begin{aligned} E &= \int_{t=0}^{2\pi/\omega} c \frac{dz}{dt} \cdot dz = \int_0^{2\pi/\omega} \{-c \omega Z \sin(\omega t - \phi_1)\} \{-\omega Z \sin(\omega t - \phi_1)\} dt \\ &= c \omega^2 Z^2 \int_0^{2\pi/\omega} \sin^2(\omega t - \phi_1) dt = \pi c \omega Z^2 \end{aligned}$$

Since $Z = \frac{m \omega^2 Y}{\sqrt{(k - m \omega^2)^2 + (c \omega)^2}}$,

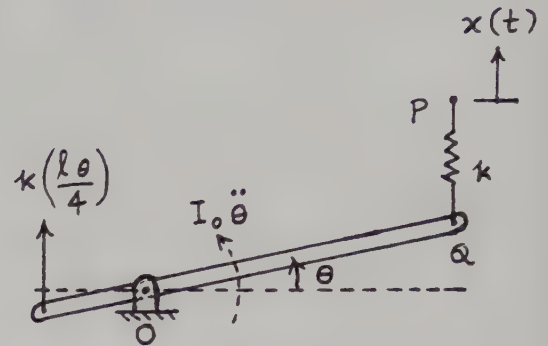
$$E = \left\{ \frac{\pi c \omega (m^2 \omega^4 Y)}{(k - m \omega^2)^2 + c^2 \omega^2} \right\}$$

For maximum power, $\frac{dE}{dc} = 0$

$$\text{i.e., } \frac{\{(k - m\omega^2)^2 + c^2\omega^2\} (\pi\omega^5 m^2 \gamma) - \pi c \omega^5 m^2 \gamma (2c\omega^2)}{\{(k - m\omega^2)^2 + c^2\omega^2\}^2} = 0$$

$$\text{i.e., } c = \left(\frac{k - m\omega^2}{\omega} \right).$$

3.39



Linear displacement of point Q due to $\theta = \frac{3\ell}{4} \theta$ and net compression of spring PQ = $\frac{3}{4} \ell \theta - x(t)$. Equation of motion:

$$I_0 \ddot{\theta} = - \frac{k \ell \theta}{4} \frac{\ell}{4} - k \left(\frac{3\ell \theta}{4} - x(t) \right) \frac{3\ell}{4} \quad (1)$$

$$\text{where } I_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4} \right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10) (1^2) = 1.4583 \text{ kg-m}^2$$

Hence Eq. (1) can be rewritten as

$$I_0 \ddot{\theta} + \left[\frac{5}{8} k \ell^2 \right] \theta = \left[\frac{3}{4} k \ell x_0 \right] \sin \omega t \quad (2)$$

Steady state angular displacement of the bar is given by Eq. (3.6):

$$\Theta = \left[\frac{3}{4} k \ell x_0 \right] / \left[\frac{5}{8} k \ell^2 - I_0 \omega^2 \right] \quad (3)$$

$$= \left[\frac{3}{4} (1000) (1) (0.01) \right] / \left[\frac{5}{8} (1000) (1^2) - 1.4583 (10^2) \right] = 0.01565 \text{ rad}$$

$$\text{and hence } \theta(t) = \Theta \sin \omega t = 0.01565 \sin 10 t \text{ rad}$$

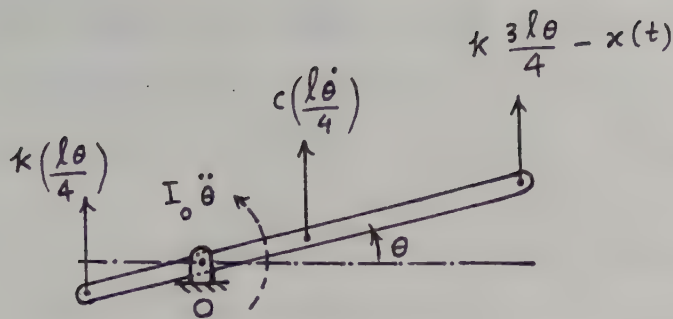
3.40

Equation of motion:

$$I_0 \ddot{\theta} = - k \frac{\ell \theta}{4} \left(\frac{\ell}{4} \right) - c \frac{\ell}{4} \dot{\theta} \left(\frac{\ell}{4} \right) - k \left(\frac{3\ell}{4} \theta - x(t) \right) \frac{3\ell}{4}$$

$$\text{i.e., } I_0 \ddot{\theta} + \frac{1}{16} c \ell^2 \dot{\theta} + \frac{5}{8} k \ell^2 \theta = \frac{3}{4} k \ell x(t) = \frac{3}{4} k \ell x_0 \sin \omega t \quad (1)$$

$$\text{where } I_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4} \right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10) (1^2) = 1.4583 \text{ kg-m}^2 \quad (2)$$



Using given data, Eq. (1) can be expressed as

$$1.4583 \ddot{\theta} + \frac{1}{16} (500) (1^2) \dot{\theta} + \frac{5}{8} (1000) (1^2) \theta = \frac{3}{4} (1000) (1) (0.01) \sin 10 t$$

i.e., $1.4583 \ddot{\theta} + 31.25 \dot{\theta} + 625.0 \theta = 7.5 \sin 10 t$ (3)

Steady state angular displacement of the bar is given by Eq. (3.28) with:

$$\Theta = \frac{7.5}{\left\{ \left[625.0 - 1.4583 (10^2) \right]^2 + 31.25^2 (10^2) \right\}^{\frac{1}{2}}} = 0.01311 \text{ rad}$$

$$\phi = \tan^{-1} \left(\frac{31.25 (10)}{625.0 - 1.4583 (10^2)} \right) = 0.5779 \text{ rad}$$

$$\therefore \theta(t) = \Theta \sin (\omega t - \phi) = 0.01311 \sin (10 t - 0.5779) \text{ rad}$$

3.41 Displacement transmissibility (T):

$$T = \frac{X}{Y} = \left[\frac{1 + (2 \zeta r)^2}{(1 - r^2)^2 + (2 \zeta r)^2} \right]^{\frac{1}{2}}$$

For maximum of T,

$$\frac{dT}{dr} = \frac{1}{2} \left[\frac{1 + 4 \zeta^2 r^2}{(1 + r^4 - 2 r^2) + 4 \zeta^2 r^2} \right]^{-\frac{1}{2}} \frac{\left[(1 - r^2)^2 + (2 \zeta r)^2 \right] (8 \zeta^2 r) - (1 + 4 \zeta^2 r^2) [4 r^3 - 4 r + 8 \zeta^2 r]}{\left[(1 - r^2)^2 + (2 \zeta r)^2 \right]^2} = 0$$

This equation can be simplified to obtain:

$$(2 \zeta^2) r^4 + r^2 - 1 = 0$$

Solution: $r^2 = \frac{-1 \pm \sqrt{1 + 8 \zeta^2}}{4 \zeta^2}$

or $r = r_m = \frac{1}{2 \zeta} \sqrt{\sqrt{1 + 8 \zeta^2} - 1}$

3.42

Equation of motion: $M \ddot{x} + c \dot{x} + k x = m e \omega^2 \sin \omega t$

where $\omega = \frac{3000 (2 \pi)}{60} = 314.16 \text{ rad/sec}$, $M = 100 \text{ kg}$, $c = 2000 \text{ N-s/m}$, $k = 10^6 \text{ N/m}$,
 $m = 0.1 \text{ kg}$ and $e = r = 0.1 \text{ m}$. Steady state response is:

$$x_p(t) = X \sin(\omega t + \phi)$$

where $X = \frac{m e \omega^2}{\left[(k - M \omega^2)^2 + (c \omega)^2 \right]^{\frac{1}{2}}}$

$$= \frac{0.1 (0.1) (314.16^2)}{\left[\left\{ 10^6 - 100 (314.16^2) \right\}^2 + (2000 (314.16))^2 \right]^{\frac{1}{2}}} = 110.9960 (10^{-6}) \text{ m}$$

and $\phi = \tan^{-1} \left(\frac{c \omega}{k - M \omega^2} \right) = \tan^{-1} \left(\frac{2000 (314.16)}{10^6 - 100 (314.16^2)} \right)$
 $= -0.07072 \text{ rad} = -4.0520^\circ$

3.43

k = spring constant of cantilever beam

$$= \frac{3 E I}{l^3} = \frac{3 (2.5 \times 10^6)}{4^3}$$

$$= 0.1172 \times 10^6 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{m_1 + 0.25 m_b}} = \sqrt{\frac{0.1172 \times 10^6}{20 + 0.25 (240)}} = 38.2753 \text{ rad/sec}$$

$$\omega = 2\pi (1500)/60 = 157.08 \text{ rad/sec}$$

$$r = \omega / \omega_n = 157.08 / 38.2753 = 4.1040, \quad r^2 = 16.8428$$

Forced response is given by Eq. (3.79):

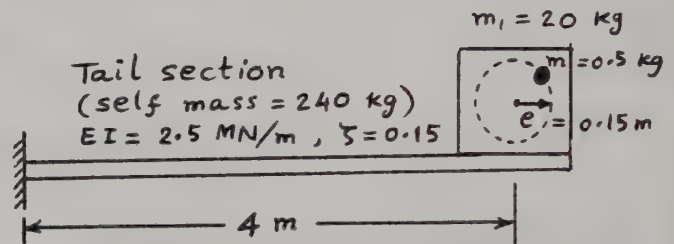
$$x_p(t) = X \sin(\omega t - \phi)$$

where

$$X = \frac{m e}{m_1} \cdot \frac{r^2}{\sqrt{(1 - r^2)^2 + (2 \zeta r)^2}}$$

$$= \frac{(0.5)(0.15)}{20} \cdot \frac{16.8428}{\sqrt{(1 - 16.8428)^2 + (2 \times 0.15 \times 4.1040)^2}}$$

$$= 3.9747 \times 10^{-3} \text{ m} = 3.9747 \text{ mm}$$



$$(3.44) \quad \delta_{st} = \frac{45}{1000} \text{ m} = \frac{Mg}{k} = \frac{380 \times 9.81}{k}$$

$$\text{i.e., } k = 82,840 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{82,840}{380}} = 14.7648 \text{ rad/sec}; \quad \omega = \frac{2\pi(1750)}{60}$$

$$r = \frac{\omega}{\omega_n} = \frac{183.26}{14.7648} = 12.412; \quad r^2 = 154.0566$$

(i) Amplitude of vibration

$$X = \frac{me}{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.15}{380} \frac{154.0566}{\sqrt{(1-154.0566)^2 + 0}}$$

$$= 3.9732 \times 10^{-4} \text{ m}$$

(ii) Force transmitted to ground

$$= kX = (82840)(3.9732 \times 10^{-4}) = 32.9140 \text{ N}$$

$$(3.45) \quad I = \frac{1}{12} (0.5) (0.1)^3 = 0.4167 \times 10^{-4} \text{ m}^4$$

$$k = \frac{192EI}{l^3} = \frac{192 (2.07 \times 10^{11}) (0.4167 \times 10^{-4})}{(5)^3} = 1.3248 \times 10^7 \text{ N/m}$$

$$(a) \quad \omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{13.248 \times 10^6}{75}} = 420.2856 \text{ rad/sec}$$

$$\omega = 2\pi(1200)/60 = 125.664 \text{ rad/sec}$$

$$r = \omega/\omega_n = 125.664/420.2856 = 0.299, \quad r^2 = 0.0894$$

Amplitude of steady-state vibration is given by Eq. (3.30) with $\zeta = 0$:

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k|(r^2 - 1)|} = \frac{5000}{(1.3248 \times 10^7)(0.9106)}$$

$$= 0.4145 \times 10^{-3} \text{ m}$$

(b) Using the effective mass due to self weight of beam (for a cantilever) to be valid here also,

$$\omega_n = \sqrt{\frac{k}{M + 0.2357 m}}$$

where M = mass of motor = 75 kg, and

$$m = \text{mass of beam} = (5 \times 0.5 \times 0.1) \left(\frac{76.5 \times 10^3}{9.81} \right) = 1949.5313 \text{ kg}$$

$$\omega_n = \sqrt{\frac{13.248 \times 10^6}{75 + (1949.5313)(0.2357)}} = 157.4339 \text{ rad/sec}$$

$$r = \omega/\omega_n = 125.664/157.4339 = 0.7982, \quad r^2 = 0.6371$$

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k |r^2 - 1|} = \frac{5000}{(1.3248 \times 10^7)(0.3629)} \\ = 1.0400 \times 10^{-3} \text{ m}$$

3.46

Let width = 0.5 m and thickness = t m.

$$I = \frac{1}{12} (0.5) t^3 = \frac{t^3}{24} \text{ m}^4$$

$$k = \frac{3EI}{l^3} = \frac{3(2.07 \times 10^{11}) \left(\frac{t^3}{24} \right)}{(5)^3} = 2.07 \times 10^8 t^3 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{M + 0.2357 m}}$$

$$\text{Where } m = \text{mass of beam} = (5 \times 0.5 \times t) \left(\frac{76.5 \times 10^3}{9.81} \right) = 19495.41 t \text{ kg}$$

$$\omega_n = \sqrt{\frac{2.07 \times 10^8 t^3}{75 + 0.2357 (19495.41 t)}}$$

$$r = \frac{\omega}{\omega_n} = 125.664 \sqrt{\frac{75 + 4595.0688 t}{2.07 \times 10^8 t^3}}$$

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k |r^2 - 1|}$$

$$\text{i.e., } 0.5 = \frac{5000}{(2.07 \times 10^8 t^3) \left\{ (125.664)^2 \left[\frac{75 + 4595.0688 t}{2.07 \times 10^8 t^3} \right] - 1 \right\}}$$

$$\text{i.e., } 1.3108 \times 10^4 t^3 - 4595.069 t - 74.367 = 0$$

By trial and error, the value of t is found as

$$t \approx 0.6 \text{ m.}$$

Since this is too large, we can start with a new width such as 1.0 m.

3.47

$$m = (600/9.81) \text{ N}, \quad \omega = 2\pi(1000)/60 = 104.72 \text{ rad/sec}$$

$$k = 6(6000) = 36,000 \text{ N/m}$$

$$\omega_n = \sqrt{k/m} = \sqrt{36000 / \left(\frac{600}{9.81} \right)} = 24.2611 \text{ rad/sec}$$

$$r = \omega/\omega_n = 104.72/24.2611 = 4.3164, \quad r^2 = 18.6311$$

$$X = \frac{F_0}{k |r^2 - 1|} = \frac{m_0 e \omega^2}{k |r^2 - 1|}$$

where m_0 = unbalanced mass
and e = eccentricity

$$\text{i.e., } 2.5 \times 10^{-3} = \frac{m_0 e (104.72)^2}{36000 |17.6311|}$$

$$\text{i.e., } m_0 e = 0.1447 \text{ kg-m}$$

$$\therefore \text{Unbalance} = W_0 e = m_0 g e = 0.1447 (9.81) = 1.4195 \text{ N-m}$$

$$(3.48) \quad m = \frac{1000}{386.4} = 2.588 \frac{\text{lb-s}^2}{\text{in}}, \quad \omega = \frac{2\pi(1500)}{60} = 157.08 \frac{\text{rad}}{\text{s}}$$

Possible isolators are: (i) $k = 45000 \text{ lb/in}$, $\zeta = 0$

(ii) $k = 90000 \text{ lb/in}$, $\zeta = 0$

(iii) $k = 45000 \text{ lb/in}$, $\zeta = 0.15$

(iv) $k = 90000 \text{ lb/in}$, $\zeta = 0.15$

We will compare the force transmissibilities of these isolators.

$$\text{Force transmissibility} = T_r = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$

$$(i) \quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{45000}{2.588}} = 131.8634 \text{ rad/sec}$$

$$r = \omega/\omega_n = 157.08/131.8634 = 1.1912, \quad r^2 = 1.4190$$

$$T_r = \frac{1}{|1 - r^2|} = \frac{1}{0.419} = 2.3866$$

$$(ii) \quad \omega_n = \sqrt{\frac{90000}{2.588}} = 186.4829 \text{ rad/sec}$$

$$r = \omega/\omega_n = 157.08/186.4829 = 0.8423, \quad r^2 = 0.7095$$

$$T_r = \frac{1}{|1 - r^2|} = \frac{1}{0.2905} = 3.4423$$

$$(iii) \quad \omega_n = \sqrt{\frac{45000}{2.588}} = 131.8634 \text{ rad/sec}$$

$$r = 1.1912, \quad r^2 = 1.4190, \quad \zeta = 0.15$$

$$T_r = \sqrt{\frac{1 + (2 \times 1.1912 \times 0.15)^2}{(1 - 1.4190)^2 + (2 \times 1.1912 \times 0.15)^2}} = 1.9282$$

$$(iv) \quad \omega_n = 186.4829 \text{ rad/sec}, \quad r = 0.8423, \quad r^2 = 0.7095, \quad \zeta = 0.15$$

$$T_r = \sqrt{\frac{1 + (2 \times 0.8423 \times 0.15)^2}{(1 - 0.7095)^2 + (2 \times 0.8423 \times 0.15)^2}} = 2.6789$$

$\therefore$ Isolation (iii) is best.

$$(3.49) \quad E_g(3.82): \quad \frac{Mx}{me} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

When $r = 1$,

$$\frac{Mx}{me} = \frac{1}{2\zeta} \quad \text{or} \quad \frac{M}{me} = \frac{1}{2\zeta X} = \frac{1}{2\zeta(0.55)} = \frac{1}{1.1\zeta} \quad (E_1)$$

When $r = \text{large}$,

$$\frac{Mx}{me} \approx 1 \quad \text{or} \quad \frac{M}{me} \approx \frac{1}{X} = \frac{1}{0.15} \quad (E_2)$$

Combining (E_1) and (E_2) , we obtain

$$\frac{M}{me} = \frac{1}{0.15} = \frac{1}{1.1\zeta}$$

$$\therefore \zeta = 0.1364$$

(3.50) For each spring,

$$k = \frac{Gd^4}{64nR^3} = \frac{(11.5385 \times 10^6)(0.25)^4}{64(8)(1.5)^3} = 26.083 \text{ lb/in}$$

$$\text{Total } k = 4(26.083) = 104.332 \text{ lb/in}$$

$$\omega = \frac{2\pi(1800)}{60} = 188.496 \text{ rad/sec}$$

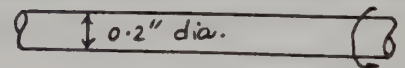
$$m = 100/386.4 \text{ lb-s}^2/\text{in}, \quad M = 750/386.4 \text{ lb-s}^2/\text{in}, \quad \zeta = 0$$

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{104.332}{(750/386.4)}} = 7.3316 \text{ rad/sec}$$

$$r = 188.496/7.3316 = 25.7102, \quad r^2 = 661.0144$$

$$X = \frac{me}{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{100(0.01)}{750} \left(\frac{661.0144}{660.0144} \right) = 1.3354 \times 10^{-3} \text{ in.}$$

$$(3.51) \quad \omega = 2\pi(1500)/60 = 157.08 \text{ rad/sec}$$



1500 rpm

(a) Force due to eccentricity of rotor

$$= me\omega^2 = \left(\frac{30}{386.4} \right) (0.01) (157.08)^2 = 19.1569 \text{ lb.}$$

(b) H.P. = (Force)(eccentricity)(angular velocity)

$$= (19.1569) \left(\frac{0.01}{12} \right) \left(\frac{157.08}{550} \right) = 0.004559 \text{ hp.}$$

3.52

$$\omega_{n, fan} = \sqrt{\frac{k}{m_{fan}}}$$

$$= \sqrt{\frac{200}{50/386.4}}$$

$$= 39.3141 \text{ rad/sec}$$

$$\omega = \frac{2\pi(750)}{60} = 78.54 \text{ rad/sec}$$

$$(J_p)_{plate+fan} = \frac{1}{3} \left(\frac{100}{386.4} \right) (40)^2 + \left(\frac{50}{386.4} \right) (5)^2 = 141.2612 \text{ lb-in-sec}^2$$

$$F_0 = m e \omega^2 = \left(\frac{50}{386.4} \right) (0.1) (78.54)^2 = 79.8205 \text{ lb}$$

Point R is subjected to the force, $F(t) = F_0 \cos \omega t = 79.8205 \cos 78.54 t$
Assume that S is not moving. lb.

Then R is displaced by:

$$x(t) = \frac{F_0 \cos \omega t}{|k - m \omega^2|} = \frac{F_0 \cos \omega t}{k \left| 1 - \left(\frac{\omega}{\omega_n} \right)^2 \right|} = \frac{79.8205 \cos \omega t}{200 \left| 1 - \left(\frac{78.54}{39.3141} \right)^2 \right|}$$

$$= 0.1334 \cos 78.54 t \text{ inch}$$

Let θ = angular displacement of plate PQ.

Displacement of S = 5θ inch

Extension of spring RS = $(5\theta - 0.1334 \cos 78.54 t)$ inch

Restoring moment of spring force about P

$$= 200 [5\theta - 0.1334 \cos 78.54 t] 5 \text{ lb-in}$$

Velocity of Q = $40 \dot{\theta}$ inch/sec

Damping force at Q = $40 \dot{\theta} (1) = 40 \dot{\theta}$ lb

Moment of damping force about P = $40 \dot{\theta} (40) = 1600 \dot{\theta}$ lb-in

Equation of motion of plate PQ:

$$J_p \ddot{\theta} + 1600 \dot{\theta} + 1000 (5\theta - 0.1334 \cos 78.54 t) = 0$$

$$\text{i.e., } 141.2612 \ddot{\theta} + 1600 \dot{\theta} + 5000 \theta = 133.4 \cos 78.54 t \quad (E_1)$$

Comparing (E_1) with Eq. (3.24), the solution of (E_1) can be expressed as $\theta_p(t) = \Theta \cos(\omega t - \phi)$

Where, from Eqs. (3.30) and (3.31), we get

$$\Theta = \frac{(133.4/5000)}{\sqrt{(1 - 174.2751)^2 + (2 \times 0.9519 \times 13.2013)^2}} = 1.5238 \times 10^{-4} \text{ rad}$$

$$\text{and } \phi = \tan^{-1}(-25.1326/173.2751) = -8.2529^\circ$$

Steady state motion of Q = $\theta_p(40)$

$$= 0.006095 \cos(78.54 t + 8.2529^\circ) \text{ inch}$$

Displacement of $S = \oplus(5)$ inch

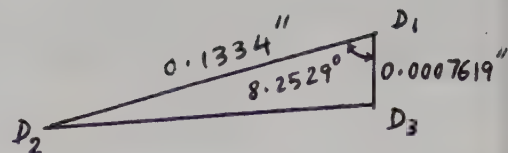
$$= (1.5238 \times 10^{-4})(5) \text{ inch}$$

$$= 0.0007619 \text{ inch}$$

$D_2 D_3$ = maximum deformation of
Spring $\approx 0.1334''$

Max. force transmitted to point $S = k(D_2 D_3)$

$$= 200(0.1334) = 26.68 \text{ lb}$$



$$\begin{aligned}
 (3.53) \quad I &= \int_0^{2\pi/\omega} \sin \omega t \cdot \cos(\omega t - \phi) dt \\
 &= \int_0^{2\pi/\omega} \sin \omega t [\cos \omega t \cdot \cos \phi + \sin \omega t \cdot \sin \phi] dt \\
 &= \int_0^{2\pi/\omega} \{ \cos \phi (\sin \omega t \cdot \cos \omega t) + \sin \phi (\sin^2 \omega t) \} dt \\
 &= \int_0^{2\pi/\omega} \left\{ \cos \phi \left(\frac{\sin 2\omega t}{2} \right) + \sin \phi \left(\frac{1 - \cos 2\omega t}{2} \right) \right\} dt \\
 &= \frac{\cos \phi}{2} \left(-\frac{\cos 2\omega t}{2\omega} \right) \Big|_0^{2\pi/\omega} + \frac{\sin \phi}{2} \left(t - \frac{\sin 2\omega t}{2} \right) \Big|_0^{2\pi/\omega} \\
 &= \frac{\pi}{\omega} \sin \phi
 \end{aligned}$$

$$\Delta W' = \omega F_0 X \cdot I = \omega F_0 X \sin \phi$$

(3.54) Let $x(t)$ = displacement of mass m
 New length of each spring, $k_1 = (l^2 + x^2)^{1/2}$
 New tension in each spring $k_1 = T = (\sqrt{l^2 + x^2} - l) k_1 + T_0$
 Horizontal component of new tension in each spring k_1

$$= \frac{T x}{\sqrt{l^2 + x^2}}$$

Vertical component of new tension in each spring $k_1 = \frac{T l}{\sqrt{l^2 + x^2}}$

Total friction force $= \mu mg + \frac{2 T l}{\sqrt{l^2 + x^2}}$

when mass moves to right:

Equation of motion of mass m :

$$m \ddot{x} + k_2 x + \frac{2 T x}{\sqrt{l^2 + x^2}} - \mu \left[mg + \frac{2 T l}{\sqrt{l^2 + x^2}} \right] = p_0 A \sin \omega t$$

where A = area of piston.

i.e., $m \ddot{x} + x \left(k_2 + 2 \frac{T_0}{l} \right) = \mu mg + 2 \mu T_0 + p_0 A \sin \omega t$

Similarly, when the mass moves to left:

$$m \ddot{x} + x \left(k_2 + 2 \frac{T_0}{l} \right) = -\mu mg - 2 \mu T_0 + p_0 A \sin \omega t$$

$$(3.55) \quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2100}{2}} = 32.403703 \text{ rad/sec}$$

$$N = \text{vertical force} = mg = 2(9.81) = 19.62 \text{ N}$$

$$\frac{\omega}{\omega_n} = \frac{2.5173268 \times 2\pi}{32.403703} = 0.4881191$$

$$X = \frac{F_0}{k} \left[\frac{1 - \left(\frac{4\mu N}{\pi F_0} \right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2} \right]^{1/2}$$

$$\text{i.e.,} \quad 0.075 = \frac{120}{2100} \left[\frac{1 - \left\{ \frac{4\mu(19.62)}{\pi(120)} \right\}^2}{(1 - 0.4881191^2)} \right]^{1/2}$$

$$\text{i.e.,} \quad 1.3125 = \left(\frac{1 - 0.04334 \mu^2}{0.5802473} \right)^{1/2}$$

$$\text{i.e.,} \quad 0.9995666 = 1 - 0.04334 \mu^2$$

$$\text{i.e.,} \quad \mu = 0.1$$

$$(3.56) \quad (\alpha) \quad k = \frac{W}{\delta_{st}} = \frac{5000}{0.05} = 10^5 \text{ N/m}$$

When $\omega = \omega_n$, Eq. (3.102) gives

$$X = \frac{F_0}{k\beta} \Rightarrow 0.1 = \frac{1000}{(10^5)\beta} \Rightarrow \beta = 0.1$$

$$(b) \quad \Delta W = \pi c_{eg} \omega X^2 = \pi \beta k X^2 \quad \text{where } c_{eg} = \frac{\beta k}{\omega} \text{ from Eq. (3.100)}$$

$$\Delta W = \pi (0.1) (10^5) (0.1)^2 = 314.16 \text{ Joules/cycle}$$

(c) Steady state amplitude at one-quarter of resonant frequency:

$$\frac{\omega}{\omega_n} = 0.25$$

$$X = \frac{F_0}{k \left[\left\{ 1 - \left(\frac{\omega}{\omega_n} \right)^2 \right\}^2 + \beta^2 \right]^{1/2}} = \frac{1000}{10^5 \left[\left\{ 1 - 0.25^2 \right\}^2 + (0.1)^2 \right]^{1/2}}$$

$$= 0.01061 \text{ m}$$

(d) Steady state amplitude at thrice the resonant frequency:

$$\frac{\omega}{\omega_n} = 3$$

$$X = \frac{1000}{10^5 \left[(1 - 3^2)^2 + (0.1)^2 \right]^{1/2}} = 0.00125 \text{ m}$$

$$\begin{aligned} \textcircled{3.57} \quad \Delta W &= \pi \beta k X^{\gamma} \\ 3.8 &= \pi \beta (60000) (0.04)^{\gamma} \\ 9.5 &= \pi \beta (60000) (0.06)^{\gamma} \end{aligned} \quad \left. \vphantom{\begin{aligned} 3.8 &= \pi \beta (60000) (0.04)^{\gamma} \\ 9.5 &= \pi \beta (60000) (0.06)^{\gamma} \end{aligned}} \right\} \Rightarrow \begin{aligned} \beta (0.04)^{\gamma} &= 0.0000202 \\ \beta (0.06)^{\gamma} &= 0.0000504 \end{aligned}$$

Taking logarithms,

$$\ln \beta + \gamma \ln (0.04) = \ln (0.0000202)$$

$$\ln \beta + \gamma \ln (0.06) = \ln (0.0000504)$$

$$\text{i.e.,} \quad \ln \beta - 3.218876 \gamma = -10.809828 \quad \text{---- (E}_1\text{)}$$

$$\ln \beta - 2.813411 \gamma = -9.895511 \quad \text{---- (E}_2\text{)}$$

$$\text{subtracting (E}_1\text{) from (E}_2\text{),} \quad 0.405465 \gamma = 0.914309$$

$$\gamma = 2.254964$$

$$\text{From (E}_1\text{),} \quad \ln \beta = -10.809828 + 3.218876 (2.254964) = -3.551378$$

$$\beta = 0.028685$$

$$\begin{aligned} \textcircled{3.58} \quad \text{Work done} &= W = \int F dx = \int F \dot{x} dt \\ \text{If } F(t) &= F_0 \cos \omega t \quad \text{and} \quad x(t) = X \cos(\omega t - \phi), \quad \text{work done in one cycle} \\ &= W = - \int_0^{2\pi/\omega} F_0 \cos \omega t \cdot \omega X \sin(\omega t - \phi) dt \\ &= - \frac{F_0 \omega X \cos \phi}{2} \left(-\frac{1}{2\omega} \cos 2\omega t \right)_0^{2\pi/\omega} + \frac{F_0 \omega X \sin \phi}{2} \left(t + \frac{1}{2\omega} \sin 2\omega t \right)_0^{2\pi/\omega} \\ &= F_0 \pi X \sin \phi \end{aligned}$$

$$\text{Given data:} \quad F_0 = 5 \text{ lb}, \quad \omega = 3\pi \frac{\text{rad}}{\text{sec}}, \quad \tau = \frac{2}{3} \text{ sec}, \quad \phi = \frac{\pi}{3}, \quad X = 0.5''$$

$$W = F_0 \pi X \sin \phi = 5\pi (0.5) \sin \frac{\pi}{3} = 6.8018 \text{ lb-in}$$

(i) In one second, it will complete $1\frac{1}{2}$ cycles.

$$W|_{1 \text{ second}} = 1.5 W = 10.2027 \text{ lb-in.}$$

(ii) In four seconds, it will complete 6 cycles.

$$W|_{4 \text{ seconds}} = 6 W = 40.8108 \text{ lb-in.}$$

$$\begin{aligned} \textcircled{3.59} \quad \text{Damping force} &= F = c(\dot{x})^n \\ \text{Energy dissipated per quarter cycle during harmonic motion } x(t) &= X \sin \omega t \\ \text{is} \quad \frac{\Delta W}{4} &= \int_0^{\pi/2\omega} c(\dot{x})^n dx = \int_0^{\pi/2\omega} c(\omega X \cos \omega t)^n dx \\ \text{But } dx &= \dot{x} dt = \omega X \cos \omega t \cdot dt \end{aligned}$$

$$\begin{aligned}
 \Delta W &= 4c \omega^{n+1} X^{n+1} \int_0^{\pi/2\omega} \cos^{n+1} \omega t \cdot dt \\
 &= 4c \omega^{n+1} X^{n+1} \left\{ \frac{1}{(n+1)\omega} \cos^n \omega t \cdot \sin \omega t \Big|_0^{\pi/2\omega} + \frac{n}{n+1} \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt \right\} \\
 &= 4c \omega^{n+1} X^{n+1} \left(\frac{n}{n+1} \right) \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt
 \end{aligned}$$

Equating this expression to $\pi c_{eq} \omega X^2$, we obtain

$$c_{eq} = \frac{4c \omega^n X^{n-1}}{\pi} \left(\frac{n}{n+1} \right) \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt \equiv c \omega^n X^{n-1} \alpha_n$$

$$\text{where } \alpha_n = \frac{4}{\pi} \left(\frac{n}{n+1} \right) \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt \quad \text{---- (E}_1\text{)}$$

For example, for $n=2$, (E₁) becomes

$$\alpha_n = \frac{4}{\pi} \left(\frac{2}{3} \right) \int_0^{\pi/2\omega} \cos \omega t \cdot dt = \frac{8}{3\pi} \left(\frac{\sin \omega t}{\omega} \right) \Big|_0^{\pi/2\omega} = \frac{8}{3\pi\omega}$$

$$\text{and hence } c_{eq} = \frac{8c\omega X}{3\pi}$$

which can be seen to be same as the expression found in Example 3.6.

For few other values of n , α_n can be found as follows:

n	1	2	3	4
α_n	$\frac{1}{\omega}$	$\frac{8}{3\pi\omega}$	$\frac{3}{4\omega}$	$\frac{32}{15\pi\omega}$

The amplitude can be found as

$$\begin{aligned}
 X &= \frac{F_0}{\sqrt{(k-m\omega^2)^2 + c_{eq}^2 \omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2 \omega^2}} \\
 &= \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c^2 \omega^{2(n+1)} X^{2(n-1)} \alpha_n^2}}
 \end{aligned}$$

- 3.60 Energy dissipated per cycle for viscous damping = $\pi c \omega X^2$
 Energy dissipated per cycle for Coulomb damping = $4\mu NX$

Equivalent viscous damping constant (c_{eq}) is given by

$$\pi c_{eq} \omega X^2 = \pi c \omega X^2 + 4\mu NX$$

$$c_{eq} = \left(c + \frac{4\mu N}{\pi \omega X} \right)$$

Amplitude X is given by

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c_{eq}^2 \omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2 \omega^2}}$$

substituting for c_{eq} , squaring and rearranging,

$$X^2 \{ k^2(1-r^2)^2 + c^2 \omega^2 \} + X \left(\frac{8\mu N c \omega}{\pi} \right) + \left(\frac{16\mu^2 N^2}{\pi^2} - F_0^2 \right) = 0$$

- (3.61) (a) Equation of motion $m\ddot{x} \pm \mu N + c(\dot{x})^3 + kx = F_0 \cos \omega t$
Thus the system has combined Coulomb and velocity-cubed damping.

For Coulomb damping, $c_{eq1} = \frac{4\mu N}{\pi \omega X}$ (E₁)

For velocity-cubed damping, the equivalent viscous damping coefficient can be obtained from the solution of problem 3.59:

$$c_{eq2} = c \omega^3 X^2 \alpha_3$$
 (E₂)

where

$$\alpha_3 = \frac{4}{\pi} \left(\frac{3}{4} \right) \int_0^{\pi/2\omega} \cos^2 \omega t \, dt = \frac{3}{4\omega}$$
 (E₃)

$$\therefore c_{eq2} = \frac{3}{4} c \omega^2 X^2$$
 (E₄)

and $c_{eq} = c_{eq1} + c_{eq2} = \frac{4\mu N}{\pi \omega X} + \frac{3}{4} c \omega^2 X^2$ (E₅)

- (b) Steady state amplitude under harmonic force:

$$X = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2 \omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + \left\{ \frac{4\mu N}{\pi \omega X} + \frac{3}{4} c \omega^2 X^2 \right\}^2 \omega^2}}$$
 (E₆)

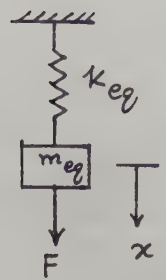
- (c) Amplitude ratio:

$$\begin{aligned} \frac{X}{\delta_{st}} &= \frac{X}{(F_0/k)} = \frac{1}{\sqrt{(1-r^2)^2 + \left(\frac{c_{eq}^2 \omega^2}{k^2} \right)}} \\ &= \frac{1}{\sqrt{(1-r^2)^2 + \left\{ \frac{4\mu N}{\pi X k} + \frac{3}{4} \frac{c \omega^3 X^2}{k} \right\}^2}} \end{aligned}$$
 (E₇)

At resonance, $r=1$ and Eq. (E₇) reduces to

$$\left. \frac{X}{\delta_{st}} \right|_{\text{resonance}} = \frac{1}{\left\{ \frac{4\mu N}{\pi X k} + \frac{3}{4k} c \omega^3 X^2 \right\}}$$
 (E₈)

- (3.62) Model the pipe as a single degree of freedom system with m_{eq} = equivalent mass at end
 $= \frac{33}{140} m$ (m = mass of pipe; see Problem 2.38) and $k_{eq} = \frac{3EI}{\ell^3}$. Slope of pipe at end:



$$\theta = \frac{F \ell^2}{2 E I} = \frac{F \ell^3}{3 E I} \left(\frac{3}{2 \ell} \right) = \frac{3 x}{2 \ell}$$

where x = end deflection of the cantilever pipe under a transverse load F . Force induced due to fluid velocity v is $\rho A v^2$. Force acting on the single degree of freedom system (in vertical direction):

$$F = \rho A v^2 \sin \theta \approx \rho A v^2 \theta = \rho A v^2 \frac{3 x}{2 \ell}$$

$$\begin{aligned} \text{Equation of motion: } m_{eq} \ddot{x} + k_{eq} x &= F \\ \text{or } \frac{33}{140} m \ddot{x} + \left(\frac{3 E I}{\ell^3} - \frac{3 \rho A v^2}{2 \ell} \right) x &= 0 \end{aligned}$$

$$\text{Instability occurs when } \frac{3 E I}{\ell^3} - \frac{3 \rho A v^2}{2 \ell} < 0 \text{ or } v > \sqrt{\frac{2 E I}{\rho A \ell^2}}$$

3.63

Assume Reynolds number (Re) greater than 1000. Strouhal number (St) for vortex shedding is taken as: $St = \frac{f d}{V} = 0.21$ where f = frequency of vortex shedding, d = diameter of cylinder and V = velocity of fluid (air). At 50 mph speed,

$$V = \frac{50 (1760) (36)}{3600} = 880 \text{ in/sec and } f = \frac{0.21 V}{d} = \frac{184.8}{d} \text{ Hz (d in inches)}$$

For the three sections of the antenna, the vortex frequencies are:

$$\begin{aligned} f_1 &= \frac{184.8}{0.3} = 616.0 \text{ Hz} \\ f_2 &= \frac{184.8}{0.2} = 924.0 \text{ Hz} \\ f_3 &= \frac{184.8}{0.1} = 1848.0 \text{ Hz} \end{aligned}$$

At 75 mph speed,

$$V = \frac{75 (1760) (36)}{3600} = 1320 \text{ in/sec and } f = \frac{0.21 V}{d} = \frac{277.2}{d} \text{ Hz (d in inches)}$$

For the three sections of the antenna, the frequencies are:

$$\begin{aligned} f_1 &= \frac{277.2}{0.3} = 924.0 \text{ Hz} \\ f_2 &= \frac{277.2}{0.2} = 1386.0 \text{ Hz} \end{aligned}$$

$$f_3 = \frac{277.2}{0.1} = 2772.0 \text{ Hz}$$

Since the natural frequencies are much smaller, no instability occurs.

3.64

- (a) Equivalent mass of single d.o.f. system = $m_{eq} = M + \frac{33}{140} m$ where m = mass of cylindrical part of the sign post:

$$m = \frac{\pi}{4} (D^2 - d^2) h \rho = \frac{\pi}{4} (0.25^2 - 0.2^2) (10) \left(\frac{76500}{9.81} \right) = 1378.0527 \text{ kg}$$

$$\therefore m_{eq} = 200 + \frac{33}{140} (1378.0527) = 524.8267 \text{ kg}$$

$$I = \frac{\pi}{64} (D^4 - d^4) = \frac{\pi}{64} (0.25^4 - 0.2^4) = 113.208 (10^{-6}) \text{ m}^4$$

Equivalent stiffness of the system:

$$k_{eq} = \frac{3 E I}{h^3} = \frac{3 (207 (10^9)) (113.208 (10^{-6}))}{10^3} = 70,302.168 \text{ N/m}$$

Natural frequency of transverse vibration of sign post:

$$\omega_1 = \left(\frac{k_{eq}}{m_{eq}} \right)^{\frac{1}{2}} = \left(\frac{70302.168}{524.8267} \right)^{\frac{1}{2}} = 11.5738 \text{ rad/sec} = 1.8420 \text{ Hz}$$

- (b) Wind velocity corresponding to maximum vibration of sign post (V) is given by:

$$St = 0.21 = \frac{f_1 D}{V} \quad \text{or} \quad V = \frac{f_1 D}{0.21} = \frac{(1.8420) (0.25)}{0.21} = 2.1929 \text{ m/s}$$

- (c) Maximum force acting on the system due to wind velocity:

$$F(t) = F_0 \sin \omega t = \frac{1}{2} c \rho V^2 A \sin \omega t = \frac{1}{2} (1) (1.2215) (2.1929^2) (8.0) \sin \omega t \text{ N}$$

$$= 23.4958 \sin \omega t \text{ N}$$

where $c = 1$ for a cylinder, ρ = density of air = 1.2215 kg/m^3 , A = projected area of cylindrical part = $(0.8)(10) = 8.0 \text{ m}^2$, and ω = frequency of wind force.
Equation of motion:

$$m_{eq} \ddot{x} + c_{eq} \dot{x} + k_{eq} x = F(t)$$

and the maximum steady state displacement of the sign post occurs when $\omega = \omega_1$ and is given by Eq. (3.34):

$$X = \frac{\delta_{st}}{2 \zeta} = \frac{F_0}{k_{eq} (2) \zeta} = \frac{23.4958}{2 (0.1) (70302.168)} = 0.001671 \text{ m}$$

3.65

(a) Equation of motion

$$m \ddot{x} + c \dot{x} + kx = F_0 x$$

or

$$m \ddot{x} + c \dot{x} + (k - F_0) x = 0 \quad (E_1)$$

Assuming the solution

$$x(t) = C e^{st} \quad (E_2)$$

where C is a constant, Eq. (E₁) gives the auxiliary equation

$$s^2 + \frac{c}{m} s + \left(\frac{k - F_0}{m} \right) = 0 \quad (E_3)$$

Roots of (E₃) are

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m} \right)^2 - \left(\frac{k - F_0}{m} \right)} \quad (E_4)$$

First consider the case of positive stiffness ($k > F_0$). For this case, following possibilities exist.

1. If $\left(\frac{c}{2m} \right)^2 > \left(\frac{k - F_0}{m} \right)$:

Both s_1 and s_2 will be real and negative and hence

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \quad (E_5)$$

will be stable.

2. If $\left(\frac{c}{2m} \right)^2 = \left(\frac{k - F_0}{m} \right)$:

Both s_1 and s_2 will be identical, real and negative.

Solution

$$x(t) = (C_1 + C_2 t) e^{s_1 t} \quad (E_6)$$

will be stable since $e^{s_1 t} \rightarrow 0$ as $t \rightarrow \infty$.

3. If $\left(\frac{c}{2m} \right)^2 < \left(\frac{k - F_0}{m} \right)$:

Here s_1 and s_2 will be complex conjugates and solution will be

$$x(t) = C e^{-\left(\frac{c}{2m} \right) t} \sin \left(\sqrt{\left\{ -\left(\frac{c}{2m} \right)^2 + \left(\frac{k - F_0}{m} \right) \right\} t + \phi} \right) \quad (E_7)$$

This represents a converging oscillatory motion and hence the system will be stable.

Next consider the case of negative stiffness ($k < F_0$). Here

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m} \right)^2 + \left(\frac{F_0 - k}{m} \right)} \quad (E_8)$$

Thus s_1 will be positive and s_2 will be negative, and the solution becomes

$$x(t) = C_1 e^{+s_1 t} + C_2 e^{-|s_2| t} \quad (E_9)$$

This solution can be seen to diverge as $t \rightarrow \infty$.

(b) Equation of motion

$$m \ddot{x} + c \dot{x} + kx = F_0 \dot{x}$$

or

$$\ddot{x} + \left(\frac{c - F_0}{m} \right) \dot{x} + \frac{k}{m} x = 0 \quad (E_{10})$$

Assuming $x(t) = C e^{st}$ the auxiliary equation becomes

$$s^2 + \left(\frac{c - F_0}{m}\right)s + \frac{k}{m} = 0 \quad (E_{11})$$

and hence

$$s_{1,2} = -\left(\frac{c - F_0}{2m}\right) \pm \sqrt{\left(\frac{c - F_0}{2m}\right)^2 - \frac{k}{m}} \quad (E_{12})$$

First consider the case of positive damping ($c > F_0$) in (E₁₀). For this case, it can be seen that the system will be stable for all possible values of $\left\{ \left(\frac{c - F_0}{2m}\right)^2 - \frac{k}{m} \right\}$.

Next, consider the case of negative damping ($c < F_0$). Depending on the sign of the quantity under the radical in Eq. (E₁₂), we will have three types of solution.

1. $\left(\frac{c - F_0}{2m}\right)^2 > \frac{k}{m}$. Here both s_1 and s_2 are real and positive and hence $x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$ (E₁₃)

This denotes a diverging nonoscillatory motion; so the system is unstable.

2. $\left(\frac{c - F_0}{2m}\right)^2 = \frac{k}{m}$. Here s_1 and s_2 are identical and are real and positive. Hence $x(t) = (C_1 + C_2 t) e^{s_1 t}$ (E₁₄)

This represents a diverging nonoscillatory solution; so the system will be unstable.

3. $\left(\frac{c - F_0}{2m}\right)^2 < \frac{k}{m}$. Here s_1 and s_2 are complex conjugates and hence

$$s_{1,2} = \left(\frac{F_0 - c}{2m}\right) \pm i \sqrt{\frac{k}{m} - \left(\frac{c - F_0}{2m}\right)^2} \quad (E_{15})$$

The solution becomes

$$x(t) = X e^{\left(\frac{F_0 - c}{2m}\right)t} \sin \left(\sqrt{\frac{k}{m} - \left(\frac{c - F_0}{2m}\right)^2} t + \phi \right) \quad (E_{16})$$

Since the exponent is positive, Eq. (E₁₆) denotes a diverging oscillatory motion and hence the system is unstable.

Thus the condition for dynamic stability of the system can be stated as

$$F_0 \leq c \quad (E_{17})$$

(c) Equation of motion $m\ddot{x} + c\dot{x} + kx = F_0 \ddot{x}$

or $(m - F_0)\ddot{x} + c\dot{x} + kx = 0$ (E₁₈)

With the solution $x(t) = C e^{st}$ (E₁₉)

the auxiliary equation will be

$$s^2 + \left(\frac{c}{m - F_0}\right)s + \left(\frac{k}{m - F_0}\right) = 0$$
 (E₂₀)

The roots are

$$s_{1,2} = -\frac{c}{2(m - F_0)} \pm \sqrt{\left\{\frac{c}{2(m - F_0)}\right\}^2 - \left(\frac{k}{m - F_0}\right)}$$
 (E₂₁)

First consider the case of positive mass ($m > F_0$) in (E₁₈).

In this case, the system will be stable for all values of

$$\left[\left\{ \frac{c}{2(m - F_0)} \right\}^2 - \left(\frac{k}{m - F_0} \right) \right].$$

Next consider the case of negative mass ($m < F_0$) in (E₁₈). For this case s_1 and s_2 can be expressed as

$$s_{1,2} = \frac{c}{2(F_0 - m)} \pm \sqrt{\left\{ \frac{c}{2(F_0 - m)} \right\}^2 + \left(\frac{k}{F_0 - m} \right)}$$
 (E₂₂)

This shows that s_1 will be positive and s_2 will be negative; thus the solution will be divergent.

3.66 The main program which calls HARESP and the output are given below.

```

C =====
C MAIN PROGRAM WHICH CALLS HARESP
C =====
C FOLLOWING 2 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION X(20),XD(20),XDD(20)
  DATA XM,XC,XK,F0,DM,N,IC/6.0,210.0,14000.0,450.0,10.0,20,0/
C END OF PROBLEM-DEPENDENT DATA
  CALL HARESP (XM,XC,XK,F0,DM,IC,N,X,XD,XDD,XAMP,XPHI)
  WRITE (21,100)
100  FORMAT (//,40H STEADY STATE RESPONSE OF AN UNDERDAMPED,/,
2 53H SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE)
  WRITE (21,200) XM,XC,XK,F0,DM,IC,N
200  FORMAT (//,12H GIVEN DATA://,5H XM =,E15.8,/,5H XC =,E15.8,/,
2 5H XK =,E15.8,/,5H F0 =,E15.8,/,5H DM =,E15.8,/,5H IC =,I2,/,
3 5H N =,I2)
  WRITE (21,300)
300  FORMAT (//,10H RESPONSE://,5H 1 ,3X,5H X(I),12X,6H XD(I),
2 11X,7H XDD(I),/)
  DO 400 I=1,N
400  WRITE (21,500) I,X(I),XD(I),XDD(I)
500  FORMAT (I4,2X,E15.8,2X,E15.8,2X,E15.8)
  STOP
  END

```

STEADY STATE RESPONSE OF AN UNDERDAMPED
SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE

GIVEN DATA:

XM = 0.60000000E+01
 XC = 0.21000000E+03
 XK = 0.14000000E+05
 F0 = 0.45000000E+03
 DM = 0.10000000E+02
 IC = 0
 N = 20

RESPONSE:

I	X(I)	XD(I)	XDD(I)
1	0.52434015E-02	0.32760188E+00	-0.52434015E+00
2	0.15110249E-01	0.29536480E+00	-0.15110248E+01
3	0.23497989E-01	0.23421521E+00	-0.23497992E+01
4	0.29585576E-01	0.15013890E+00	-0.29585576E+01
5	0.32777105E-01	0.51365890E-01	-0.32777107E+01
6	0.32760169E-01	-0.52435201E-01	-0.32760170E+01
7	0.29536424E-01	-0.15110357E+00	-0.29536424E+01
8	0.23421437E-01	-0.23498075E+00	-0.23421438E+01

9	0.15013780E-01	-0.29585633E+00	-0.15013779E+01
10	0.51364703E-02	-0.32777125E+00	-0.51364702E+00
11	-0.52436423E-02	-0.32760152E+00	0.52436423E+00
12	-0.15110461E-01	-0.29536372E+00	0.15110462E+01
13	-0.23498155E-01	-0.23421355E+00	0.23498156E+01
14	-0.29585691E-01	-0.15013662E+00	0.29585693E+01
15	-0.32777146E-01	-0.51363401E-01	0.32777145E+01
16	-0.32760132E-01	0.52437652E-01	0.32760131E+01
17	-0.29536312E-01	0.15110572E+00	0.29536314E+01
18	-0.23421254E-01	0.23498255E+00	0.23421254E+01
19	-0.15013559E-01	0.29585743E+00	0.15013559E+01
20	-0.51362254E-02	0.32777163E+00	0.51362258E+00

3.67 Eg. (3.35) gives the complete solution

$$x(t) = X_0 e^{-\gamma \omega_n t} \cos(\omega_d t + \phi_0) + X \cos(\omega t - \phi) \quad (E.1)$$

Differentiation gives

$$\begin{aligned} \dot{x}(t) = & -\gamma \omega_n X_0 e^{-\gamma \omega_n t} \cos(\omega_d t + \phi_0) - \omega_d X_0 e^{-\gamma \omega_n t} \sin(\omega_d t + \phi_0) \\ & - \omega X \sin(\omega t - \phi) \end{aligned} \quad (E.2)$$

$$\begin{aligned} \ddot{x}(t) = & X_0 e^{-\gamma \omega_n t} (\gamma^2 \omega_n^2 - \omega_d^2) \cos(\omega_d t + \phi_0) \\ & + 2\gamma \omega_n \omega_d X_0 e^{-\gamma \omega_n t} \sin(\omega_d t + \phi_0) - \omega^2 X \cos(\omega t - \phi) \end{aligned} \quad (E.3)$$

Let $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ be known. Then

$$x_0 = X_0 \cos \phi_0 + X \cos \phi \quad (E.4)$$

$$\dot{x}_0 = -\gamma \omega_n X_0 \cos \phi_0 - \omega_d X_0 \sin \phi_0 + \omega X \sin \phi \quad (E.5)$$

Eqs. (E.4) and (E.5) give

$$\begin{aligned} X_0 = & [(X_0 \cos \phi_0)^2 + (X_0 \sin \phi_0)^2]^{1/2} \\ = & \left[(x_0 - X \cos \phi)^2 + \left\{ \frac{-\dot{x}_0 - \gamma \omega_n x_0 + \gamma \omega_n X \cos \phi + \omega X \sin \phi}{\omega_d} \right\}^2 \right]^{1/2} \end{aligned} \quad (E.6)$$

$$\phi_0 = \tan^{-1} \left\{ \frac{-\dot{x}_0 - \gamma \omega_n x_0 + \gamma \omega_n X \cos \phi + \omega X \sin \phi}{\omega_d (x_0 - X \cos \phi)} \right\} \quad (E.7)$$

Now the computer program TOTALR can be written. In addition to the arguments used in section 3.12, the following arguments are also used:

XO = value of $x(t)$ at $t=0$. Input

XDO = value of $\frac{dx}{dt}(t)$ at $t=0$. Input.

The program and the output are given below.

```

C =====
C
C SOLUTION OF PROBLEM 3.67
C MAIN PROGRAM WHICH CALLS TOTALR
C HARMONIC FORCE IS ASSUMED TO BE  $F(T)=F_0\cos(\omega T)$ 
C DATA OF PROBLEM 3.14 IS USED IN MAIN PROGRAM
C
C =====
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION X(20),XD(20),XDD(20)
  DATA XM,XC,XK,F0,OM,N/10.0,45.0,2500.0,180.0,21.9912,20/
  DATA X0,XD0/0.015,5.0/
C END OF PROBLEM-DEPENDENT DATA
  CALL TOTALR (XM,XC,XK,F0,OM,N,X,XD,XDD,XAMP,XPHI,X0,XD0)
  WRITE (21,100)
100  FORMAT (//,33H TOTAL RESPONSE OF AN UNDERDAMPED,/,
2 53H SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE)
  WRITE (21,200) XM,XC,XK,F0,OM,N
200  FORMAT (//,12H GIVEN DATA://,5H XM =,E15.8,/,5H XC =,E15.8,/,
2 5H XK =,E15.8,/,5H F0 =,E15.8,/,5H OM =,E15.8,/,
3 5H N =,12)
  WRITE (21,250) X0,XD0
250  FORMAT (/,20H INITIAL CONDITIONS://,6H X0 =,E15.8,/,
2 6H XD0 =,E15.8)
  WRITE (21,300)
300  FORMAT (//,10H RESPONSE://,5H I ,3X,5H X(1),12X,6H XD(1),
2 11X,7H XDD(1),/)
  DO 400 I=1,N
400  WRITE (21,500) I,X(I),XD(I),XDD(I)
500  FORMAT (I4,2X,E15.8,2X,E15.8,2X,E15.8)
  STOP
  END
C =====
C
C SUBROUTINE TOTALR
C
C =====
  SUBROUTINE TOTALR (XM,XC,XK,F0,OM,N,X,XD,XDD,XAMP,XPHI,X0,XD0)
  DIMENSION X(N),XD(N),XDD(N)
  OMN=SQRT(XK/XM)
  XA1=XC/(2.0*XM*OMN)
  OMD=OMN*SQRT(1.0-XA1**2)
  DST=F0/XK
  R=OM/OMN
  XAMP=DST/SQRT((1.0-R**2)**2+(2.0*XA1*R)**2)
  XPHI=ATAN(2.0*XA1*R/(1.0-R**2))
  DELT=2.0*3.1416/(OMD*REAL(N))
  XCOS=X0-XAMP*COS(XPHI)
  XSIN=(-XD0-XA1*OMN*X0+XA1*OMN*XAMP*COS(XPHI)+OM*XAMP*SIN
2 (XPHI))/OMD
  XZ=SQRT(XCOS**2+XSIN**2)
  PZ=ATAN(XSIN/XCOS)
  TIME=0.0
  DO 10 I=1,N
  IIME=TIME+DELT

```

```

EX1=EXP(-XAI*OMN*TIME)
EC1=COS(OMD*TIME+PZ)
EC2=COS(OM*TIME-XPHI)
ES1=SIN(OMD*TIME+PZ)
ES2=SIN(OM*TIME-XPHI)
X(I)=XZ*EX1*EC1+XAMP*EC2
XD(I)=-XAI*OMN*XZ*EX1*EC1-OMD*XZ*EX1*ES1-OM*XAMP*ES2
10 XDD(I)=XZ*EX1*EC1*((XAI*OMN)**2-OMD**2)+2.0*XAI*OMN*OMD*XZ*
2 EX1*ES1-(OM**2)*XAMP*EC2
RETURN
END

```

TOTAL RESPONSE OF AN UNDERDAMPED
SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE

GIVEN DATA:

```

XM = 0.10000000E+02
XC = 0.45000000E+02
XK = 0.25000000E+04
FO = 0.18000000E+03
OM = 0.21991199E+02
N = 20

```

INITIAL CONDITIONS:

```

X0 = 0.15000000E-01
XDO = 0.50000000E+01

```

RESPONSE:

I	X(I)	XD(I)	XDD(I)
1	-0.10720424E-01	-0.62618327E+01	0.14583885E+02
2	-0.13126461E+00	-0.56405468E+01	0.46769913E+02
3	-0.23313771E+00	-0.44189043E+01	0.73777733E+02
4	-0.30557248E+00	-0.27362981E+01	0.92193657E+02
5	-0.34123138E+00	-0.79179698E+00	0.99568497E+02
6	-0.33717862E+00	0.11797444E+01	0.94842743E+02
7	-0.29529962E+00	0.29382858E+01	0.78578743E+02
8	-0.22207454E+00	0.42714534E+01	0.52945923E+02
9	-0.12771350E+00	0.50244703E+01	0.21447788E+02
10	-0.24761723E-01	0.51222720E+01	-0.11575108E+02
11	0.73627755E-01	0.45800018E+01	-0.41590317E+02
12	0.15549700E+00	0.34999285E+01	-0.64561966E+02
13	0.21168864E+00	0.20550721E+01	-0.77567314E+02
14	0.23700854E+00	0.46208435E+00	-0.79236084E+02
15	0.23077171E+00	-0.10522565E+01	-0.69936806E+02
16	0.19665717E+00	-0.22854314E+01	-0.51678036E+02
17	0.14191106E+00	-0.30888264E+01	-0.27741650E+02
18	0.76043457E-01	-0.33878446E+01	-0.21128597E+01
19	0.92454441E-02	-0.31893401E+01	0.21192802E+02
20	-0.49206819E-01	-0.25754893E+01	0.38788284E+02

3.68

The main program and output are given.

```

=====
C MAIN PROGRAM WHICH CALLS HARESP
C =====
C FOLLOWING 2 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION X(20),XD(20),XDD(20)
  DATA XM,XC,XK,F0,OM,N,IC/10.0,45.0,2500.0,180.0,20.0,20,1/
C END OF PROBLEM-DEPENDENT DATA
  CALL HARESP (XM,XC,XK,F0,OM,IC,N,X,XD,XDD,XAMP,XPHI)
  WRITE (21,100)

100  FORMAT (//,40H STEADY STATE RESPONSE OF AN UNDERDAMPED,/,
2 53H SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE)
  WRITE (21,200) XM,XC,XK,F0,OM,IC,N
200  FORMAT (//,12H GIVEN DATA://,5H XM =,E15.8,/,5H XC =,E15.8,/,
2 5H XK =,E15.8,/,5H F0 =,E15.8,/,5H OM =,E15.8,/,5H IC =,12,/,
3 5H N =,12)
  WRITE (21,300)
300  FORMAT (//,10H RESPONSE://,5H I ,3X,5H X(1),12X,6H XD(1),
2 11X,7H XDD(1),/)
  DO 400 I=1,N
400  WRITE (21,500) I,X(1),XD(1),XDD(1)
500  FORMAT (I4,2X,E15.8,2X,E15.8,2X,E15.8)
  STOP
  END

```

STEADY STATE RESPONSE OF AN UNDERDAMPED
SINGLE DEGREE OF FREEDOM SYSTEM UNDER HARMONIC FORCE

GIVEN DATA:

XM = 0.10000000E+02
XC = 0.45000000E+02
XK = 0.25000000E+04
F0 = 0.18000000E+03
OM = 0.20000000E+02
IC = 1
N = 20

RESPONSE:

I	X(I)	XD(I)	XDD(I)
1	0.67556962E-01	-0.15523261E+01	-0.27022785E+02
2	0.40265664E-01	-0.18938755E+01	-0.16106266E+02
3	0.90328613E-02	-0.20500383E+01	-0.36131446E+01
4	-0.23084141E-01	-0.20055280E+01	0.92336569E+01
5	-0.52941486E-01	-0.17647020E+01	0.21176596E+02
6	-0.77616550E-01	-0.13511336E+01	0.31046618E+02
7	-0.94693922E-01	-0.80530620E+00	0.37877567E+02
8	-0.10250194E+00	-0.18064982E+00	0.41000774E+02

9	-0.10027631E+00	0.46169037E+00	0.40110523E+02
10	-0.88234901E-01	0.10588365E+01	0.35293961E+02
11	-0.67556389E-01	0.15523360E+01	0.27022554E+02
12	-0.40264975E-01	0.18938812E+01	0.16105991E+02
13	-0.90321312E-02	0.20500395E+01	0.36128526E+01
14	0.23084890E-01	0.20055246E+01	-0.92339563E+01
15	0.52942146E-01	0.17646941E+01	-0.21176857E+02
16	0.77617034E-01	0.13511224E+01	-0.31046814E+02
17	0.94694205E-01	0.80529296E+00	-0.37877682E+02
18	0.10250200E+00	0.18063448E+00	-0.41000801E+02
19	0.10027614E+00	-0.46170485E+00	-0.40110458E+02
20	0.88234514E-01	-0.10588492E+01	-0.35293804E+02

3.69

Equation of motion is $m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$

When $y(t) = Y \sin \omega t$, $x_p(t) = X \cos(\omega t - \phi_1 - \phi_2)$

Complete solution can be expressed as

$$x(t) = X_0 e^{-\gamma \omega_n t} \cos(\omega_d t + \phi_0) + X \cos(\omega t - \phi_1 - \phi_2) \quad (E.1)$$

$$\text{with } X = Y \left[\frac{1 + (2\gamma r)^2}{(1 - r^2)^2 + (2\gamma r)^2} \right]^{1/2},$$

$$\phi_1 = \tan^{-1} \left(\frac{2\gamma r}{1 - r^2} \right), \quad \phi_2 = \tan^{-1} \left(\frac{1}{2\gamma r} \right), \quad r = \frac{\omega}{\omega_n}.$$

If the initial conditions are known $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$,

$$x_0 = X_0 \cos \phi_0 + X \cos(\phi_1 + \phi_2)$$

$$\text{and } \dot{x}_0 = -\gamma \omega_n X_0 \sin \phi_0 - \omega_d X_0 \sin \phi_0 - \omega X \sin(-\phi_1 - \phi_2)$$

Hence

$$X_0 = \left[\{x_0 - X \cos(\phi_1 + \phi_2)\}^2 + \left\{ \frac{-\dot{x}_0 - \gamma \omega_n x_0 + \gamma \omega_n X \cos(\phi_1 + \phi_2) + \omega X \sin(\phi_1 + \phi_2)}{\omega_d} \right\}^2 \right]^{1/2}$$

$$\phi_0 = \tan^{-1} \left[\frac{-\dot{x}_0 - \gamma \omega_n x_0 + \gamma \omega_n X \cos(\phi_1 + \phi_2) + \omega X \sin(\phi_1 + \phi_2)}{\omega_d \{x_0 - X \cos(\phi_1 + \phi_2)\}} \right]$$

If necessary, the velocity $\dot{x}(t)$ and acceleration $\ddot{x}(t)$ can be found from Eq. (E.1). The computer program and output are given below.

```

C =====
C
C SOLUTION OF PROBLEM 3.69
C MAIN PROGRAM WHICH CALLS BASEX
C RESPONSE OF A SINGLE D.O.F. SYSTEM SUBJECTED TO BASE EXCITATION,
C Y(T)=Y*SIN(OM*T)
C
C =====

```

```

C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(20),XD(20),XDD(20)
      DATA XM,XC,XK,Y,OM,N/2.0,10.0,100.0,0.1,25.0,20/
      DATA X0,XD0/0.01,5.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL BASEX (XM,XC,XK,Y,OM,N,X,XD,XDD,X0,XD0)
      PRINT 100
100  FORMAT (//,33H TOTAL RESPONSE OF AN UNDERDAMPED,/,
2    52H SINGLE D.O.F. SYSTEM UNDER HARMONIC BASE EXCITATION)
      PRINT 200, XM,XC,XK,Y,OM,N
200  FORMAT (//,12H GIVEN DATA:,,5H XM =,E15.8,,5H XC =,E15.8,,
2    5H XK =,E15.8,,5H Y =,E15.8,,5H OM =,E15.8,,5H N =,12)
      PRINT 300,X0,XD0
300  FORMAT (/ ,20H INITIAL CONDITIONS:,,6H X0 =,E15.8,,6H XD0 =,
2    E15.8)
      PRINT 400
400  FORMAT (//,10H RESPONSE:,,5H 1 ,3X,5H X(1),12X,6H XD(1),
2    11X,7H XDD(1),/)
      DO 500 I=1,N
500  PRINT 600, I,X(I),XD(I),XDD(I)
600  FORMAT (I4,2X,E15.8,2X,E15.8,2X,E15.8)
      STOP
      END

C =====
C
C SUBROUTINE BASEX
C
C =====
      SUBROUTINE BASEX (XM,XC,XK,Y,OM,N,X,XD,XDD,X0,XD0)
      DIMENSION X(N),XD(N),XDD(N)
      OMN=SQRT(XK/XM)
      XAI=XC/(2.0*XM*OMN)
      OMD=OMN*SQRT(1.0-XAI**2)
      R=OM/OMN
      DELT=2.0*3.1416/(OMD*REAL(N))
      XAMP=Y*SQRT(1.0+(2.0*XAI*R)**2/((1.0-R**2)**2+(2.0*XAI*R)**2))
      PHI1=ATAN(2.0*XAI*R/(1.0-R*R))
      PHI2=ATAN(1.0/(2.0*XAI*R))
      XCC=XC
      TIME=0.0
      DO 10 I=1,N
      TIME=TIME+DELT
      XC=X0-XAMP*COS(PHI1+PHI2)
      XS=(-XD0-XAI*OMN*XC+OM*XAMP*SIN(PHI1+PHI2))/OMD
      XZ=SQRT(XC**2+XS**2)
      PZ=ATAN(XS/XC)
      EX=EXP(-XAI*OMN*TIME)
      CS=COS(OMD*TIME+PZ)
      SI=SIN(OMD*TIME+PZ)
      CS12=COS(OM*TIME-PHI1-PHI2)
      SI12=SIN(OM*TIME-PHI1-PHI2)
      X(I)=XZ*EX*CS+XAMP*CS12
      XD(I)=-XAI*OMN*XZ*EX*CS-OMD*XZ*EX*SI-OM*XAMP*SI12
10   XDD(I)=XZ*EX*CS*((XAI*OMN)**2-OMD**2)+2.0*XAI*OMN*OMD*XZ*EX*SI
2     -(OM**2)*XAMP*CS12

```

```

XC=XCC
RETURN
END

```

TOTAL RESPONSE OF AN UNDERDAMPED
SINGLE D.O.F. SYSTEM UNDER HARMONIC BASE EXCITATION

GIVEN DATA:

```

XM = 0.20000000E+01
XC = 0.10000000E+02
XK = 0.10000000E+03
Y = 0.10000000E+00
DM = 0.25000000E+02
N = 20

```

INITIAL CONDITIONS:

```

X0 = 0.99999998E-02
XD0 = 0.50000000E+01

```

RESPONSE:

I	X(I)	XD(I)	XDD(I)
1	-0.50330855E-01	-0.57580528E+01	-0.10305269E+02
2	-0.30772516E+00	-0.44899216E+01	0.62558521E+02
3	-0.43412885E+00	-0.65008521E+00	0.85064461E+02
4	-0.38465756E+00	0.22720275E+01	0.28117821E+02
5	-0.27407524E+00	0.18295681E+01	-0.40405914E+02
6	-0.24066399E+00	-0.41657627E+00	-0.39765869E+02
7	-0.28432682E+00	-0.90879965E+00	0.23411983E+02
8	-0.27970907E+00	0.14345939E+01	0.64174423E+02
9	-0.14624044E+00	0.38781688E+01	0.26183165E+02
10	0.39102390E-01	0.33305802E+01	-0.47345394E+02
11	0.12840362E+00	0.26650119E+00	-0.67513680E+02
12	0.80883794E-01	-0.18088942E+01	-0.10970811E+02
13	0.96553117E-02	-0.68976790E+00	0.50778744E+02
14	0.38462169E-01	0.18209940E+01	0.40712753E+02
15	0.14735566E+00	0.22107751E+01	-0.27525837E+02
16	0.19996722E+00	-0.31552225E+00	-0.66972618E+02
17	0.11764607E+00	-0.28227291E+01	-0.26468327E+02
18	-0.18114097E-01	-0.23140993E+01	0.45068512E+02
19	-0.63696094E-01	0.51316518E+00	0.59701672E+02
20	0.10478826E-01	0.21295214E+01	0.43720150E+00

3.70

Unbalanced force in vertical direction = $m e \omega^2 \sin \omega t$ (E₁)

Unbalanced force in horizontal direction = 0

Let M = total mass of the shaker

Equation of motion is $M \ddot{x} + c \dot{x} + k x = m e \omega^2 \sin \omega t$ (E₂)

Steady state solution of (E₂) is

$$x(t) = X \sin(\omega t - \phi) \quad (E_3)$$

where

$$X = \frac{m e r^2}{M \left[(1-r^2)^2 + (2\zeta r)^2 \right]^{1/2}} \quad (E_4)$$

and $\phi = \tan^{-1} \left(\frac{2\zeta r}{1-r^2} \right) \quad (E_5)$

where $r = \frac{\omega}{\omega_n} = \omega \sqrt{M/k} \quad (E_6)$

Frequency range: 20 Hz to 30 Hz
i.e., 125.664 rad/sec to 188.496 rad/sec

$$\therefore 125.664 \text{ rad/sec} \leq \omega \leq 188.496 \text{ rad/sec} \quad (E_7)$$

$$0.1'' \leq X \leq 0.2'' \text{ where } X \text{ is given by } (E_4).$$

Mean power output over a time period τ is given by

$$P = \frac{1}{\tau} \int_0^{\tau} F(\tau) \frac{dx}{dt}(\tau) d\tau \quad (E_9)$$

where $\tau = \frac{2\pi}{\omega}$,

$$F(\tau) = m e \omega^2 \sin \omega t, \text{ and}$$

$$\frac{dx}{dt} = \omega X \cos(\omega t - \phi)$$

$$P \geq 1 \text{ hp} \quad (E_{10})$$

$$\frac{M}{m} \geq 50$$

Procedure: (E_{11})

Find ω , e , M , m , k and c

so as to satisfy the requirements stated in
(E_7), (E_8), (E_{10}) and (E_{11}).

$$(3.71) \quad m = \frac{10^5}{386.4} = 258.7992 \text{ lb-s}^2/\text{in}, \quad l = 600'', \quad E = 30 \times 10^6 \text{ psi}$$

$$k = \frac{3EI}{l^3} = \frac{3(30 \times 10^6)}{(600)^3} \cdot \frac{\pi}{64} (D^4 - d^4) = 0.020453 (D^4 - d^4) \frac{\text{lb}}{\text{in}}$$

$$\omega = 2\pi(15) = 94.248 \text{ rad/sec}; \quad \zeta = 0.15$$

Ground acceleration $\ddot{y}(t) = 193.2 \sin 94.248 t \text{ in/s}^2 \quad (E_1)$

Equation of motion:

$$m \ddot{z} + c \dot{z} + k z = -m \ddot{y} = -50000 \sin 94.248 t \quad (E_2)$$

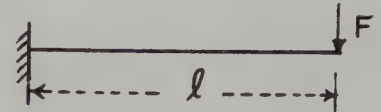
where $z = x - y =$ relative displacement.

$$\text{Let } z(t) = Z \sin(\omega t - \phi) = Z \sin(94.248 t - \phi) \quad (E_3)$$

$$\text{with } Z = \frac{-50000}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad (E_4)$$

$$\text{and } \phi = \tan^{-1} \left(\frac{c\omega}{k - m\omega^2} \right) \quad (E_5)$$

$$y_{\max} = \frac{F l^3}{3EI} \Rightarrow \frac{3 y_{\max}}{l^2} = \frac{F l}{EI}$$



Max. bending moment $= M = F l$

$$\text{Max. bending stress} = \frac{M (D/2)}{EI} = \frac{F l}{EI} \cdot \frac{D}{2} = \frac{3 D}{2 l^2} y_{\max}$$

If maximum relative displacement, $y_{\max} = Z$, is known, max. bending stress (σ_1) is given by

$$\sigma_1 = \frac{3 D}{2 l^2} \cdot Z$$

Direct compressive stress (σ_2) due to weight of water tank is

$$\sigma_2 = \frac{m g}{\frac{\pi}{4} (D^2 - d^2)} = \frac{4 \times 10^5}{\pi (D^2 - d^2)}$$

$$\text{Total stress} = \sigma_1 + \sigma_2 \leq 30000 \text{ psi}$$

i.e.,

$$\frac{3 D}{2 l^2} Z + \frac{4 \times 10^5}{\pi (D^2 - d^2)} \leq 30000 \quad (E_6)$$

Natural frequency of water tank is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.020453 (D^4 - d^4)}{258.7992}} \geq 30 \pi \frac{\text{rad}}{\text{sec}}$$

i.e.,

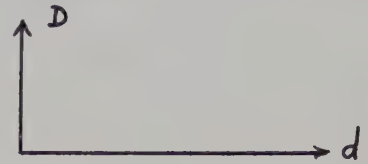
$$D^4 - d^4 \geq 1.1240 \times 10^8 \quad (E_7)$$

Weight of steel column is

$$\begin{aligned} W_s &= \frac{\pi}{4} (D^2 - d^2) l \rho_{\text{weight}} = \frac{\pi}{4} (D^2 - d^2) (600) (0.283) \\ &= 133.3609 (D^2 - d^2) \text{ lb.} \end{aligned} \quad (E_8)$$

Problem: Find $\begin{Bmatrix} D \\ d \end{Bmatrix}$ to minimize W_s subject to restrictions of (E_6) and (E_7) . Also use the conditions :
 $D \geq d$, $D \geq 0$ and $d \geq 0$.

Procedure: Plot the inequalities (E_6) and (E_7) in the D - d space and draw contours of W_s and identify the minimum weight design.



Chapter 4

Vibration Under General Forcing Conditions

4.1 $F(t) = \frac{F_0}{\pi} + \frac{F_0}{2} \sin \omega t - \frac{2F_0}{\pi} \sum_{n=2,4,6,\dots} \frac{1}{(n^2-1)} \cos n\omega t$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{\pi k} + \frac{F_0}{2k} \frac{1}{\sqrt{(1-r^2)^2 + (2\gamma r)^2}} \sin(\omega t - \phi_1) - \frac{2F_0}{\pi k} \sum_{n=2,4,6,\dots} \frac{1}{(n^2-1)} \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\gamma nr)^2}} \cos(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$, $\phi_1 = \tan^{-1} \left(\frac{2\gamma r}{1-r^2} \right)$ and $\phi_n = \tan^{-1} \left(\frac{2\gamma nr}{1-n^2r^2} \right)$

4.2 From the solution of problem 1.63,

$$F(t) = \begin{cases} (2F_0 t/\tau) ; & 0 \leq t \leq \tau/2 \\ -(2F_0 t/\tau) + 2F_0 ; & \tau/2 \leq t \leq \tau \end{cases}$$

Fourier series representation of $F(t)$ is

$$F(t) = \frac{F_0}{2} - \frac{4F_0}{\pi^2} \sum_{n=1,3,5,\dots} \frac{1}{n^2} \cos n\omega t$$

$$\therefore x(t) = \frac{F_0}{2k} - \frac{4F_0}{\pi^2 k} \sum_{n=1,3,5,\dots} \frac{1}{n^2} \frac{\cos(n\omega t - \phi_n)}{\sqrt{(1-n^2r^2)^2 + (2\gamma nr)^2}}$$

where $r = \frac{\omega}{\omega_n}$ and $\phi_n = \tan^{-1} \left(\frac{2\gamma nr}{1-n^2r^2} \right)$.

4.3 $F(t) = \frac{8F_0}{\pi^2} \sum_{n=1,3,5,\dots} (-1)^{\frac{n-1}{2}} \sin \frac{n\omega t}{n^2}$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{8F_0}{\pi^2 k} \sum_{n=1,3,5,\dots} \frac{1}{n^2} (-1)^{\frac{n-1}{2}} \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\gamma nr)^2}} \sin(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1} \left(\frac{2\gamma nr}{1-n^2r^2} \right)$

4.4 $F(t) = \frac{F_0}{2} + \frac{F_0}{\pi} \sum_{n=1,2,\dots} \frac{1}{n} \sin n\omega t$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{2k} + \frac{F_0}{\pi k} \sum_{n=1,2,\dots}^{\infty} \frac{1}{n} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\gamma nr)^2}} \sin(n\omega t - \phi_n)$$

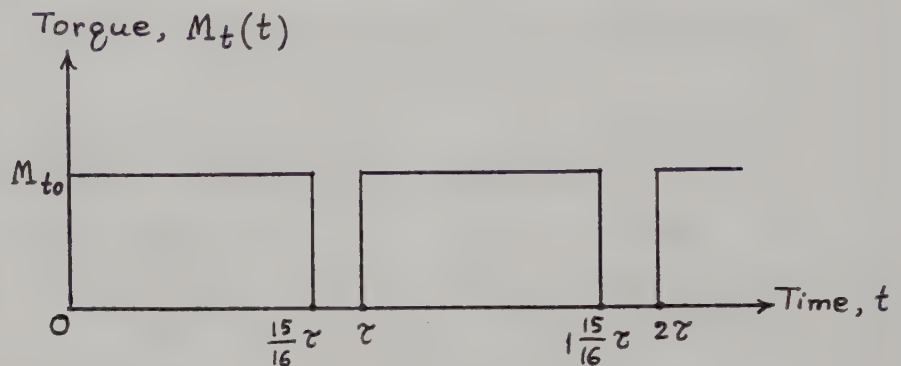
$$\text{where } r = \omega/\omega_n \text{ and } \phi_n = \tan^{-1} \left(\frac{2\gamma nr}{1-n^2r^2} \right)$$

4.5 From Example 1.12, $F(t) = \frac{F_0}{\pi} \left[\frac{\pi}{2} - \sum_{n=1,2,3,\dots} \frac{1}{n} \sin n\omega t \right]$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{2k} - \frac{F_0}{\pi k} \sum_{n=1,2,3,\dots} \frac{1}{n} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\gamma nr)^2}} \sin(n\omega t - \phi_n)$$

$$\text{where } r = \omega/\omega_n \text{ and } \phi_n = \tan^{-1} \left(\frac{2\gamma nr}{1-n^2r^2} \right)$$

4.6



Torque transmitted to driven gear is shown in the figure. It can be expressed as:

$$M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n \omega t + b_n \sin n \omega t)$$

$$\text{where } \omega = 2\pi \left(\frac{1000}{60} \right) = 104.72 \text{ rad/sec}$$

$$\tau = \frac{2\pi}{\omega} = 0.06 \text{ sec} ; \quad \frac{15}{16} \tau = 0.05625 \text{ sec}$$

$$M_t(t) = \begin{cases} M_{t0} = 1000 \text{ N-m} ; & 0 \leq t \leq \frac{15}{16} \tau \\ 0 ; & \frac{15}{16} \tau \leq t \leq \tau \end{cases}$$

$$a_0 = \frac{2}{\tau} \int_0^{\tau} M_t(t) dt = \frac{2}{0.06} \int_0^{0.05625} (1000) dt = \frac{2000}{0.06} (0.05625) = 1875.0 \text{ N-m}$$

$$\begin{aligned} a_n &= \frac{2}{\tau} \int_0^{\tau} M_t(t) \cos n \omega t dt = \frac{2}{\tau} M_{t0} \int_0^{0.05625} \cos n \omega t dt \\ &= \frac{2 M_{t0}}{\tau} \left(\frac{\sin n \omega t}{n \omega} \right)_0^{0.05625} = \frac{318.3091}{n} \sin 5.8905 n \text{ N-m} \end{aligned}$$

$$b_n = \frac{2}{\tau} \int_0^{\tau} M_t(t) \sin n \omega t dt = \frac{2 M_{t0}}{\tau} \int_0^{0.05625} \sin n \omega t dt$$

$$= \frac{2 M_{t0}}{\tau} \left[-\frac{\cos n \omega t}{n \omega} \right]_0^{0.05625} = \frac{318.3091}{n} (1 - \cos 5.8905 n) \text{ N-m}$$

$$k_t = \frac{GJ}{\ell} = G \left(\frac{\pi d^4}{4} \right) \frac{1}{\ell} = (80 (10^9)) \left(\frac{\pi (0.05)^4}{4} \right) \frac{1}{1} = 392,700 \text{ N-m/rad}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{392700}{0.1}} = 1981.6660 \text{ rad/sec}$$

Equation of motion:

$$J_0 \ddot{\theta} + k_t \theta = M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n \omega t + b_n \sin n \omega t)$$

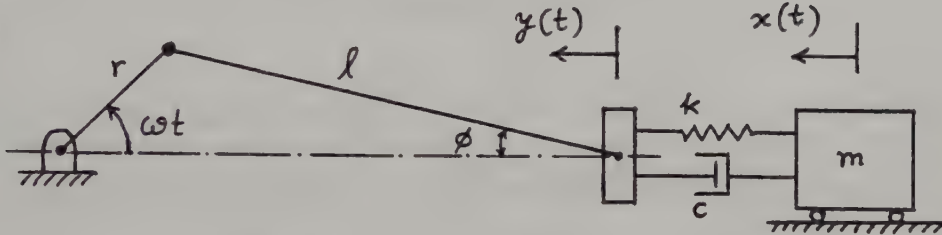
Response:

$$\theta(t) = \frac{a_0}{2 k_t} + \sum_{n=1}^{\infty} \left\{ \frac{a_n \cos n \omega t + b_n \sin n \omega t}{k_t - J_0 (n \omega)^2} \right\}$$

$$= 0.0023873$$

$$+ \sum_{n=1}^{\infty} \left\{ \frac{318.3091 \sin 5.8905 n \cos \omega t + 318.3091 (1 - \cos 5.8905 n) \sin n \omega t}{n (392700.0 - 1096.6278 n^2)} \right\} \text{ rad}$$

4.7



Base motion is given by:

$$y(t) = r + \ell - r \cos \omega t - \ell \cos \phi = r + \ell - r \cos \omega t - \ell \sqrt{1 - \sin^2 \phi} \quad (1)$$

Using $\ell \sin \phi = r \sin \omega t$, Eq. (1) becomes

$$y(t) = r + \ell - r \cos \omega t - \ell \sqrt{1 - \frac{r^2}{\ell^2} \sin^2 \omega t} \quad (2)$$

Using the approximation:

$$\sqrt{1 - \frac{r^2}{\ell^2} \sin^2 \omega t} \approx 1 - \frac{r^2}{2 \ell^2} \sin^2 \omega t \quad (3)$$

Eq. (2) can be expressed as

$$y(t) = r + \ell - r \cos \omega t - \ell \left(1 - \frac{1}{2} \frac{r^2}{\ell^2} \sin^2 \omega t \right)$$

$$= r - r \cos \omega t + \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 - \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 \cos 2 \omega t \quad (4)$$

Equation of motion:

$$\begin{aligned}
 m \ddot{x} + c \dot{x} + k x &= k y + c \dot{y} \\
 &= k r - k r \cos \omega t + \frac{k \ell}{4} \left(\frac{r}{\ell} \right)^2 - \frac{k \ell}{4} \left(\frac{r}{\ell} \right)^2 \cos 2 \omega t + \dots \\
 &\quad + c r \omega \sin \omega t + \frac{c \ell}{4} \left(\frac{r}{\ell} \right)^2 (2 \omega) \sin 2 \omega t + \dots
 \end{aligned} \tag{5}$$

Solution of Eq. (5) can be found by adding the solutions due to each term on the right hand side of Eq. (5).

Solution due to constant term, F_0 (terms 1 and 3 on the r.h.s. of Eq. (5)):

$$\begin{aligned}
 x(t) &= \frac{F_0}{k} \left[1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \cos(\omega_d t - \phi) \right] \\
 \text{where } \phi &= \tan^{-1} \left(\frac{\zeta}{\sqrt{1 - \zeta^2}} \right)
 \end{aligned} \tag{6}$$

Solution due to sinusoidal term, $F_0 \sin \Omega t$ (terms 5 and 6 on the r.h.s. of Eq. (5)):

$$x(t) = X \sin(\Omega t - \phi_0) \tag{7}$$

$$\text{where } X = \frac{F_0}{\left[(k - m \Omega^2)^2 + c^2 \Omega^2 \right]^{\frac{1}{2}}} \quad \text{and} \quad \phi_0 = \tan^{-1} \left(\frac{c \Omega}{k - m \Omega^2} \right) \tag{8}$$

Solution due to cosine term, $F_0 \cos \Omega t$ (terms 2 and 4 in Eq. (5)):

$$x(t) = X \cos(\Omega t - \phi_0) \tag{9}$$

$$\text{where } X = \frac{F_0}{\left[(k - m \Omega^2)^2 + c^2 \Omega^2 \right]^{\frac{1}{2}}} \quad \text{and} \quad \phi_0 = \tan^{-1} \left(\frac{c \Omega}{k - m \Omega^2} \right) \tag{10}$$

For given data, $\zeta = \frac{c}{2 \sqrt{m k}} = \frac{10}{2 \sqrt{1(100)}} = 0.5$, $\frac{r}{\ell} = 0.1$, $\omega = 100$, $2 \omega = 200$, etc. and the solution of Eq. (5) can be obtained by using Eqs. (6) to (8) suitably.

4.8

Base motion can be represented by Fourier series as (from Example 1.12):

$$y(t) = \frac{Y}{\pi} \left[\frac{\pi}{2} - \left\{ \sin \omega t + \frac{1}{2} \sin 2 \omega t + \frac{1}{3} \sin 3 \omega t + \dots \right\} \right] \tag{1}$$

Equation of motion of mass:

$$m \ddot{x} + c (\dot{x} - \dot{y}) + k (x - y) = 0 \quad (2)$$

Since $y(t)$ is composed of several terms, the solution of Eq. (2) can be found by superposing the solutions corresponding to each of the terms appearing in Eq. (1). When $y(t) = Y/2$, constant, equation of motion becomes:

$$m \ddot{x} + c \dot{x} + k x = \frac{k Y}{2} = \text{constant} \quad (3)$$

The steady state solution of Eq. (3) is given by (see Example 4.3):

$$x(t) = \frac{Y}{2} \left[1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \cosh(\omega_d t - \phi) \right] \quad (4)$$

$$\text{where } \phi = \tan^{-1} \left(\frac{\zeta}{\sqrt{1 - \zeta^2}} \right) \quad (5)$$

When $y(t) = A \sin \Omega t$, the steady state solution of Eq. (2) is given by Eq. (3.67):

$$x(t) = A \sin(\Omega t - \phi) \quad (6)$$

$$\text{where } A = \left[\frac{1 + (2 \zeta r)^2}{(1 - r^2)^2 + (2 \zeta r)^2} \right]^{\frac{1}{2}} \quad (7)$$

$$\phi = \tan^{-1} \left(\frac{2 \zeta r^3}{1 + r^2 (4 \zeta^2 - 1)} \right) \quad (8)$$

$$\text{and } r = \frac{\Omega}{\omega_n}$$

4.9

From solution of Problem 4.7, we can express the base motion as:

$$y(t) = r - r \cos \omega t + \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 - \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 \cos 2 \omega t + \dots \quad (1)$$

Equation of motion:

$$m \ddot{x} + k (x - y) \pm \mu N = 0$$

or

$$m \ddot{x} + k x \pm \mu N = k y = \left\{ k r + \frac{k \ell}{4} \left(\frac{r}{\ell} \right)^2 \right\} - k r \cos \omega t - \frac{k \ell}{4} \left(\frac{r}{\ell} \right)^2 \cos 2 \omega t + \dots \quad (2)$$

For given numerical data, Eq. (2) becomes:

$$\ddot{x} + 100 x \pm 0.981 = 100 y$$

$$= \left\{ 100 (0.1) + \frac{100 (1)}{4} \left(\frac{0.1}{1} \right)^2 \right\} - (100) (0.1) \cos 100 t - \frac{100 (1)}{4} \left(\frac{0.1}{1} \right)^2 \cos 200 t - \dots$$

$$= 10.25 - 10 \cos 100 t - 0.25 \cos 200 t - \dots \quad (3)$$

Using the definition of equivalent damping constant, c_{eq} , the solution of Eq. (3) can be found by superposition.

$$c_{eq} = \frac{4 \mu N}{\pi \Omega X} = \frac{4 \mu m g}{\pi \Omega X} \quad (4)$$

where Ω is the frequency of the harmonic force and X is the amplitude of the mass.

Steady state solution due to constant term, F_0 , on the r.h.s. of Eq. (3) (from Example 4.3):

$$x(t) = \frac{F_0}{k} = \left\{ k r + \frac{k \ell}{4} \left(\frac{r}{\ell} \right)^2 \right\} = r + \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 \quad (5)$$

Steady state solution due to harmonic term, $F_0 \cos \Omega t$, on the r.h.s. of Eq. (3) (from Eq. (3.93)):

$$x(t) = X \cos (\Omega t - \phi) \quad (6)$$

$$\text{where } X = \frac{F_0}{k} \left[\frac{1 - \left(\frac{4 \mu N}{\pi F_0} \right)^2}{\left(1 - \frac{\Omega^2}{\omega_n^2} \right)^2} \right]^{\frac{1}{2}} \quad \text{and} \quad \phi = \tan^{-1} \left\{ \frac{\pm \frac{4 \mu N}{\pi F_0}}{\left[1 - \left(\frac{4 \mu N}{\pi F_0} \right)^2 \right]^{\frac{1}{2}}} \right\} \quad (7)$$

$N = m g = 1 (9.81) = 9.81$ Newtons, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{1}} = 10$ rad/sec, and + sign is to be used in Eq. (7) for $\Omega < \omega_n$ and - sign for $\Omega > \omega_n$. Equations (5) and (6) can be superposed suitably to find the complete steady state solution of Eq. (3).

4.10 Base motion can be represented by Fourier series as (see solution of Problem 4.8 or Example 1.12):

$$y(t) = \frac{Y}{\pi} \left\{ \frac{\pi}{2} - (\sin \omega t + \frac{1}{2} \sin 2 \omega t + \frac{1}{3} \sin 3 \omega t + \dots) \right\} \quad (1)$$

Equation of motion of mass:

$$m \ddot{x} + k ((x - y) \pm \mu N) = 0$$

or

$$m \ddot{x} + k x \pm \mu N = k y$$

$$= \frac{k Y}{\pi} \left[\frac{\pi}{2} - \left\{ \sin \omega t + \frac{1}{2} \sin 2 \omega t + \frac{1}{3} \sin 3 \omega t + \dots \right\} \right] \quad (2)$$

Using the definition of equivalent viscous damping constant:

$$c_{eq} = \frac{4 \mu N}{\pi \Omega X} = \frac{4 \mu m g}{\pi \Omega X} \quad (3)$$

where Ω is the frequency of the harmonic force and X is the amplitude of the mass, the solution of Eq. (2) can be determined using the superposition principle.

Steady state solution due to constant term, F_0 , on the r.h.s. of Eq. (2) (from Example 4.3):

$$x(t) = \frac{F_0}{k} = \frac{k Y}{2 k} = \frac{Y}{2} \quad (4)$$

Steady state solution due to harmonic term, $F_0 \sin \Omega t$, on the r.h.s. of Eq. (2) (from Eq. (3.93)):

$$x(t) = X \sin (\Omega t - \phi)$$

$$\text{where } X = \frac{F_0}{k} \left[\frac{1 - \left(\frac{4 \mu N}{\pi F_0} \right)^2}{\left(1 - \frac{\Omega^2}{\omega_n^2} \right)^2} \right]^{\frac{1}{2}} \quad \text{and } \phi = \tan^{-1} \left\{ \frac{\pm \frac{4 \mu N}{\pi F_0}}{\left[1 - \left(\frac{4 \mu N}{\pi F_0} \right)^2 \right]^{\frac{1}{2}}} \right\} \quad (6)$$

and + sign is to be used in Eq. (6) for $\Omega < \omega_n$ and - sign for $\Omega > \omega_n$. Equations (4) and (5) can be superposed suitably to find the complete steady state solution of Eq. (2).

4.11 From solution of problem 1.70,

$$F(t) = 9.9584 - 20.1587 \cos 10.472 t + 23.5253 \sin 10.472 t$$

$$+ 3.3099 \cos 20.944 t + 12.2646 \sin 20.944 t$$

$$+ 3.7719 \cos 31.416 t - 0.4064 \sin 31.416 t \quad (E_1)$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{15000}{1}} = 122.4745 \text{ rad/sec}; \quad \zeta = 0.1; \quad r = 0.0855$$

The steady state solution is given by Eq. (4.13):

$$\begin{aligned}
 x_p(t) = & \frac{9.9584}{k} - \frac{20.1587}{k} \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \cos(10.472t - \phi_1) \\
 & + \frac{23.5253}{k} \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(10.472t - \phi_1) \\
 & + \frac{3.3099}{k} \frac{1}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}} \cos(20.944t - \phi_2) \\
 & + \frac{12.2646}{k} \frac{1}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}} \sin(20.944t - \phi_2) \\
 & + \frac{3.7719}{k} \frac{1}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}} \cos(31.416t - \phi_3) \\
 & - \frac{0.4064}{k} \frac{1}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}} \sin(31.416t - \phi_3) \quad (E_2)
 \end{aligned}$$

where

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(0.017226) = 0.0172 \text{ rad}$$

$$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1-4r^2}\right) = \tan^{-1}(0.035229) = 0.0352 \text{ rad}$$

$$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1-9r^2}\right) = \tan^{-1}(0.054913) = 0.0549 \text{ rad}$$

Noting that $2\zeta r = 2(0.1)(0.0855) = 0.0171$

and $(1-r^2) = 0.9927$, E_2 can be rewritten as

$$\begin{aligned}
 x_p(t) = & [6.6389 - 13.7821 \cos(10.472t - 0.0172) \\
 & + 15.7965 \sin(10.472t - 0.0172) + 2.2715 \cos(20.944t - 0.0352) \\
 & + 8.4168 \sin(20.944t - 0.0352) + 2.6876 \cos(31.416t - 0.0549) \\
 & - 0.2896 \sin(31.416t - 0.0549)] \times 10^{-4} \text{ m.}
 \end{aligned}$$

4.12 From problem 1.69,

$$\begin{aligned}
 F(t) = & 1137.5 - 414.9436 \cos 523.6t + 150.3139 \sin 523.6t \\
 & + 28.6058 \cos 1047.2t - 146.1706 \sin 1047.2t \\
 & + 35.7278 \cos 1570.8t + 55.1546 \sin 1570.8t \quad (E_1)
 \end{aligned}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8000}{0.5}} = 126.4911 \text{ rad/sec} ; \quad \zeta = 0.06$$

$$r = \frac{\omega}{\omega_n} = \frac{523.6}{126.4911} = 4.1394 ; \quad r^2 = 17.1348 ; \quad 1-r^2 = -16.1348$$

$$2\zeta r = 2(0.06)(4.1394) = 0.4967$$

$$\begin{aligned}
 x_p(t) = & \frac{1137.5}{k} - \frac{414.9436}{k} \frac{\cos(523.6t - \phi_1)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \\
 & + \frac{150.3139}{k} \frac{\sin(523.6t - \phi_1)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} + \frac{28.6058}{k} \frac{\cos(1047.2t - \phi_2)}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}} \\
 & - \frac{146.1706}{k} \frac{\sin(1047.2t - \phi_2)}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}} + \frac{35.7278}{k} \frac{\cos(1570.8t - \phi_3)}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}} \\
 & + \frac{55.1546}{k} \frac{\sin(1570.8t - \phi_3)}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}} \quad (E_2)
 \end{aligned}$$

Since $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(-0.0308) = -0.0308$

$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1-4r^2}\right) = \tan^{-1}(-0.0147) = -0.0147$

$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1-9r^2}\right) = \tan^{-1}(-0.00973) = -0.00973$

the steady-state solution, $E_g(E_2)$, can be expressed as

$$\begin{aligned}
 x_p(t) = & 0.1422 - 0.003211 \cos(523.6t + 0.0308) \\
 & + 0.001163 \sin(523.6t + 0.0308) + 5.2921 \times 10^{-5} \cos(1047.2t + 0.0147) \\
 & - 2.7042 \times 10^{-4} \sin(1047.2t + 0.0147) + 2.9145 \times 10^{-5} \cos(1570.8t + 0.00973) \\
 & + 4.4992 \times 10^{-5} \sin(1570.8t + 0.00973) \quad m
 \end{aligned}$$

4.13 $\omega_n = \sqrt{\frac{5 \times 10^6}{10 \times 10^3}} = 22.3607 \frac{\text{rad}}{\text{sec}}$

$\tau = 0.15 \text{ sec}$

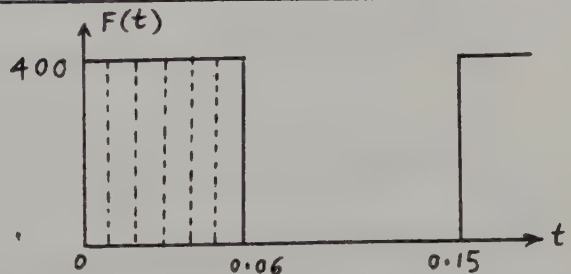
$\omega = \frac{2\pi}{\tau} = 41.888 \text{ rad/sec}$

$r = \frac{\omega}{\omega_n} = 1.8733, \quad r^2 = 3.5093$

$\zeta = 0$

Following data is used in Program # 1 to find Fourier coefficients in the expansion of $F(t)$:

t, sec	0.01	0.02	0.03	0.04	0.05	0.06	0.07	...	0.15
F(t), *N	400	400	400	400	400	400	0	...	0



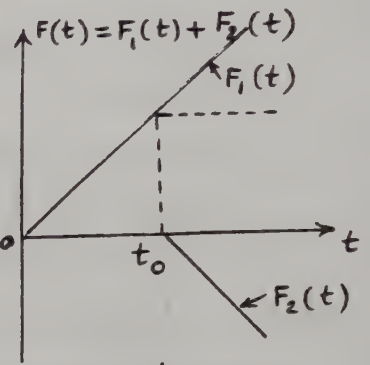
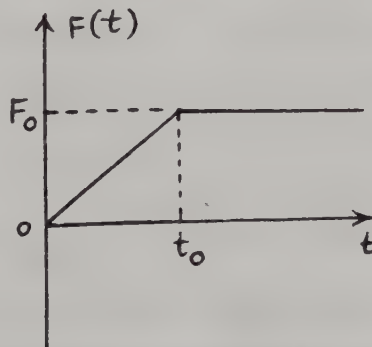
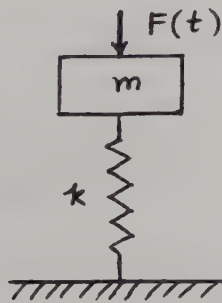
Result is:

$$F(t) = 160.0 + 25.5002 \cos 41.888t + 242.6276 \sin 41.888t \\ - 75.3884 \cos 83.776t + 16.0237 \sin 83.776t \\ + 16.4806 \cos 125.664t + 50.7237 \sin 125.664t \\ - 62.3538 \cos 167.552t + 27.7604 \sin 167.552t \\ + \dots \quad \text{kN}$$

Since $\zeta = 0$ and all $\phi_j = 0$, $j = 1, 2, \dots$, the steady-state response of the water tank, Eq. (4.13), becomes

$$x_p(t) = 0.032 + 2.0325 \times 10^{-3} \cos 41.888t \\ + 19.3383 \times 10^{-3} \sin 41.888t - 1.1566 \times 10^{-3} \cos 83.776t \\ + 0.2458 \times 10^{-3} \sin 83.776t + 0.1078 \times 10^{-3} \cos 125.664t \\ + 0.3317 \times 10^{-3} \sin 125.664t - 0.2334 \times 10^{-3} \cos 167.552t \\ + 0.1007 \times 10^{-3} \sin 167.552t + \dots \quad \text{m}$$

4.14



Forcing function can be considered as the sum of two ramp functions, $F_1(t) = \frac{F_0 t}{t_0}$ and $F_2(t) = -\frac{F_0(t-t_0)}{t_0}$.

Response of the casting (undamped spring-mass system) to F_1 is given by

$$x_1(t) = \frac{F_0}{k} \left(\frac{t}{t_0} - \frac{\sin \omega_n t}{\omega_n t_0} \right) \quad \text{for } t \geq 0 \quad (E_1)$$

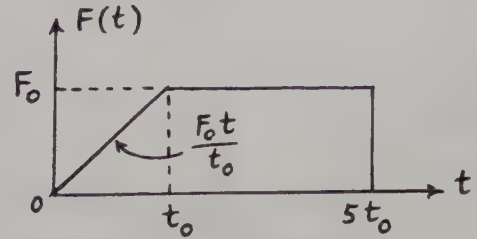
Response due to F_2 can be obtained from Eq. (E₁) by replacing t by $t-t_0$ and F_0 by $-F_0$:

$$x_2(t) = -\frac{F_0}{k} \left\{ \frac{t-t_0}{t_0} - \frac{\sin \omega_n(t-t_0)}{\omega_n t_0} \right\} \quad \text{for } t \geq t_0 \quad (E_2)$$

Total response of the casting is given by

$$x(t) = x_1(t) + x_2(t) = \frac{F_0}{k} \left\{ 1 + \frac{\sin \omega_n(t-t_0) - \sin \omega_n t}{\omega_n t_0} \right\} \quad \text{for } t \geq t_0 \quad (E_3)$$

4.15 $F(t) = \begin{cases} (F_0 t/t_0) & ; 0 \leq t \leq t_0 \\ F_0 & ; t_0 \leq t \leq 5t_0 \\ 0 & ; t > 5t_0 \end{cases} \quad \text{--- (E1)}$



Response of the anvil is given

by [Eq. (4.33) for an undamped system]:

$$x(t) = \frac{1}{m \omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau \quad (E_2)$$

For $0 \leq t \leq t_0$:

$$\begin{aligned} x(t) &= \frac{F_0}{m \omega_n t_0} \int_0^t \tau \sin \omega_n(t-\tau) d\tau \\ &= \frac{F_0}{k t_0} \left(t - \frac{1}{\omega_n} \sin \omega_n t \right) \end{aligned} \quad (E_3)$$

For $t_0 \leq t \leq 5t_0$:

$$\begin{aligned} x(t) &= \frac{1}{m \omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau \\ &= \frac{F_0}{m \omega_n t_0} \int_0^{t_0} \tau \sin \omega_n(t-\tau) d\tau + \frac{F_0}{m \omega_n} \int_{t_0}^t \sin \omega_n(t-\tau) d\tau \\ &= \frac{F_0}{m \omega_n t_0} \left[\sin \omega_n t \left\{ \frac{1}{\omega_n^2} \cos \omega_n t_0 + \frac{t_0}{\omega_n} \sin \omega_n t_0 - \frac{1}{\omega_n^2} \right\} \right. \\ &\quad \left. - \cos \omega_n t \left\{ \frac{1}{\omega_n^2} \sin \omega_n t_0 - \frac{t_0}{\omega_n} \cos \omega_n t_0 \right\} \right] \\ &\quad + \frac{F_0}{m \omega_n} \left\{ \frac{1}{\omega_n} \cos(\omega_n t - \omega_n \tau) \right\}_{t_0}^t \\ &= \frac{F_0}{k \omega_n t_0} \left[\sin \omega_n(t-t_0) - \sin \omega_n t + \omega_n t_0 \right] \end{aligned} \quad (E_4)$$

For $t > 5t_0$:

$$\begin{aligned}
 x(t) &= \frac{1}{m\omega_n} \left[\int_0^{t_0} \frac{F_0 \tau}{t_0} \sin \omega_n(t-\tau) d\tau + F_0 \int_{t_0}^{5t_0} \sin \omega_n(t-\tau) d\tau \right] \\
 &= \frac{F_0}{m\omega_n^2 t_0} \left[\frac{1}{\omega_n} \sin \omega_n(t-t_0) + t_0 \cos \omega_n(t-t_0) - \frac{1}{\omega_n} \sin \omega_n t \right] \\
 &\quad + \frac{F_0}{m\omega_n} \left[\frac{1}{\omega_n} \cos \omega_n(t-\tau) \right]_{t_0}^{5t_0} \\
 &= \frac{F_0}{k\omega_n t_0} \left[\sin \omega_n(t-t_0) - \sin \omega_n t + t_0 \omega_n \cos \omega_n(t-5t_0) \right] \quad (E_5)
 \end{aligned}$$

4.16 From Eq. (4.33), $x(t) = \frac{1}{m\omega_d} \int_0^t F_0 e^{-\alpha\tau} e^{-\gamma\omega_n(t-\tau)} \sin \omega_d(t-\tau) d\tau$

$$\begin{aligned}
 &= \frac{F_0}{m\omega_d} e^{-\gamma\omega_n t} \int_0^t e^{-(\alpha-\gamma\omega_n)\tau} \sin \omega_d(t-\tau) d\tau \\
 x(t) &= \frac{F_0 e^{-\gamma\omega_n t}}{m\omega_d} \int_0^t e^{-(\alpha-\gamma\omega_n)\tau} \{ \sin \omega_d t \cdot \cos \omega_d \tau - \cos \omega_d t \cdot \sin \omega_d \tau \} d\tau \\
 &= \frac{F_0 e^{-\gamma\omega_n t}}{m\omega_d} \sin \omega_d t \left[\frac{e^{-(\alpha-\gamma\omega_n)\tau}}{(\alpha-\gamma\omega_n)^2 + \omega_d^2} \{ -(\alpha-\gamma\omega_n) \cos \omega_d \tau + \omega_d \sin \omega_d \tau \} \right]_0^t \\
 &\quad - \frac{F_0 e^{-\gamma\omega_n t}}{m\omega_d} \cos \omega_d t \left[\frac{e^{-(\alpha-\gamma\omega_n)\tau}}{(\alpha-\gamma\omega_n)^2 + \omega_d^2} \{ -(\alpha-\gamma\omega_n) \sin \omega_d \tau - \omega_d \cos \omega_d \tau \} \right]_0^t \\
 &= \frac{F_0 e^{-\gamma\omega_n t}}{m\omega_d [(\alpha-\gamma\omega_n)^2 + \omega_d^2]} \left\{ \omega_d e^{-(\alpha-\gamma\omega_n)t} + (\alpha-\gamma\omega_n) \sin \omega_d t - \omega_d \cos \omega_d t \right\} \\
 &= \frac{F_0 e^{-\alpha t}}{m [(\alpha-\gamma\omega_n)^2 + \omega_d^2]} + \frac{F_0 e^{-\gamma\omega_n t}}{m\omega_d [(\alpha-\gamma\omega_n)^2 + \omega_d^2]} \sin (\omega_d t - \phi) \\
 &\quad \text{where } \phi = \tan^{-1} \left(\frac{\omega_d}{\alpha - \gamma\omega_n} \right).
 \end{aligned}$$

4.17 Equation of motion:

$$m\ddot{x} + kx = A p(t)$$

$$\text{or } 10\ddot{x} + 1000x = \frac{\pi}{4} (0.1)^2 (50) (1 - e^{-3t}) = 0.3927 - 0.3927 e^{-3t}$$

Solution:

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{0.3927}{k} - \frac{0.3927}{k + m(3^2)} e^{-3t}$$

where $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10 \text{ rad/sec}$. Thus $x(t)$ becomes

$$x(t) = C_1 \cos 10t + C_2 \sin 10t + 39.27 (10^{-5}) - 36.0275 (10^{-5}) e^{-3t} \text{ m}$$

where C_1 and C_2 can be determined from the initial conditions. As $t \rightarrow \infty$, $e^{-3t} \rightarrow 0$ and the steady state response becomes

$$x(t) = C_1 \cos 10t + C_2 \sin 10t + 39.27 (10^{-5}) \text{ m}$$

4.18

For $0 \leq t \leq \frac{\pi}{\omega}$:

Equation of motion: $m\ddot{x} + kx = \frac{F_0}{2} - \frac{F_0}{2} \cos \omega t$

$$x(t) = A \cos \omega_n t + B \sin \omega_n t + \frac{F_0}{2k} - \frac{F_0}{2k(1 - \frac{\omega^2}{\omega_n^2})} \cos \omega t$$

$$x(0) = 0 \quad \therefore A + \frac{F_0}{2k} - \frac{F_0}{2k(1 - \frac{\omega^2}{\omega_n^2})} = 0, \quad A = \frac{F_0 (\frac{\omega}{\omega_n})^2}{2k \{1 - (\frac{\omega}{\omega_n})^2\}}$$

$$\dot{x}(0) = 0 \quad \therefore B = 0$$

$$x(t) = \frac{F_0}{2k \{1 - (\frac{\omega}{\omega_n})^2\}} \left[1 - \cos \omega t - \left(\frac{\omega}{\omega_n}\right)^2 (1 - \cos \omega_n t) \right]$$

$$\text{At } t = \frac{\pi}{\omega}, \quad x\left(\frac{\pi}{\omega}\right) = \frac{F_0}{2k(1 - \frac{\omega^2}{\omega_n^2})} \left[2 - \frac{\omega^2}{\omega_n^2} (1 - \cos \frac{\omega_n \pi}{\omega}) \right]$$

For $t > \frac{\pi}{\omega}$: Eq. (4.33) gives $x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau$

$$x(t) = \frac{1}{m\omega_n} \int_0^{\pi/\omega} F(\tau) \sin \omega_n(t-\tau) d\tau + \frac{1}{m\omega_n} \int_{\pi/\omega}^t F(\tau) \sin \omega_n(t-\tau) d\tau$$

$$= x\left(t = \frac{\pi}{\omega}\right) + \frac{F_0}{m\omega_n} \int_{\pi/\omega}^t \sin \omega_n(t-\tau) d\tau$$

$$= x\left(t = \frac{\pi}{\omega}\right) + \frac{F_0}{m\omega_n^2} \left[\cos \omega_n(t-\tau) \right]_{\tau=\pi/\omega}^t$$

$$= \frac{F_0}{2k(1 - \frac{\omega^2}{\omega_n^2})} \left[2 - \frac{\omega^2}{\omega_n^2} (1 - \cos \frac{\omega_n \pi}{\omega}) \right] + \frac{F_0}{k} \left[1 - \cos \omega_n(t - \frac{\pi}{\omega}) \right]$$

4.19

For an undamped system Eq. (4.33) gives

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau$$

$$F(\tau) = \begin{cases} F_0 & \text{for } 0 \leq \tau \leq t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$$

$$\text{For } 0 \leq t \leq t_0: \quad x(t) = \frac{F_0}{m\omega_n} \int_0^t \sin \omega_n(t-\tau) d\tau = \frac{F_0}{m\omega_n^2} (1 - \cos \omega_n t) \\ = \frac{F_0}{k} (1 - \cos \omega_n t)$$

$$\text{For } t > t_0: \quad x(t) = \frac{F_0}{m\omega_n} \int_0^{t_0} \sin \omega_n(t-\tau) d\tau = \frac{F_0}{m\omega_n} \left[\frac{1}{\omega_n} \cos \omega_n(t-\tau) \right]_{\tau=0}^{t_0} \\ = \frac{F_0}{k} [\cos \omega_n(t-t_0) - \cos \omega_n t]$$

$$(4.20) \quad F(\tau) = \begin{cases} F_0 \left(\frac{\tau}{t_0} \right) & \text{for } 0 \leq \tau \leq t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$$

$$\text{For } 0 \leq t \leq t_0: \quad x(t) = \frac{F_0}{m\omega_n t_0} \int_0^t \tau \sin \omega_n(t-\tau) d\tau$$

$$\text{i.e. } x(t) = \frac{F_0}{m\omega_n t_0} \left[\int_0^t (t-\tau) \sin \omega_n(t-\tau) (-d\tau) - t \int_0^t \sin \omega_n(t-\tau) (-d\tau) \right] \\ = \frac{F_0}{m\omega_n t_0} \left[\frac{1}{\omega_n^2} \sin \omega_n(t-\tau) - \frac{(t-\tau)}{\omega_n} \cos \omega_n(t-\tau) \right]_{\tau=0}^t \\ + \frac{F_0 t}{m\omega_n t_0} \left[\frac{\cos \omega_n(t-\tau)}{\omega_n} \right]_{\tau=0}^t \\ = \frac{F_0}{k t_0} \left(t - \frac{1}{\omega_n} \sin \omega_n t \right)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{m\omega_n t_0} \int_0^{t_0} \tau \sin \omega_n(t-\tau) d\tau$$

$$= \frac{F_0}{m\omega_n t_0} \left[\frac{1}{\omega_n^2} \sin \omega_n(t-\tau) - \frac{(t-\tau)}{\omega_n} \cos \omega_n(t-\tau) \right]_{\tau=0}^{t_0} \\ + \frac{F_0 t}{m\omega_n t_0} \left[\frac{\cos \omega_n(t-\tau)}{\omega_n} \right]_{\tau=0}^{t_0}$$

$$= \frac{F_0}{k t_0} \left[\frac{1}{\omega_n} \sin \omega_n(t-t_0) + t_0 \cos \omega_n(t-t_0) - \frac{1}{\omega_n} \sin \omega_n t \right]$$

$$(4.21) \quad F(t) = \begin{cases} F_0 \left(1 - \cos \frac{\pi t}{2t_0} \right) & ; 0 \leq t \leq t_0 \\ 0 & ; t > t_0 \end{cases}$$

(E₁)

For an undamped system, Eq. (4.33) gives

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau$$

For $0 \leq t \leq t_0$:

$$x(t) = \frac{F_0}{m \omega_n} \int_0^t \left(1 - \cos \frac{\pi \tau}{2t_0}\right) \sin \omega_n(t - \tau) d\tau \quad (E_2)$$

Noting that

$$\int_0^t \sin(\omega_n t - \omega_n \tau) d\tau = \left[\frac{1}{\omega_n} \cos(\omega_n t - \omega_n \tau) \right]_0^t = \frac{1}{\omega_n} (1 - \cos \omega_n t)$$

and

$$\begin{aligned} & \int_0^t \cos \frac{\pi \tau}{2t_0} (\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau) d\tau \\ &= \sin \omega_n t \left[\frac{\sin \left(\frac{\pi}{2t_0} - \omega_n \right) \tau}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{\sin \left(\frac{\pi}{2t_0} + \omega_n \right) \tau}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \right]_0^t \\ & \quad - \cos \omega_n t \left[-\frac{\cos \left(\omega_n - \frac{\pi}{2t_0} \right) \tau}{2 \left(\omega_n - \frac{\pi}{2t_0} \right)} - \frac{\cos \left(\omega_n + \frac{\pi}{2t_0} \right) \tau}{2 \left(\omega_n + \frac{\pi}{2t_0} \right)} \right]_0^t \\ &= \frac{-1}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} \left[\cos \frac{\pi t}{2t_0} - \cos \omega_n t \right] + \frac{1}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \left[\cos \frac{\pi t}{2t_0} - \cos \omega_n t \right], \end{aligned}$$

Eq. (E₂) can be simplified as

$$x(t) = \frac{F_0}{k} (1 - \cos \omega_n t) + \frac{(F_0/m)}{\left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \left[\cos \frac{\pi t}{2t_0} - \cos \omega_n t \right] \quad (E_3)$$

For $t > t_0$:

$$\begin{aligned} x(t) &= \frac{F_0}{m \omega_n} \int_0^{t_0} \left(1 - \cos \frac{\pi \tau}{2t_0}\right) \sin \omega_n(t - \tau) d\tau \quad (E_4) \\ &= \frac{F_0}{m \omega_n} \left[\frac{1}{\omega_n} \cos \omega_n(t - t_0) - \frac{1}{\omega_n} \cos \omega_n t \right] - \frac{F_0}{m \omega_n} \left\{ \sin \omega_n t * \right. \\ & \quad \left[\frac{\sin \left(\frac{\pi}{2t_0} - \omega_n \right) t_0}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{\sin \left(\frac{\pi}{2t_0} + \omega_n \right) t_0}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \right] + \cos \omega_n t * \\ & \quad \left[\frac{\cos \left(\frac{\pi}{2t_0} - \omega_n \right) t_0}{2 \left(\omega_n - \frac{\pi}{2t_0} \right)} + \frac{\cos \left(\frac{\pi}{2t_0} + \omega_n \right) t_0}{2 \left(\omega_n + \frac{\pi}{2t_0} \right)} - \frac{1}{2 \left(\omega_n - \frac{\pi}{2t_0} \right)} - \frac{1}{2 \left(\omega_n + \frac{\pi}{2t_0} \right)} \right] \left. \right\} \end{aligned}$$

$$\text{i.e., } x(t) = \frac{F_0}{k} \left[\cos \omega_n(t - t_0) - \cos \omega_n t \right]$$

$$\begin{aligned}
& - \frac{F_0}{2m\omega_n \left(\frac{\pi}{2t_0} - \omega_n \right)} \sin \omega_n(t-t_0) - \frac{F_0}{2m\omega_n \left(\frac{\pi}{2t_0} + \omega_n \right)} \sin \omega_n(t-t_0) \\
& + \frac{F_0}{2m\omega_n} \cos \omega_n t \cdot \left\{ - \frac{1}{\left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{1}{\left(\frac{\pi}{2t_0} + \omega_n \right)} \right\} \quad (E_5)
\end{aligned}$$

4.22 Base displacement = $y(s) = Y \sin \frac{\pi s}{\delta}$ (E₁)

i.e., $y(t) = Y \sin \frac{\pi t}{t_0}$ (E₂)

steady-state relative displacement can be found from Eq. (4.36) as

$$z(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau) e^{-\zeta \omega_n(t-\tau)} \sin \omega_d(t-\tau) d\tau \quad (E_3)$$

where $\ddot{y}(\tau) = -Y \left(\frac{\pi}{t_0} \right)^2 \sin \frac{\pi \tau}{t_0}$ (E₄)

$$z(t) = \frac{Y}{\omega_d} \left(\frac{\pi}{t_0} \right)^2 \int_0^t e^{-\zeta \omega_n t} e^{\zeta \omega_n \tau} \sin \frac{\pi \tau}{t_0} \sin \omega_d(t-\tau) d\tau \quad (E_5)$$

But $\sin \frac{\pi \tau}{t_0} \sin \omega_d(t-\tau) = \frac{1}{2} \cos \left(\frac{\pi \tau}{t_0} - \omega_d t + \omega_d \tau \right) - \frac{1}{2} \cos \left(\frac{\pi \tau}{t_0} + \omega_d t - \omega_d \tau \right)$

$$\begin{aligned}
& = \frac{1}{2} \left[\cos \left(\frac{\pi}{t_0} + \omega_d \right) \tau \cdot \cos \omega_d t + \sin \left(\frac{\pi}{t_0} + \omega_d \right) \tau \cdot \sin \omega_d t \right] \\
& - \frac{1}{2} \left[\cos \left(\frac{\pi}{t_0} - \omega_d \right) \tau \cdot \cos \omega_d t - \sin \left(\frac{\pi}{t_0} - \omega_d \right) \tau \cdot \sin \omega_d t \right] \quad (E_6)
\end{aligned}$$

Eqs. (E₅) and (E₆) give

$$\begin{aligned}
z(t) &= \frac{1}{2} \frac{Y}{\omega_d} \left(\frac{\pi}{t_0} \right)^2 e^{-\zeta \omega_n t} \left[\cos \omega_d t \cdot \int_0^t e^{\zeta \omega_n \tau} \cos \left(\frac{\pi}{t_0} + \omega_d \right) \tau \cdot d\tau \right. \\
&+ \sin \omega_d t \cdot \int_0^t e^{\zeta \omega_n \tau} \cdot \sin \left(\frac{\pi}{t_0} + \omega_d \right) \tau \cdot d\tau \\
&- \cos \omega_d t \cdot \int_0^t e^{\zeta \omega_n \tau} \cdot \cos \left(\frac{\pi}{t_0} - \omega_d \right) \tau \cdot d\tau \\
&\left. + \sin \omega_d t \cdot \int_0^t e^{\zeta \omega_n \tau} \cdot \sin \left(\frac{\pi}{t_0} - \omega_d \right) \tau \cdot d\tau \right] \quad (E_7)
\end{aligned}$$

Eq. (E₇) can be simplified

$$\begin{aligned}
z(t) &= \frac{Y}{2\omega_d} \left(\frac{\pi}{t_0} \right)^2 \frac{1}{(\zeta \omega_n)^2 + \left(\frac{\pi}{t_0} + \omega_d \right)^2} \left\{ \zeta \omega_n \cos \frac{\pi t}{t_0} \right. \\
&+ \left(\frac{\pi}{t_0} + \omega_d \right) \sin \frac{\pi t}{t_0} - \zeta \omega_n e^{-\zeta \omega_n t} \cos \omega_d t \\
&+ \left(\frac{\pi}{t_0} + \omega_d \right) e^{-\zeta \omega_n t} \sin \omega_d t \left. \right\} \\
&+ \frac{Y}{2\omega_d} \left(\frac{\pi}{t_0} \right)^2 \frac{1}{(\zeta \omega_n)^2 + \left(\frac{\pi}{t_0} - \omega_d \right)^2} \left\{ -\zeta \omega_n \cos \frac{\pi t}{t_0} \right.
\end{aligned}$$

$$- \left(\frac{\pi}{t_0} - \omega_d \right) \sin \frac{\pi t}{t_0} + \gamma \omega_n e^{-\gamma \omega_n t} \cos \omega_d t + \left(\frac{\pi}{t_0} - \omega_d \right) e^{-\gamma \omega_n t} \sin \omega_d t \} \quad (E_8)$$

4.23

Base displacement:

$$y(t) = \begin{cases} \frac{Y v t}{\delta} ; 0 \leq t \leq t_0 = \frac{\delta}{v} \\ 0 ; t > t_0 = \frac{\delta}{v} \end{cases} \quad (1)$$

Equation of motion of vehicle:

$$m \ddot{x} + k (x - y) = 0 \quad (2)$$

Using Eq. (1), Eq. (2) can be expressed as

$$m \ddot{x} + k x = k y = \begin{cases} \frac{k v Y t}{\delta} ; 0 \leq t \leq t_0 \\ 0 ; t > t_0 \end{cases} \quad (3)$$

Steady state solution of Eq. (3) from Example 4.6:

$$x(t) = \begin{cases} \frac{v Y}{\delta \omega_n k} \left\{ \omega_n t - \sin \omega_n t \right\} ; 0 \leq t \leq t_0 \\ 0 ; t > t_0 \end{cases} \quad (4)$$

Note that the homogeneous solution,

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t \quad (5)$$

is to be added to Eq. (4) to obtain the complete solution. The constants C_1 and C_2 are to be evaluated from the initial conditions (at $t = 0$). In fact, the resulting complete solution is valid for all values of t , including values of $t > t_0$.

4.24

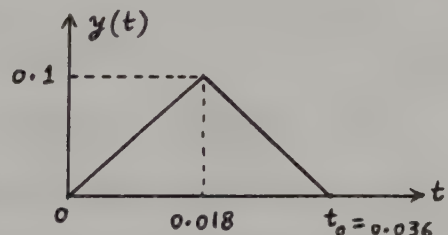
Speed of automobile = 50 km/hr

Excitation frequency = $\left(\frac{50 \times 1000}{3600} \right) \frac{1}{0.5} = 27.7778 \text{ Hz}$

Natural frequency = $f_n = 1.0 \text{ Hz} \Rightarrow \omega_n = 2\pi \text{ rad/sec}$

$$t_0 = \frac{0.5 \times 3600}{50 \times 1000} = 0.036 \text{ sec}$$

$$y(t) = \begin{cases} \frac{0.2 t}{t_0} & ; 0 \leq t \leq t_0/2 \\ -\frac{0.2 t}{t_0} + 0.2 & ; \frac{t_0}{2} \leq t \leq t_0 \\ 0 & ; t > t_0 \end{cases} \quad (E_1)$$



Equation of motion (for undamped case):

$$m\ddot{x} + k(x-y) = 0 \quad \text{or} \quad m\ddot{x} + kx = ky = F(t) \quad (E_2)$$

$$\text{Where } F(t) = ky(t) \quad (E_3)$$

Solution of Eq. (E₂) is: $x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau \quad (E_4)$

For $0 \leq t \leq \frac{t_0}{2}$:

$$x(t) = \frac{k}{m\omega_n} \int_0^t \left(\frac{0.2}{t_0}\right) \tau \sin \omega_n(t-\tau) d\tau \quad (E_5)$$

Since $\int_0^t \tau \sin \omega_n(t-\tau) d\tau = \left(\frac{t}{\omega_n} - \frac{1}{\omega_n^2} \sin \omega_n t\right)$,

Eq. (E₅) becomes

$$x(t) = 5.5556 (t - 0.1592 \sin 6.2832 t) \text{ m} ; 0 \leq t \leq 0.018 \text{ sec} \quad (E_6)$$

For $\frac{t_0}{2} \leq t \leq t_0$:

$$x(t) = \frac{k}{m\omega_n} \left\{ \int_0^{t_0/2} \frac{0.2\tau}{t_0} \sin \omega_n(t-\tau) d\tau + \int_{t_0/2}^t \left(-\frac{0.2\tau}{t_0} + 0.2\right) \sin \omega_n(t-\tau) d\tau \right\}$$

But

$$\frac{0.2k}{m\omega_n t_0} \int_0^{t_0/2} \tau \sin \omega_n(t-\tau) d\tau = \frac{0.2k}{m\omega_n t_0} \left\{ \sin \omega_n t \left[\frac{1}{\omega_n^2} \cos \omega_n \tau + \frac{\tau}{\omega_n} \sin \omega_n \tau \right]_0^{t_0/2} - \cos \omega_n t \left[\frac{1}{\omega_n^2} \sin \omega_n \tau - \frac{\tau}{\omega_n} \cos \omega_n \tau \right]_0^{t_0/2} \right\}$$

$$= 5.5556 \left[0.1592 \sin 6.2832 (t - 0.018) \right]$$

$$+ 0.0180 \cos 6.2832 (t - 0.018) - 0.1592 \sin 6.2832 t \text{ m} \quad (E_7)$$

Since $t_0 = 0.036$.

$$- \frac{0.2k}{m\omega_n t_0} \int_{t_0/2}^t \tau \sin \omega_n(t-\tau) d\tau = - \frac{0.2k}{m\omega_n t_0} \left\{ \sin \omega_n t \left[\frac{1}{\omega_n^2} \cos \omega_n \tau + \frac{\tau}{\omega_n} \sin \omega_n \tau \right]_{t_0/2}^t - \cos \omega_n t \left[\frac{1}{\omega_n^2} \sin \omega_n \tau - \frac{\tau}{\omega_n} \cos \omega_n \tau \right]_{t_0/2}^t \right\}$$

$$= -5.5556 \left[t - 0.1592 \sin 6.2832 (t - 0.018) \right]$$

$$- 0.0180 \cos 6.2832 (t - 0.018) \text{ m} \quad (E_8)$$

$$\begin{aligned}
\frac{0.2 k}{m \omega_n} \int_{t_0/2}^t \sin \omega_n(t-\tau) d\tau &= \frac{0.2 k}{m \omega_n} \left[\frac{1}{\omega_n} \cos \omega_n(t-\tau) \right]_{t_0/2}^t \\
&= \frac{0.2 k}{m \omega_n^2} \left[1 - \cos \omega_n \left(t - \frac{t_0}{2} \right) \right] \\
&= 0.2 \left[1 - \cos 6.2832 (t - 0.018) \right] \text{ m} \quad (E_9)
\end{aligned}$$

Hence the solution can be expressed as

$$\begin{aligned}
x(t) &= \left[1.7689 \sin 6.2832 (t - 0.018) - 0.8845 \sin 6.2832 t \right. \\
&\quad \left. - 5.5556 t + 0.2 \right] \text{ m} ; \quad 0.018 \leq t \leq 0.036 \text{ sec} \quad (E_{10})
\end{aligned}$$

For $t > t_0$:

$$\begin{aligned}
x(t) &= \frac{0.2 k}{m \omega_n t_0} \int_0^{t_0/2} \tau \sin \omega_n(t-\tau) d\tau - \frac{0.2 k}{m \omega_n t_0} \int_{t_0/2}^{t_0} \tau \sin \omega_n(t-\tau) d\tau \\
&\quad + \frac{0.2 k}{m \omega_n} \int_{t_0/2}^{t_0} \sin \omega_n(t-\tau) d\tau \quad (E_{11})
\end{aligned}$$

The first term on the right side of (E₁₁) is given by (E₇).

Second term on the right side of (E₁₁) is

$$\begin{aligned}
& - \frac{0.2 k}{m \omega_n t_0} \int_{t_0/2}^{t_0} \tau \sin \omega_n(t-\tau) d\tau = - \frac{0.2 k}{m \omega_n t_0} \left\{ \sin \omega_n t \cdot \left[\frac{1}{\omega_n^2} \cos \omega_n \tau \right. \right. \\
& \quad \left. \left. + \frac{\tau}{\omega_n} \sin \omega_n \tau \right]_{t_0/2}^{t_0} - \cos \omega_n t \cdot \left[\frac{1}{\omega_n^2} \sin \omega_n \tau - \frac{\tau}{\omega_n} \cos \omega_n \tau \right]_{t_0/2}^{t_0} \right\} \\
& = -5.5556 \left[0.1592 \sin 6.2832 (t - 0.036) + 0.0360 \cos 6.2832 (t - 0.036) \right. \\
& \quad \left. - 0.1592 \sin 6.2832 (t - 0.018) - 0.0180 \cos 6.2832 (t - 0.018) \right] \text{ m} \quad (E_{12})
\end{aligned}$$

The third term on the right side of (E₁₁) is:

$$\begin{aligned}
& \frac{0.2 k}{m \omega_n} \int_{t_0/2}^{t_0} \sin \omega_n(t-\tau) d\tau = \frac{0.2 k}{m \omega_n} \left[\frac{1}{\omega_n} \cos \omega_n(t-\tau) \right]_{t_0/2}^{t_0} \\
& = \frac{0.2 k}{m \omega_n^2} \left[\cos \omega_n(t-t_0) - \cos \omega_n \left(t - \frac{t_0}{2} \right) \right] \\
& = 0.2 \left[\cos 6.2832 (t - 0.036) - \cos 6.2832 (t - 0.018) \right] \text{ m} \quad (E_{13})
\end{aligned}$$

$\therefore x(t)$ is given by the sum of E_7 , (E_{12}) and (E_{13}) , which can be simplified as

$$x(t) = 1.7689 \sin 6.2832(t - 0.018) - 0.8845 \sin 6.2832 t - 0.8845 \sin 6.2832(t - 0.036) \text{ m ; } t > 0.036 \text{ sec } (E_{14})$$

4.25

When the container strikes the floor, the velocity of the mass is given by

$$mgh = \frac{1}{2} m v^2 \quad \text{or} \quad v = \sqrt{2gh} \quad (E_1)$$

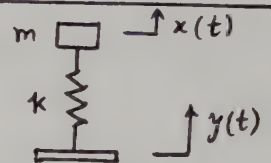
The displacement of the camcorder subjected to an initial velocity $\dot{x}_0 = v$ is given by Eq. (2.72), with $x_0 = 0$ and $\zeta < 1$,

$$x(t) = e^{-\zeta \omega_n t} \cdot \frac{\dot{x}_0}{\omega_n \sqrt{1-\zeta^2}} \cdot \sin \sqrt{1-\zeta^2} \omega_n t \quad (E_2)$$

4.26

System can be modeled as a spring-mass system subjected to base motion:

$$y(t) = \begin{cases} (\gamma t^2/t_0^2) & ; 0 \leq t \leq t_0 \\ 0 & ; t > t_0 \end{cases} \quad \text{--- } (E_1)$$



Relative displacement of mass, $z = x - y$, is given by Eq. (4.36):

$$z(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau) e^{-\zeta \omega_n(t-\tau)} \sin \omega_d(t-\tau) d\tau \quad (E_2)$$

$$\text{where } \left. \begin{aligned} y(\tau) &= (\gamma \tau^2/t_0^2) \\ \ddot{y}(\tau) &= (2\gamma/t_0^2) \end{aligned} \right\} ; 0 \leq \tau \leq t_0 \quad (E_3)$$

$$\text{and } y(\tau) = 0 ; \tau > t_0$$

For $0 \leq t \leq t_0$:

Since the system is undamped, $\omega_d = \omega_n$ and $\zeta = 0$, and (E_2) reduces to

$$z(t) = -\frac{2\gamma}{\omega_n t_0^2} \int_0^t \sin \omega_n(t-\tau) d\tau \quad (E_4)$$

Here

$$\begin{aligned} \int_0^t \sin \omega_n(t-\tau) d\tau &= \int_0^t (\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau) d\tau \\ &= \sin \omega_n t \int_0^t \cos \omega_n \tau d\tau - \cos \omega_n t \int_0^t \sin \omega_n \tau d\tau \end{aligned}$$

$$\begin{aligned}
&= \sin \omega_n t \left(\frac{1}{\omega_n} \sin \omega_n \tau \right)_0^t - \cos \omega_n t \left(-\frac{1}{\omega_n} \cos \omega_n \tau \right)_0^t \\
&= \left(\frac{1 - \cos \omega_n t}{\omega_n} \right) \quad (E_5)
\end{aligned}$$

Thus Eq. (E₄) gives

$$x(t) = \frac{\gamma t^2}{t_0^2} - \frac{2\gamma}{t_0^2 \omega_n^2} (1 - \cos \omega_n t) \quad ; \quad 0 \leq t \leq t_0 \quad (E_6)$$

For $t > t_0$:

Eq. (E₂) gives

$$z(t) = -\frac{1}{\omega_n} \int_0^{t_0} \frac{2\gamma}{t_0^2} \sin \omega_n(t-\tau) d\tau \quad (E_7)$$

But

$$\int_0^{t_0} \sin \omega_n(t-\tau) d\tau = \frac{1}{\omega_n} \{ \cos \omega_n(t-t_0) - \cos \omega_n t \}$$

$$\therefore z(t) = x(t) - y(t) = -\frac{2\gamma}{\omega_n^2 t_0^2} \{ \cos \omega_n(t-t_0) - \cos \omega_n t \} \quad ; \quad t > t_0 \quad (E_8)$$

4.27
$$\begin{aligned}
I_1 &= \int_0^t (t-\tau) e^{-\gamma \omega_n(t-\tau)} \sin \omega_d(t-\tau) (-d\tau) \\
&= \left[\frac{(t-\tau) e^{-\gamma \omega_n(t-\tau)}}{(\gamma \omega_n)^2 + \omega_d^2} \left\{ -\gamma \omega_n \sin \omega_d(t-\tau) - \omega_d \cos \omega_d(t-\tau) \right\} \right. \\
&\quad \left. - \frac{e^{-\gamma \omega_n(t-\tau)}}{(\gamma \omega_n)^2 + \omega_d^2} \left\{ (\gamma^2 \omega_n^2 - \omega_d^2) \sin \omega_d(t-\tau) + 2\gamma \omega_n \omega_d \cos \omega_d(t-\tau) \right\} \right]_{\tau=0}^t \\
&= -\frac{1}{\omega_n^4} (2\gamma \omega_n \omega_d) + \frac{t e^{-\gamma \omega_n t}}{\omega_n^2} (\gamma \omega_n \sin \omega_d t + \omega_d \cos \omega_d t) \\
&\quad + \frac{e^{-\gamma \omega_n t}}{\omega_n^4} \{ \omega_n^2 (2\gamma^2 - 1) \sin \omega_d t + 2\gamma \omega_n \omega_d \cos \omega_d t \} \\
I_2 &= \int_0^t e^{-\gamma \omega_n(t-\tau)} \sin \omega_d(t-\tau) (-d\tau) \\
&= \left[\frac{e^{-\gamma \omega_n(t-\tau)}}{\gamma^2 \omega_n^2 + \omega_d^2} \left\{ -\gamma \omega_n \sin \omega_d(t-\tau) - \omega_d \cos \omega_d(t-\tau) \right\} \right]_{\tau=0}^t \\
&= -\frac{\omega_d}{\omega_n^2} + \frac{e^{-\gamma \omega_n t}}{\omega_n^2} (\gamma \omega_n \sin \omega_d t + \omega_d \cos \omega_d t)
\end{aligned}$$

$$\begin{aligned}
 x(t) &= \frac{\delta F}{m \omega_d} \cdot I_1 - \frac{\delta F \cdot t}{m \omega_d} \cdot I_2 \\
 &= \frac{\delta F}{k} \left[t - \frac{2\gamma}{\omega_n} + e^{-\gamma \omega_n t} \left\{ \frac{2\gamma}{\omega_n} \cos \omega_d t - \left(\frac{\omega_d^2 - \omega_n^2}{\omega_n^2 \omega_d} \right) \sin \omega_d t \right\} \right]
 \end{aligned}$$

4.28

(1) $m \ddot{x} + c \dot{x} + kx = F(t)$

where $m(t) = M - m_0 t = (2000 - 10t) \quad \text{kg}$

and $F(t) = m_0 v = 10(2000) = 20000 \quad \text{N}$

(2) With $m = M - \frac{1}{2} m_0 t_0 = 2000 - \frac{1}{2}(10)(100) = 1500 \quad \text{kg}$,
equation of motion becomes

$$1500 \ddot{x} + 0.1 \times 10^6 \dot{x} + 7.5 \times 10^6 x = 20000 = \text{constant}$$

Maximum steady state displacement is

$$x_p(t) = \frac{F}{k} = \frac{20000}{7.5 \times 10^6} = 0.002667 \text{ m}$$

4.29

From Eq. (4.32), the response to unit step function can be obtained by setting $F(\tau) = 1$ as

$$h(t) = \int_0^t g(t-\tau) d\tau \quad \text{---- (E}_1\text{)}$$

By differentiating this equation with respect to t , we obtain

$$\frac{dh}{dt}(t) = g(t)$$

4.30

Equation (4.32) gives $x(t) = \int_0^t F(\tau) \cdot g(t-\tau) \cdot d\tau$

But $g(t-\tau) = \frac{dh}{d\tau}(t-\tau)$ from problem 4.29.

$$x(t) = \int_0^t F(\tau) \frac{dh}{d\tau}(t-\tau) d\tau$$

Integration by parts gives

$$x(t) = -F(t) \cdot h(t-\tau) \Big|_{\tau=0}^t + \int_0^t \frac{dF}{d\tau} \cdot h(t-\tau) d\tau$$

$$= -F(t) h(0) + F(0) h(t) + \int_0^t \frac{dF}{d\tau} h(t-\tau) d\tau$$

But $h(0) = 0$ from Eq. (E₁) of problem 4.29.

$$\therefore x(t) = F(0) h(t) + \int_0^t \frac{dF}{d\tau}(\tau) \cdot h(t-\tau) \cdot d\tau$$

4.31

Equation of motion for rotation about O:

$$J_0 \ddot{\theta} + M \ddot{x}(\ell) + k_1 a^2 \theta + k_2 b^2 \theta = F_0 \ell e^{-t} \quad (1)$$

where $\ddot{x} = \ell \ddot{\theta}$ and

$$J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{2} \right)^2 = \frac{1}{3} m \ell^2$$

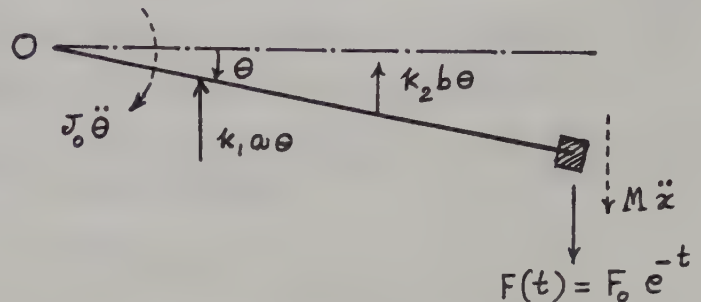
Eq. (1) can be rewritten as:

$$\left(\frac{1}{3} m \ell^2 + M \ell^2 \right) \ddot{\theta} + (k_1 a^2 + k_2 b^2) \theta = F_0 \ell e^{-t} \quad (2)$$

For given data, Eq. (2) takes the form:

$$53.3333 \ddot{\theta} + 1562.5 \theta = 500 e^{-t} \quad (3)$$

Noting that the system is undamped with $\omega_n = \sqrt{\frac{1562.5}{53.3333}} = 5.4127$ rad/sec and



the forcing term as $500 e^{-t}$, the convolution integral, Eq. (4.33), can be used to find the steady state response as:

$$\begin{aligned} \theta(t) &= \frac{1}{(53.3333)(5.4127)} \int_0^t 500 e^{-\tau} \sin 5.4127 (t - \tau) d\tau \\ &= 1.7320 \int_0^t e^{-t} e^{(t-\tau)} \sin 5.4127 (t - \tau) d\tau \\ &= -1.7320 e^{-t} \int_0^t e^{(t-\tau)} \sin 5.4127 (t - \tau) (-d\tau) \end{aligned} \quad (4)$$

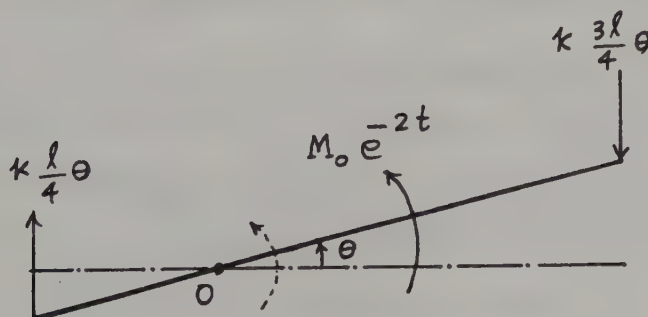
Using the formula:

$$\int e^{ax} \sin bx dx = \frac{1}{a^2 + b^2} e^{ax} [a \sin bx - b \cos bx] \quad (5)$$

Eq. (4) can be expressed as

$$\begin{aligned} \theta(t) &= -1.7320 e^{-t} \left[\frac{e^{(t-\tau)}}{1^2 + 5.4127^2} \left\{ \sin 5.4127 (t - \tau) - 5.4127 \cos 5.4127 (t - \tau) \right\} \right]_{\tau=0}^{\tau=t} \\ &= 0.3094 e^{-t} + 0.05717 \sin 5.4127 t - 0.3094 \cos 5.4127 t \text{ radian} \end{aligned}$$

4.32



$$J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4} \right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10) (1^2) = 1.4583 \text{ kg-m}^2$$

Equation of motion for rotation about O:

$$J_0 \ddot{\theta} + k \frac{\ell^2}{16} \theta + k \frac{9}{16} \ell^2 \theta = M_0 e^{-2t}$$

$$\text{or } J_0 \ddot{\theta} + \frac{5}{8} k \ell^2 \theta = M_0 e^{-2t}$$

$$\text{or } 1.4583 \ddot{\theta} + 3125.0 \theta = 100 e^{-2t} \quad (1)$$

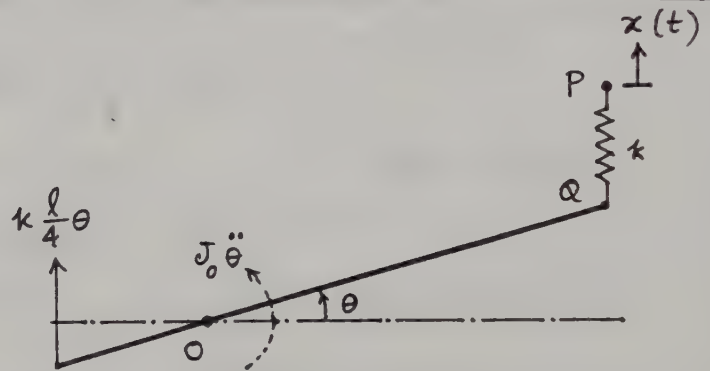
Noting that the system is undamped with $\omega_n = \sqrt{\frac{3125.0}{1.4583}} = 46.2915 \text{ rad/sec}$, the convolution integral, Eq. (4.33), can be used to find the steady state response as:

$$\begin{aligned} \theta(t) &= \frac{1}{(1.4583)(46.2915)} \int_0^t (100 e^{-2\tau}) \sin 46.2915 (t - \tau) d\tau \\ &= -1.4813 e^{-2t} \int_0^t e^{2(t-\tau)} \sin 46.2915 (t - \tau) (-d\tau) \end{aligned} \quad (2)$$

Using Eq. (5) in the solution of Problem 4.31, Eq. (2) can be expressed as:

$$\begin{aligned} \theta(t) &= -1.4813 e^{-2t} \left[\frac{e^{2(t-\tau)}}{2^2 + 46.2915^2} \left\{ 2 \sin 46.2915 (t - \tau) - 46.2915 \cos 46.2915 (t - \tau) \right\} \right]_{\tau=0}^{\tau=t} \\ &= 0.03194 e^{-2t} + 13.7994 (10^{-4}) \sin 46.2915 t - 0.03194 \cos 46.2915 t \text{ radian} \end{aligned} \quad (4)$$

4.33



Net compression of spring PQ = $\frac{3 \ell \theta}{4} - x(t)$. Equation of motion for rotation about O:

$$\begin{aligned} J_0 \ddot{\theta} &= -k \frac{\ell \theta}{4} \left(\frac{\ell}{4} \right) - k \left(\frac{3 \ell \theta}{4} - x(t) \right) \left(\frac{3 \ell}{4} \right) \\ \text{or } J_0 \ddot{\theta} + \frac{5}{8} k \ell^2 \theta &= \frac{3 k \ell}{4} x(t) = \frac{3 k \ell}{4} x_0 e^{-t} \end{aligned} \quad (1)$$

For given data, $J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4} \right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10) (1^2) = 1.4583 \text{ kg-m}^2$,
 $\frac{5}{8} k \ell^2 = \frac{5}{8} (5000) (1^2) = 3125 \text{ N/m}$, and Eq. (1) becomes:

$$1.4583 \ddot{\theta} + 3125.0 \theta = 37.5 e^{-t} \quad (2)$$

Noting that the system is undamped with

$$\omega_n = \sqrt{\frac{3125.0}{1.4583}} = 46.2915 \text{ rad/sec}$$

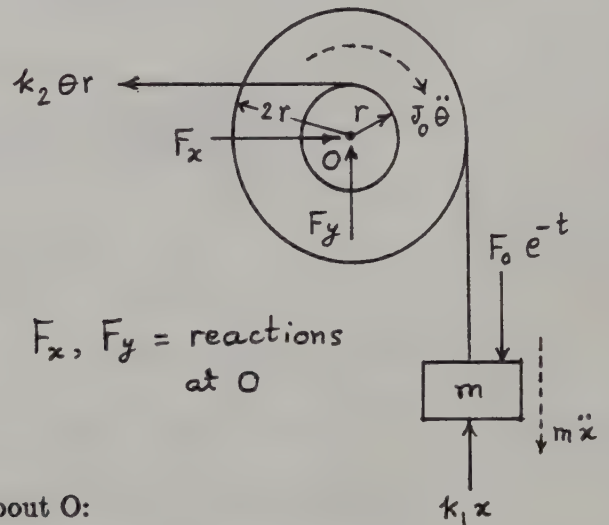
the convolution integral, Eq. (4.33), can be used to find the steady state response as:

$$\theta(t) = \frac{1}{(1.4583)(46.2915)} \int_0^t (37.5 e^{-\tau}) \sin 46.2915 (t - \tau) d\tau \quad (3)$$

Using Eq. (5) in the solution of Problem 4.31, Eq. (3) can be expressed as

$$\begin{aligned} \theta(t) &= -0.5555 e^{-t} \left[\frac{e^{(t-\tau)}}{1^2 + 46.2915^2} \left\{ \sin 46.2915 (t - \tau) - 46.2915 \cos (t - \tau) \right\} \right]_{\tau=0}^t \\ &= 0.01199 e^{-t} + 2.591 (10^{-4}) \sin 46.2915 t - 0.01199 \cos 46.2915 t \text{ radian} \quad (4) \end{aligned}$$

4.34



Equation of motion for rotation of pulley about O:

$$J_0 \ddot{\theta} + m \ddot{x} (2r) + k_1 x (2r) + k_2 (\theta r) r = 2r F_0 e^{-t} \quad (1)$$

where $\theta = \frac{x}{2r}$. Eq. (1) can be rewritten in terms of x only as:

$$\left(\frac{J_0}{2r} + 2mr \right) \ddot{x} + \left(2k_1 r + \frac{1}{2} k_2 r \right) x = 2r F_0 e^{-t} \quad (2)$$

For given data, Eq. (2) becomes

$$11.0 \ddot{x} + 112.5 x = 5 e^{-t} \quad (3)$$

Noting that the system is undamped with $\omega_n = \sqrt{\frac{112.5}{11.0}} = 3.1980 \text{ rad/sec}$, the convolution integral, Eq. (4.33), can be used to find the steady state response as:

$$x(t) = -0.1421 e^{-t} \int_0^t e^{(t-\tau)} \sin 3.1980 (t - \tau) (-d\tau) \quad (4)$$

Using Eq. (5) in the solution of Problem 4.31, Eq. (4) can be expressed as

$$x(t) = -\frac{0.1421 e^{-t}}{1^2 + 3.1980^2} \left[e^{(t-\tau)} \left\{ \sin 3.1980 (t-\tau) - 3.1980 \cos 3.1980 (t-\tau) \right\} \right]_{\tau=0}^t$$

$$= 0.04048 e^{-t} + 0.01266 \sin 3.1980 t - 0.04048 \cos 3.1980 t \text{ m} \quad (5)$$

4.35 $F(t) = \begin{cases} F_0 & ; 0 \leq t \leq t_0 \\ 0 & ; t > t_0 \end{cases} \quad (E_1)$

Eq. (4.33) gives, for an undamped system,

$$x(t) = \frac{1}{m \omega_n} \int_0^t F(\tau) \sin \omega_n (t-\tau) d\tau \quad (E_2)$$

For $0 \leq t \leq t_0$:

$$x(t) = \frac{F_0}{m \omega_n} \int_0^t \sin \omega_n (t-\tau) d\tau = \frac{F_0}{k} (1 - \cos \omega_n t) \quad (E_3)$$

using Eq. (E₅) in the solution of problem 4.26.

For $t > t_0$:

$$x(t) = \frac{F_0}{m \omega_n} \int_0^{t_0} \sin \omega_n (t-\tau) d\tau$$

Using the relation

$$\int_0^{t_0} \sin \omega_n (t-\tau) d\tau = \frac{1}{\omega_n} \{ \cos \omega_n (t-t_0) - \cos \omega_n t \}$$

the solution can be expressed as

$$x(t) = \frac{F_0}{k} [\cos \omega_n (t-t_0) - \cos \omega_n t] \quad (E_4)$$

Response spectrum:

For $0 \leq t \leq t_0$, $x(t) = \frac{F_0}{k} (1 - \cos \omega_n t) \quad (E_5)$

$$\frac{dx}{dt} = \frac{F_0 \omega_n}{k} \sin \omega_n t = 0 \Rightarrow \omega_n t_{\max} = \pi$$

$$\therefore x_{\max} = x(t=t_{\max}) = \frac{F_0}{k} (1 - \cos \omega_n t_{\max}) = \frac{2 F_0}{k} \quad (E_6)$$

For $t > t_0$,

$$x(t) = \frac{F_0}{k} \{ \cos \omega_n (t-t_0) - \cos \omega_n t \} \quad (E_7)$$

$$\frac{dx}{dt} = -\frac{F_0 \omega_n}{k} \{ \sin \omega_n (t-t_0) - \sin \omega_n t \} = 0$$

$$\Rightarrow \sin \omega_n (t_{\max} - t_0) = \sin \omega_n t_{\max}$$

$$\text{i.e., } \tan \omega_n t_{\max} = \left(\frac{\sin \omega_n t_0}{\cos \omega_n t_0 - 1} \right) \quad (E_8)$$

$$\text{i.e., } \sin \omega_n t_{\max} = \frac{\sin \omega_n t_0}{\sqrt{2(1 - \cos \omega_n t_0)}} \quad (E_9)$$

$$\text{and } \cos \omega_n t_{\max} = \frac{\cos \omega_n t_0 - 1}{\sqrt{2(1 - \cos \omega_n t_0)}} \quad (E_{10})$$

$$\therefore x_{\max} = x(t = t_{\max}) = \frac{F_0}{k} \left\{ \cos \omega_n t_{\max} \cdot \cos \omega_n t_0 + \sin \omega_n t_{\max} \cdot \sin \omega_n t_0 - \cos \omega_n t_{\max} \right\}$$

$$= \frac{F_0}{k} \left\{ \frac{(\cos \omega_n t_0 - 1)^2}{\sqrt{2(1 - \cos \omega_n t_0)}} + \frac{\sin^2 \omega_n t_0}{\sqrt{2(1 - \cos \omega_n t_0)}} \right\}$$

$$= \frac{2 F_0}{k} \sin \frac{\omega_n t_0}{2} \quad (E_{11})$$

Plotting: $\omega_n = \frac{2\pi}{\tau_n}$

$$\text{For } 0 \leq t \leq t_0, \quad \omega_n t_{\max} = \pi \quad \text{or} \quad \frac{t_{\max}}{\tau_n} = \frac{1}{2}$$

$$\text{When } t \leq t_0, \quad t_{\max} = \frac{\tau_n}{2} \leq t_0$$

$$\text{i.e., } \frac{t_0}{\tau_n} \geq \frac{1}{2}$$

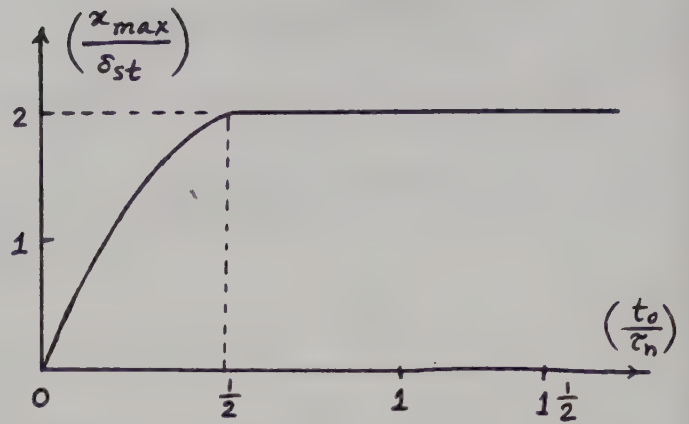
$$\therefore \frac{x_{\max}}{\delta_{st}} = 2 \quad \text{for} \quad \frac{t_0}{\tau_n} \geq \frac{1}{2} \quad (E_{12})$$

$$\text{For } t > t_0, \quad \frac{t_0}{\tau_n} < \frac{1}{2}$$

$$\frac{x_{\max}}{\delta_{st}} = 2 \sin \frac{\omega_n t_0}{2} = 2 \sin \frac{2\pi t_0}{2\tau_n}$$

$$\therefore \frac{x_{\max}}{\delta_{st}} = 2 \sin \frac{\pi t_0}{\tau_n} \quad \text{for} \quad \frac{t_0}{\tau_n} < \frac{1}{2} \quad (E_{13})$$

Eg. (E_{12}) and (E_{13}) are plotted in the figure.



Response spectrum for a rectangular pulse-type load

4.36

The response is found in problem 4.21.

For $0 \leq t \leq t_0$:

$$x(t) = \frac{F_0}{k} (1 - \cos \omega_n t) + \frac{F_0}{2m\omega_n} \cdot \frac{2\omega_n}{\left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \cdot \left\{ \cos \frac{\pi t}{2t_0} - \cos \omega_n t \right\} \quad (E_1)$$

For $x_{\max}$, $\frac{dx}{dt} = 0$

$$\text{i.e., } \frac{F_0 \omega_n}{k} \sin \omega_n t + \frac{F_0}{m \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \left(-\frac{\pi}{2t_0} \sin \frac{\pi t}{2t_0} + \omega_n \sin \omega_n t \right) = 0$$

which can be reduced to the form

$$\sin \omega_n t_{\max} = \left[\frac{\pi}{2\omega_n t_0 \left\{ m \left(\frac{\pi}{2t_0} \right)^2 - m\omega_n^2 \right\} + k} \right] \sin \frac{\pi t_{\max}}{2t_0} \quad (E_2)$$

Once $t_{\max}$ is known from (E_2) , Eq. (E_1) can be used to find $x_{\max}$ as:

$$x_{\max} = x(t = t_{\max}) = \frac{F_0}{k} (1 - \cos \omega_n t_{\max}) + \frac{F_0}{m \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \left(\cos \frac{\pi t_{\max}}{2t_0} - \cos \omega_n t_{\max} \right) \quad (E_3)$$

For $t > t_0$:

$$\begin{aligned}
 x(t) = & \frac{F_0}{k} \cos \omega_n(t - t_0) \\
 & - \sin \omega_n(t - t_0) \left\{ \frac{F_0}{2m \omega_n \left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{F_0}{2m \omega_n \left(\frac{\pi}{2t_0} + \omega_n \right)} \right\} \\
 & + \cos \omega_n t \left\{ -\frac{F_0}{k} - \frac{F_0}{2m \omega_n \left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{F_0}{2m \omega_n \left(\frac{\pi}{2t_0} + \omega_n \right)} \right\}
 \end{aligned}$$

For $t_{\max}$, $\frac{dx}{dt} = 0$ (E₄)

i.e.,

$$\begin{aligned}
 & -\frac{F_0 \omega_n}{k} \sin \omega_n(t_{\max} - t_0) - \frac{F_0 \pi \omega_n \cos \omega_n(t_{\max} - t_0)}{2m \omega_n t_0 \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \\
 & + \frac{F_0 \left(\frac{\pi}{2t_0} \right)^2 \omega_n}{k \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \sin \omega_n t_{\max} = 0
 \end{aligned}$$

(E₅)

Once $t_{\max}$ is found by solving Eq. (E₅), $x_{\max}$ can be found from Eq. (E₄) as

$$\begin{aligned}
 x_{\max} = x(t = t_{\max}) = & \frac{F_0}{k} \cos \omega_n(t_{\max} - t_0) \\
 & - \frac{\pi F_0}{2 \omega_n m t_0 \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \sin \omega_n(t_{\max} - t_0) \\
 & - \frac{F_0 \left(\frac{\pi}{2t_0} \right)^2}{k \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \cos \omega_n t_{\max}
 \end{aligned}$$

(E₆)

Eqs. (E₃) and (E₆) can be used to plot $x_{\max}$ versus ω_n to get the displacement response spectrum.

4.37 Base acceleration = $\ddot{y}(t) = a_0 \left(1 - \sin \frac{\pi t}{2t_0} \right)$ (E₁)

For an undamped system, the relative displacement is given by Eq. (4.36):

$$\begin{aligned}
 z(t) = & -\frac{1}{\omega_n} \int_0^t \ddot{y}(\tau) \sin \omega_n(t - \tau) d\tau \\
 = & -\frac{1}{\omega_n} \left[\int_0^t a_0 \sin \omega_n(t - \tau) d\tau - \int_0^t a_0 \sin \frac{\pi \tau}{2t_0} \sin \omega_n(t - \tau) d\tau \right]
 \end{aligned}$$

(E₂)

Here

$$\int_0^t \sin \omega_n (t-\tau) d\tau = \left(\frac{1 - \cos \omega_n t}{\omega_n} \right) \quad (E_3)$$

and

from Eq. (E₃) in the solution of problem 4.26.

$$\begin{aligned} & \int_0^t \sin \frac{\pi \tau}{2t_0} (\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau) d\tau \\ &= \sin \omega_n t \left\{ -\frac{\cos \left(\frac{\pi}{2t_0} - \omega_n \right) \tau}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} - \frac{\cos \left(\frac{\pi}{2t_0} + \omega_n \right) \tau}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \right\}_0^t \\ & - \cos \omega_n t \left\{ \frac{\sin \left(\frac{\pi}{2t_0} - \omega_n \right) \tau}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} - \frac{\sin \left(\frac{\pi}{2t_0} + \omega_n \right) \tau}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \right\}_0^t \end{aligned} \quad (E_4)$$

Thus the solution, Eq. (E₂), can be finally expressed as

$$\begin{aligned} z(t) = & -\frac{a_0}{\omega_n} \left(1 - \frac{\cos \omega_n t}{\omega_n} \right) - \frac{a_0}{\omega_n} \left\{ \sin \omega_n t \left[\frac{\cos \left(\frac{\pi}{2t_0} - \omega_n \right) t}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} \right. \right. \\ & \left. \left. + \frac{\cos \left(\frac{\pi}{2t_0} + \omega_n \right) t}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} - 2 \right] + \cos \omega_n t \left[\frac{\sin \left(\frac{\pi}{2t_0} - \omega_n \right) t}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} - \frac{\sin \left(\frac{\pi}{2t_0} + \omega_n \right) t}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \right] \right\} \end{aligned} \quad (E_5)$$

4.38

During $0 \leq t \leq t_0$:

$$x(t) = \frac{F_0}{k} \left(1 - \frac{t}{t_0} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t \right) \quad (E_1)$$

$$\dot{x}(t) = 0 \text{ gives } \omega_n t_0 \sin \omega_n t_m = 1 - \cos \omega_n t_m$$

$$\text{i.e. } \omega_n t_m = 2 \tan^{-1}(\omega_n t_0) \quad (E_2)$$

(E₁) becomes

$$\frac{x_m}{(F_0/k)} = 1 - \frac{t_m}{t_0} - \cos \omega_n t_m + \frac{1}{\omega_n t_0} \sin \omega_n t_m \quad (E_3)$$

where t_m is given by (E₂).

During $t > t_0$:

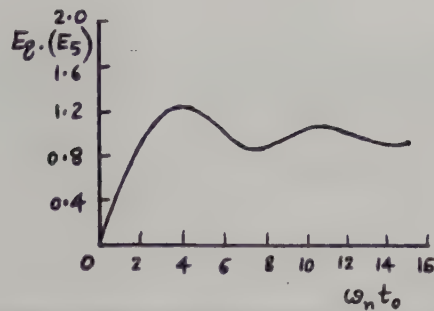
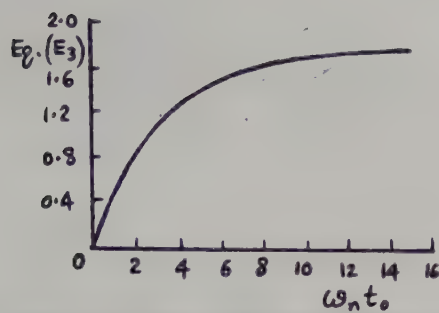
$$x(t) = \frac{F_0}{k \omega_n t_0} \left[(1 - \cos \omega_n t_0) \sin \omega_n t - (\omega_n t_0 - \sin \omega_n t_0) \cos \omega_n t \right] \quad (E_4)$$

$$\text{i.e. } \frac{x(t) k \omega_n t_0}{F_0} = \tilde{x}(t) = A \sin \omega_n t + B \cos \omega_n t$$

$$\text{where } A = 1 - \cos \omega_n t_0 ; B = -(\omega_n t_0 - \sin \omega_n t_0)$$

$$\text{Since } \tilde{x}|_{\max} = \sqrt{A^2 + B^2},$$

$$\frac{x_m}{(F_0/k)} = \frac{1}{\omega_n t_0} \left[(1 - \cos \omega_n t_0)^2 + (\omega_n t_0 - \sin \omega_n t_0)^2 \right]^{1/2} \quad (E_5)$$



4.39

From Example 4.7, the response of the building frame is given by

$$x(t) = \frac{F_0}{k} \left[1 - \frac{t}{t_0} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t \right], \quad 0 \leq t \leq t_0 \quad (E_1)$$

and

$$x(t) = \frac{F_0}{k \omega_n t_0} \left[(1 - \cos \omega_n t_0) \sin \omega_n t - (\omega_n t_0 - \sin \omega_n t_0) \cos \omega_n t \right], \quad t > t_0 \quad (E_2)$$

(i) For $0 \leq t \leq t_0$:

For $x(t)$ to be maximum, the quantity inside square brackets in (E_1) must be maximum. This implies that

$$\frac{d}{dt} \left[1 - \frac{t}{t_0} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t \right] = 0$$

$$\text{i.e.,} \quad \omega_n t_0 \sin \omega_n t = 1 - \cos \omega_n t$$

$$\text{i.e.,} \quad \omega_n t_0 \cos \frac{\omega_n t}{2} = \sin \frac{\omega_n t}{2}$$

$$\text{i.e.,} \quad \tan \frac{\omega_n t}{2} = \omega_n t_0 \quad (E_3)$$

Thus, if $x(t)$ attains its maximum value at $t = t_{\max}$, $E_3(E_3)$ gives

$$t_{\max} = \frac{2}{\omega_n} \tan^{-1}(\omega_n t_0) \quad (E_4)$$

Once $t_{\max}$ is known from (E_4) , (E_1) gives

$$x_{\max} = x(t = t_{\max}) = \frac{F_0}{k} \left\{ 1 - \frac{t_{\max}}{t_0} - \cos \omega_n t_{\max} + \frac{1}{\omega_n t_0} \sin \omega_n t_{\max} \right\} \quad (E_5)$$

(ii) For $t > t_0$:

For $x(t)$ to be maximum, the quantity inside square brackets in $E_2(E_2)$ must be maximum. This implies that

$$\frac{d}{dt} \left[(1 - \cos \omega_n t_0) \sin \omega_n t - (\omega_n t_0 - \sin \omega_n t_0) \cos \omega_n t \right] = 0$$

i.e.,

$$(1 - \cos \omega_n t_0) \omega_n \cos \omega_n t + (\omega_n t_0 - \sin \omega_n t_0) \omega_n \sin \omega_n t = 0$$

i.e.,

$$\tan \omega_n t = - \left(\frac{1 - \cos \omega_n t_0}{\omega_n t_0 - \sin \omega_n t_0} \right) \quad (E_6)$$

If $x(t)$ attains its maximum at $t = t_{\max}$, Eq. (E₆) gives

$$t_{\max} = \frac{1}{\omega_n} \tan^{-1} \left(\frac{-1 + \cos \omega_n t_0}{\omega_n t_0 - \sin \omega_n t_0} \right) \quad (E_7)$$

Once $t_{\max}$ is computed from (E₇), (E₂) gives

$$x_{\max} = x(t = t_{\max}) = \frac{F_0}{k \omega_n t_0} \left[(1 - \cos \omega_n t_0)^2 + (\omega_n t_0 - \sin \omega_n t_0)^2 \right]^{\frac{1}{2}} \quad (E_8)$$

Given data:

$$m = 5000 \text{ kg}, \quad F_0 = 4 \times 10^6 \text{ N}, \quad t_0 = 0.4 \text{ sec}, \quad x_{\max} \leq 0.01 \text{ m}.$$

$$\omega_n = \sqrt{k/m} = 0.01414 \sqrt{k}$$

Procedure:

1. Assume a series of values of k .
2. Find $t_{\max}$ using Eqs. (E₄) and (E₇).
3. Find $x_{\max}$ using Eqs. (E₅) and (E₈).
4. Select k such that $x_{\max} \leq 0.01 \text{ m}$ in Eq. (E₅) or (E₈).

Sample computer program and results are shown below.

```

XM=5000.0
FO=4.0E+6
TO=0.4
XK=0.0
DO 10 I=1,100
XK=XK+1.0E+7
OMN=0.01414*SQRT(XK)
TMAX1=(2.0/OMN)*ATAN(OMN*TO)
XMAX1=(FO/XK)*((1.0-(TMAX1/TO)-COS(OMN*TMAX1)+SIN(OMN*TMAX1)/
2 (OMN*TO))
XNR=-(1.0-COS(OMN*TO))
XDR=(OMN*TO-SIN(OMN*TO))
TMAX2=ATAN(XNR/XDR)/OMN
X1=(1.0-COS(OMN*TO))**2
X2=(OMN*TO-SIN(OMN*TO))**2
X3=(X1+X2)**0.5
XMAX2=X3*FO/(XK*OMN*TO)
PRINT 5, I, XK, TMAX1, XMAX1, TMAX2, XMAX2
5 FORMAT (I5, 2X, E15.4, 2X, 2E12.4, 2X, 2E12.4)
10 CONTINUE
STOP
END

```

1	0.1000E+08	0.6776E-01	0.7322E+00	-0.5134E-03	0.4184E+00
2	0.2000E+08	0.4843E-01	0.3758E+00	-0.8204E-05	0.1797E+00
3	0.3000E+08	0.3973E-01	0.2534E+00	-0.3859E-04	0.1357E+00
4	0.4000E+08	0.3450E-01	0.1914E+00	-0.4108E-03	0.1027E+00
5	0.5000E+08	0.3092E-01	0.1538E+00	-0.4234E-03	0.7857E-01
⋮					

These results indicate that the required stiffness is

$$k = 8 \times 10^8 \text{ N/m.}$$

4.40

Let d = thickness of bracket. Then, from Example 4.12, self weight of beam = $w = 0.5 d$ lb, total weight at free end of beam = $W = 0.5 d + 0.4$ lb, moment of inertia of beam cross section = $I = 0.04167 d^3 \text{ in}^4$, static deflection of beam under W :

$$\delta_{st} = \frac{W \ell^3}{3 E I} = \frac{(0.5 d + 0.4)}{d^3} 7.9994 (10^{-4}) \text{ in}$$

We need to use a trial and error procedure to find the correct value of d .

Let $d = 1 \text{ in}$:

$$w = 0.5 \text{ lb, } W = 0.9 \text{ lb, } I = 0.04167 \text{ in}^4, \delta_{st} = 0.9 (7.9994) (10^{-4}) = 7.19946 (10^{-4}) \text{ in,}$$

$$\tau_n = 2 \pi \sqrt{\frac{\delta_{st}}{g}} = \sqrt{\frac{7.19946 (10^{-4})}{386.4}} = 0.008577 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.008577} = 11.6591$$

From Fig. 4.49, shock amplification factor (A_a) corresponding to $t_0/\tau_n = 11.6591$ is $A_a \approx 2.0$.

$$\text{Dynamic load at end of cantilever} = P_d = A_a M a_s = (2.0) (0.9/g) (100g) = 180.0 \text{ lb.}$$

$$\sigma_{\max} = \frac{M_b c}{I} = \frac{(180 (10)) (1.0/2)}{0.04167} = 21598.2721 \text{ lb/in}^2$$

Since this is smaller than the permissible value of 26000 psi, we choose a smaller value of d next.

Let $d = 0.9 \text{ in}$:

$$w = 0.45 \text{ lb, } W = 0.85 \text{ lb, } I = 0.03038 \text{ in}^4, \delta_{st} = \frac{0.85 (7.9994 (10^{-4}))}{0.9^3} = 9.3271 (10^{-4}) \text{ in}$$

$$\tau_n = 2 \pi \sqrt{\frac{\delta_{st}}{g}} = 2 \pi \sqrt{\frac{9.3271 (10^{-4})}{386.4}} = 0.009762 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.009762} = 10.2438$$

$$A_a \approx 2.0, P_d = A_a M a_s = (2.0) (0.85/g) (100g) = 170.0 \text{ lb}$$

$$\sigma_{\max} = \frac{M_b c}{I} = \frac{(170 (10)) (0.9/2)}{0.03038} = 25181.0402 \text{ lb/in}^2$$

Since this stress is close to the maximum permissible value, we take $d = 0.9 \text{ in}$.

4.41

Let d = thickness of bracket. Then from Example 4.12, self weight of beam = $w = 0.5 d$ lb, total weight at free end of beam = $W = 0.5 d + 0.4$ lb, moment of inertia of beam cross section = $I = 0.04167 d^3 \text{ in}^4$, static deflection of beam under W :

$$\delta_{st} = \left(\frac{0.5 d + 0.4}{d^3} \right) 7.9994 (10^{-4}) \text{ in}$$

We need to use a trial and error procedure to find the correct value of d . However, since the shock amplification factor, for large values of t_0/τ_n , for the triangular pulse of Fig. 4.50 is similar to that of the pulse shown in Fig. 4.11, we start with $d = 0.6$ in. This gives:

$$w = 0.3 \text{ lb}, W = 0.7 \text{ lb}, I = (0.04167) (0.216) = 0.009001 \text{ in}^4,$$

$$\delta_{st} = \frac{0.7}{0.6^3} (7.9994 (10^{-4})) = 25.9240 (10^{-4}) \text{ in}, \tau_n = 2\pi \sqrt{\frac{25.9240 (10^{-4})}{386.4}} =$$

0.01627 sec, $t_0/\tau_n = (0.1/0.01627) = 6.1445$. From Fig. 4.50, we find shock amplification factor as $A_a \approx 1.1$, dynamic load at end of beam = $P_d = A_a M a_a = (1.1) (0.7/g) (100g) = 77.0$ lb, maximum bending stress at root of beam = $\sigma_{max} = \frac{M_b c}{I} = (77(10)) (0.6/2) / (0.009001) = 25663.8151$ psi. Since this stress is very close to the maximum permissible value, we select $d = 0.6$ in as the design.

4.42

Let d = thickness of bracket (beam). Self weight of beam = $w = d (1/2) (16) (0.1) = 0.8 d$ lb, total load at middle of beam = $W = (1 + 0.8 d)$ lb, area moment of inertia of beam cross section = $I = \frac{1}{12} \left(\frac{1}{2} \right) d^3 \text{ in}^4$. Static deflection of beam at middle due to W :

$$\delta_{st} = \frac{W \ell^3}{192 E I} = \frac{(1 + 0.8 d) (16^3)}{192 (10^7) (0.04167 d^3)} = (1 + 0.8 d) (51.1959 (10^{-6})) \text{ in}$$

We need to use a trial and error procedure to determine the correct value of d .

Let $d = 0.4$ in:

$$w = 0.32 \text{ lb}, W = 1.32 \text{ lb}, I = 0.002667 \text{ in}^4, \delta_{st} = 67.5786 (10^{-6}) \text{ in},$$

$$\tau_n = 2\pi \sqrt{\frac{67.5786 (10^{-6})}{386.4}} = 0.002628 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.002628} = 38.0569$$

From Fig. 4.11(b), $A_a \approx 1.1$, and the dynamic load on beam is given by $P_d = A_a M a_s$ where M = total mass of beam and a_s = acceleration due to shock = 100 g. Thus $P_d = (1.1) (1.32/g) (100 g) = 145.2$ lb. Maximum bending moment in a fixed-fixed beam due to load (F) at the middle is given by $M_b = \frac{F \ell}{8}$ so that

$$\sigma_{max} = \frac{M_b c}{I} = \frac{\left(\frac{145.2 (16)}{8} \right) \left(\frac{0.4}{2} \right)}{0.002667} = 21777.2778 \text{ lb/in}^2$$

Since this stress is smaller than the maximum permissible value of 26000 psi, we next select a smaller value of d.

Let $d = 0.35$ in:

$$w = 0.28 \text{ lb}, W = 1.28 \text{ lb}, I = 0.001787 \text{ in}^4, \delta_{st} = 65.5307 (10^{-6}) \text{ in},$$

$$\tau_n = 2 \pi \sqrt{\frac{65.5307 (10^{-6})}{386.4}} = 0.002587 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.002587} = 38.6469$$

From Fig. 4.11(b), $A_a \approx 1.1$, $P_d = (1.1) (1.28/g) (100g) = 140.8 \text{ lb}$

$$\sigma_{\max} = \frac{M_b c}{I} = \frac{\left(\frac{140.8 (16)}{8} \right) \left(\frac{0.35}{2} \right)}{0.001787} = 27576.9446 \text{ lb/in}^2$$

Since this stress exceeds the permissible value, we increase the value of d.

Let $d = 0.37$ in:

$$w = 0.296 \text{ lb}, W = 1.296 \text{ lb}, I = 0.002111 \text{ in}^4, \delta_{st} = 66.3499 (10^{-6}) \text{ in}$$

$$\tau_n = 2 \pi \sqrt{\frac{66.3499 (10^{-6})}{386.4}} = 0.002604 \text{ sec}$$

$$\frac{t_0}{\tau_n} = 38.4024$$

From Fig. 4.11(b), $A_a \approx 1.1$, $P_d = (1.1) (1.296/g) (100 g) = 142.56 \text{ lb}$, and

$$\sigma_{\max} = \frac{M_b c}{I} = \frac{\left(\frac{142.56 (16)}{8} \right) \left(\frac{0.37}{2} \right)}{0.002111} = 24986.8309 \text{ lb/in}^2$$

Since this stress is close to the maximum permissible value, we select the design as $d = 0.37$ in.

4.43 $W = m g = 100000 \text{ lb}, \zeta = 0.05, \sigma_y = 30000 \text{ psi},$

$$\sigma_{\max} = \text{maximum permissible stress} = \frac{\sigma_y}{2} = \frac{30000}{2} = 15000 \text{ psi},$$

$$\tau_n = \frac{2 \pi}{\omega_n} = 2 \pi \sqrt{\frac{m}{k}} = 2 \pi \sqrt{\frac{100000}{386.4 k}} = \frac{101.0793}{\sqrt{k}} \text{ sec}$$

We need to use a trial and error procedure to find k.

Let $k = 10000 \text{ lb/in}$:

$$k = 10^4 = \frac{3 E I}{\ell^3} = \frac{3 (30 (10^6)) I}{600^3}$$

$$I = 24000 \text{ in}^4 = \frac{\pi}{64} d_o^4 (0.5904) \text{ with } \frac{d_i}{d_o} = 0.8$$

$$d_o^4 = 82.8121 (10^4) \text{ in}^4$$

$$d_o = 30.1664 \text{ in} ; d_i = 24.1331 \text{ in}$$

$$\tau_n = \frac{101.0793}{\sqrt{10^4}} \approx 1 \text{ sec}$$

From Fig. 4.14, for $\tau_n = 1 \text{ sec}$ and $\zeta = 0.05$, we find $S_v = 25 \text{ in/sec}$, $S_d = 4.2 \text{ in}$ and $S_a = 0.42 \text{ g}$.

Maximum shear force in column:

$$F_{\max} = \frac{W}{g} S_a = \frac{100000}{g} (0.42g) = 42000 \text{ lb}$$

Maximum bending moment:

$$M_b = F_{\max} h = (42000) (50 (12)) = 25.2 (10^6) \text{ lb-in}$$

Maximum bending stress:

$$\sigma_b = \frac{M_b c}{I} = \frac{(25.2 (10^6)) (30.1664/2)}{24 (10^3)} = 15837.36 \text{ lb/in}^2$$

Since this stress is slightly smaller than the maximum permissible value, we choose a larger value of k .

Let $k = 20000 \text{ lb/in}$:

$$k = 2 (10^4) = \frac{3 (30 (10^6)) I}{600^3}$$

$$I = 48000 \text{ in}^4 = \frac{\pi}{64} d_o^4 (0.5904)$$

$$d_o^4 = 165.6243 (10^4) \text{ in}^4$$

$$d_o = 35.8741 \text{ in} ; d_i = 28.6993 \text{ in}$$

$$\tau_n = \frac{101.0793}{\sqrt{20000}} = 0.7147 \text{ sec}$$

From Fig. 4.14, we find

$$S_v = 26 \text{ in/sec}, S_d = 3 \text{ in}, S_a = 0.6 \text{ g}$$

Maximum shear force in column:

$$F_{\max} = \frac{W}{g} S_a = \frac{10^5}{g} (0.6g) = 60000 \text{ lb}$$

Maximum bending moment:

$$M_b = F_{\max} h = (60000) (600) = 36 (10^6) \text{ lb-in}$$

Maximum bending stress:

$$\sigma_b = \frac{M_b c}{I} = \frac{(36 (10^6)) (35.8741/2)}{48 (10^3)} = 13452.7875 \text{ lb/in}^2$$

Since this stress is less than the maximum permissible value, we choose the inner and outer diameters of the column as $d_i = 28.6993 \text{ in}$ and $d_o = 35.8741 \text{ in}$.

4.44

$m g = 5000 \text{ lb}$, $\zeta = 0.02$. From Fig. 4.15, in order to have $S_a \approx 1 g$, we need to have $\tau_n = 0.2 \text{ sec}$. Thus

$$\tau_n = 0.2 = \frac{2\pi}{\omega_n} ; \omega_n = 31.416 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

$$k = (31.416)^2 m = (31.416^2) (5000/386.4) = 12771.2870 \text{ lb/in}$$

4.45

Equation of motion is $\ddot{x} + \omega_n^2 x = \frac{F_0}{m} e^{i\omega t}$

For zero initial conditions

$$(s^2 + \omega_n^2) \bar{x}(s) = \frac{F_0}{m} \cdot \frac{1}{s - i\omega} ; \bar{x}(s) = \frac{F_0}{m} \cdot \frac{1}{(s^2 + \omega_n^2)} \cdot \frac{s + i\omega}{(s^2 + \omega^2)}$$

Inverse Laplace transformation gives

$$x(t) = \frac{F_0}{m} \frac{1}{(\omega_n^2 - \omega^2)} \left\{ e^{i\omega t} - \left(\cos \omega_n t + \frac{i\omega}{\omega_n} \sin \omega_n t \right) \right\}$$

The terms containing $\cos \omega_n t$ and $\sin \omega_n t$ denote the transient

part of the response and hence can be neglected. Thus the steady state response can be expressed as

$$x(t) = \frac{F_0}{k} \left(\frac{1}{1 - r^2} \right) e^{i\omega t} \quad \text{where } r = \omega/\omega_n.$$

4.46

$$\bar{F}(s) = \frac{F_0}{s}$$

$$\text{From Eq. (4.57), } \bar{x}(s) = \frac{F_0}{ms(s^2 + 2\zeta\omega_n s + \omega_n^2)} + \left(\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) x_0$$

$$+ \left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) \dot{x}_0$$

Inverse transformation gives

$$x(t) = \frac{F_0}{m} \left\{ 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin(\omega_d t + \phi_1) \right\} + \frac{x_0}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1)$$

$$+ \frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

where $\phi_1 = \cos^{-1}(\zeta)$.

4.47

The forcing function can be expressed as

$$F(t) = F_0 \{ u(t) - u(t - t_0) \}$$

where $u(t - \tau)$ is the unit step function applied at $t = \tau$. The

Laplace transform of $F(t)$ is

$$\bar{F}(s) = F_0 \left(\frac{1}{s} - \frac{1}{s} e^{-st_0} \right)$$

The equation of motion $m\ddot{x} + kx = F(t)$ gives

$$\bar{x}(s) = \frac{F_0}{s} (1 - e^{-st_0}) \frac{1}{ms^2 + k} = \frac{F_0}{m} \left\{ \frac{1}{s(s^2 + \omega_n^2)} - \frac{e^{-st_0}}{s(s^2 + \omega_n^2)} \right\}$$

Since $\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 + \omega_n^2)} \right\} = \frac{1}{\omega_n^2} (1 - \cos \omega_n t)$,

$$x(t) = \frac{F_0}{m \omega_n^2} (1 - \cos \omega_n t) \cdot u(t) - \frac{F_0}{m \omega_n^2} \{1 - \cos \omega_n(t - t_0)\} u(t - t_0)$$

Hence

$$x(t) = \begin{cases} \frac{F_0}{m \omega_n^2} (1 - \cos \omega_n t) & \text{for } 0 \leq t \leq t_0 \\ \frac{F_0}{m \omega_n^2} \{ \cos \omega_n(t - t_0) - \cos \omega_n t \} & \text{for } t \geq t_0 \end{cases}$$

4.48 $\dot{x}(t) = \frac{1}{k} \sum_{i=1}^{j-1} \Delta F_i \left\{ -(\gamma \omega_n) e^{-\gamma \omega_n(t-t_i)} \cos \omega_d(t-t_i) \right.$
 $\left. + e^{-\gamma \omega_n(t-t_i)} \omega_d \sin \omega_d(t-t_i) - (\gamma \omega_n) e^{-\gamma \omega_n(t-t_i)} \frac{\gamma \omega_n}{\omega_d} \sin \omega_d(t-t_i) \right.$
 $\left. - e^{-\gamma \omega_n(t-t_i)} \frac{\gamma \omega_n}{\omega_d} \cdot \omega_d \cdot \cos \omega_d(t-t_i) \right\}$
 $= \frac{1}{k} \sum_{i=1}^{j-1} \Delta F_i \left\{ e^{-\gamma \omega_n(t-t_i)} \left(\omega_d + \frac{\gamma^2 \omega_n^2}{\omega_d} \right) \sin \omega_d(t-t_i) \right\}$
 $\dot{x}_j = \dot{x}(t=t_j) = \frac{1}{k} \sum_{i=1}^{j-1} \Delta F_i \left(\omega_d + \frac{\gamma^2 \omega_n^2}{\omega_d} \right) e^{-\gamma \omega_n(t_j-t_i)} \sin \omega_d(t_j-t_i)$

4.49 Eq. (4.68): Differentiating Eq. (4.66),
 $\dot{x}(t) = \frac{F_j}{k} e^{-\gamma \omega_n(t-t_{j-1})} \omega_d \left(1 + \frac{\gamma^2 \omega_n^2}{\omega_d^2} \right) \sin \omega_d(t-t_{j-1}) + \omega_d e^{-\gamma \omega_n(t-t_{j-1})} \left\{ -x_{j-1} \sin \omega_d(t-t_{j-1}) \right.$
 $\left. + \frac{\dot{x}_{j-1}}{\omega_d} \cos \omega_d(t-t_{j-1}) - \frac{\gamma \omega_n}{\omega_d^2} (\dot{x}_{j-1} + \gamma \omega_n x_{j-1}) \sin \omega_d(t-t_{j-1}) \right\}$
 For $t = t_j$,
 $\dot{x}_j = \frac{F_j}{k} e^{-\gamma \omega_n \Delta t_j} \omega_d \left(1 + \frac{\gamma^2 \omega_n^2}{\omega_d^2} \right) \sin \omega_d \Delta t_j + \omega_d e^{-\gamma \omega_n \Delta t_j} \left\{ -x_{j-1} \sin \omega_d \Delta t_j \right.$
 $\left. + \frac{\dot{x}_{j-1}}{\omega_d} \cos \omega_d \Delta t_j - \frac{\gamma \omega_n}{\omega_d^2} (\dot{x}_{j-1} + \gamma \omega_n x_{j-1}) \sin \omega_d \Delta t_j \right\}$

Eq. (4.71):

$$\dot{x}(t) = \frac{\Delta F_j}{k \Delta t_j} \left[1 - \gamma \omega_n e^{-\gamma \omega_n(t-t_{j-1})} \frac{2\gamma}{\omega_n} \cos \omega_d(t-t_{j-1}) - e^{-\gamma \omega_n(t-t_{j-1})} \right. \left. \frac{2\gamma}{\omega_n} \omega_d \sin \omega_d(t-t_{j-1}) + \gamma \omega_n e^{-\gamma \omega_n(t-t_{j-1})} \left(\frac{\omega_d^2 - \gamma^2 \omega_n^2}{\omega_n^2 \omega_d} \right) \sin \omega_d(t-t_{j-1}) \right]$$

$$\begin{aligned}
& - e^{-\gamma \omega_n (t-t_{j-1})} \left(\frac{\omega_d^2 - \gamma^2 \omega_n^2}{\omega_n^2 \omega_d} \right) \omega_d \cos \omega_d (t-t_{j-1}) \Big] + \frac{F_{j-1}}{k} \left[\gamma \omega_n e^{-\gamma \omega_n (t-t_{j-1})} \right. \\
& \cos \omega_d (t-t_{j-1}) + e^{-\gamma \omega_n (t-t_{j-1})} \omega_d \sin \omega_d (t-t_{j-1}) + \gamma \omega_n e^{-\gamma \omega_n (t-t_{j-1})} \\
& \left. \left(\frac{\gamma \omega_n}{\omega_d} \right) \sin \omega_d (t-t_{j-1}) - e^{-\gamma \omega_n (t-t_{j-1})} \left(\frac{\gamma \omega_n}{\omega_d} \right) \omega_d \cos \omega_d (t-t_{j-1}) \right] \\
& - \gamma \omega_n e^{-\gamma \omega_n (t-t_{j-1})} x_{j-1} \cos \omega_d (t-t_{j-1}) - e^{-\gamma \omega_n (t-t_{j-1})} x_{j-1} \omega_d * \\
& \sin \omega_d (t-t_{j-1}) - \gamma \omega_n e^{-\gamma \omega_n (t-t_{j-1})} \left(\frac{\dot{x}_{j-1} + \gamma \omega_n x_{j-1}}{\omega_d} \right) \sin \omega_d (t-t_{j-1}) \\
& + e^{-\gamma \omega_n (t-t_{j-1})} \left(\frac{\dot{x}_{j-1} + \gamma \omega_n x_{j-1}}{\omega_d} \right) \omega_d \cos \omega_d (t-t_{j-1})
\end{aligned}$$

Simplification of this equation gives at $t = t_j$,

$$\begin{aligned}
\dot{x}_j &= \frac{\Delta F_j}{k \Delta t_j} \left[1 - e^{-\gamma \omega_n \Delta t_j} \left\{ \cos \omega_d \Delta t_j + \frac{\gamma \omega_n}{\omega_d} \sin \omega_d \Delta t_j \right\} \right] \\
&+ \frac{F_{j-1}}{k} e^{-\gamma \omega_n \Delta t_j} \frac{\omega_n^2}{\omega_d} \sin \omega_d \Delta t_j \\
&+ e^{-\gamma \omega_n \Delta t_j} \left\{ \dot{x}_{j-1} \cos \omega_d \Delta t_j - \frac{\gamma \omega_n}{\omega_d} \left(\dot{x}_{j-1} + \frac{\omega_n}{\gamma} x_{j-1} \right) \sin \omega_d \Delta t_j \right\}
\end{aligned}$$

4.50

i	t_i	$\dot{x}_i$, Eq. (4.68)	$\dot{x}_i$, Eq. (4.71)
1	0	0	0
2	0.1π	0.276010E+00	0.275640E+00
3	0.2π	0.462605E+00	0.461687E+00
4	0.3π	0.549249E+00	0.547683E+00
5	0.4π	0.536630E+00	0.534405E+00
6	0.5π	0.435845E+00	0.433036E+00
7	0.6π	0.266613E+00	0.263378E+00
8	0.7π	0.547668E-01	0.513272E-01
9	0.8π	-0.170711E+00	-0.174093E+00
10	0.9π	-0.380784E+00	-0.383830E+00
11	π	-0.549289E+00	-0.551730E+00

4.51

Method 1:
$$x_j = \frac{1}{k} \sum_{i=1}^{j-1} \Delta F_i \left\{ 1 - \cos \omega_n (t_j - t_i) \right\}$$

$$\dot{x}_j = \frac{1}{k} \sum_{i=1}^{j-1} \Delta F_i \omega_n \sin \omega_n (t_j - t_i)$$

Method 2:

$$\begin{aligned}
x_j &= \frac{F_j}{k} [1 - \cos \omega_n \Delta t_j] + x_{j-1} \cos \omega_n \Delta t_j + \frac{\dot{x}_{j-1}}{\omega_n} \sin \omega_n \Delta t_j \\
\dot{x}_j &= \frac{F_j \omega_n}{k} \sin \omega_n \Delta t_j + \omega_n \left\{ -x_{j-1} \sin \omega_n \Delta t_j + \frac{\dot{x}_{j-1}}{\omega_n} \cos \omega_n \Delta t_j \right\}
\end{aligned}$$

Method 3:

$$\begin{aligned}
x_j &= \frac{\Delta F_j}{k \Delta t_j} \left\{ \Delta t_j - \frac{1}{\omega_n} \sin \omega_n \Delta t_j \right\} - \frac{F_{j-1}}{k} (1 - \cos \omega_n \Delta t_j) + x_{j-1} \cos \omega_n \Delta t_j \\
&+ \frac{\dot{x}_{j-1}}{\omega_n} \sin \omega_n \Delta t_j
\end{aligned}$$

$$\ddot{x}_j = \frac{\Delta F_j}{k \Delta t_j} (1 - \cos \omega_n \Delta t_j) + \frac{F_{j-1}}{k} \omega_n \sin \omega_n \Delta t_j + \dot{x}_{j-1} \cos \omega_n \Delta t_j - \omega_n x_{j-1} \sin \omega_n \Delta t_j$$

VALUE OF	METHOD #1 (FIG.4.18)	METHOD #1 (FIG.4.19)	METHOD #2 (FIG.4.20)	METHOD #3 (FIG.4.21)
I	X(I)	X(I)	X(I)	X(I)
2	0.489135E-01	0.412870E-01	0.451034E-01	0.463829E-01
3	0.183327E+00	0.153639E+00	0.168413E+00	0.170863E+00
4	0.374871E+00	0.311494E+00	0.342969E+00	0.346380E+00
5	0.590262E+00	0.485757E+00	0.537537E+00	0.541621E+00
6	0.794774E+00	0.646981E+00	0.720014E+00	0.724433E+00
7	0.955998E+00	0.768556E+00	0.860896E+00	0.865287E+00
8	0.104732E+01	0.829582E+00	0.936446E+00	0.940461E+00
9	0.105081E+01	0.817133E+00	0.931268E+00	0.934605E+00
10	0.959172E+00	0.727695E+00	0.840009E+00	0.842438E+00
11	0.776638E+00	0.566437E+00	0.668027E+00	0.669410E+00
I	XD(I)	XD(I)	XD(I)	XD(I)
2	0.309017E+00	0.260676E+00	0.284772E+00	0.284646E+00
3	0.539444E+00	0.448685E+00	0.493775E+00	0.493286E+00
4	0.669917E+00	0.547974E+00	0.608327E+00	0.607283E+00
5	0.690013E+00	0.552279E+00	0.620126E+00	0.618406E+00
6	0.601222E+00	0.465650E+00	0.531993E+00	0.529561E+00
7	0.416706E+00	0.301948E+00	0.357498E+00	0.354412E+00
8	0.159909E+00	0.833536E-01	0.119506E+00	0.115921E+00
9	-0.137878E+00	-0.161957E+00	-0.152198E+00	-0.156045E+00
10	-0.440725E+00	-0.402734E+00	-0.423989E+00	-0.427799E+00
11	-0.711751E+00	-0.615408E+00	-0.661862E+00	-0.665302E+00

4.52 $\omega_n = \sqrt{k/m} = \sqrt{50/2} = 5 \text{ rad/s}; \quad \gamma = \frac{c}{2m\omega_n} = \frac{2}{2(2)(5)} = 0.1$

The problem-dependent data to be used in Program 5 and the output are given.

C FOLLOWING 11 LINES CONTAIN PROBLEM-DEPENDENT DATA

```

      DIMENSION F(11),FF(11),DELF(11),T(11),X(11),XD(11),X1(11),
2     XD1(11),X2(11),XD2(11),X3(11),XD3(11),X4(11),XD4(11)
      DATA F/-8.,-12.,-15.,-13.,-11.,-7.,-4.,3.,10.,15.,18./
      DO 5 I=1,11
5     FF(I)=F(I)
      XA1=0.1
      OMN=5.0
      DELT=0.1
      XK=50.0

```

C END OF PROBLEM-DEPENDENT DATA

VALUE OF	METHOD #1 (FIG.4.18)	METHOD #1 (FIG.4.19)	METHOD #2 (FIG.4.20)	METHOD #3 (FIG.4.21)
I	X(I)	X(I)	X(I)	X(I)

2	-0.947628E-02	-0.355361E-01	-0.284289E-01	-0.308375E-01
3	-0.415895E-01	-0.124570E+00	-0.110554E+00	-0.117557E+00
4	-0.888374E-01	-0.231948E+00	-0.224265E+00	-0.229133E+00
5	-0.129452E+00	-0.316847E+00	-0.330743E+00	-0.326279E+00
6	-0.140543E+00	-0.345356E+00	-0.391810E+00	-0.371424E+00
7	-0.106067E+00	-0.292742E+00	-0.381288E+00	-0.341045E+00
8	-0.149353E-01	-0.146366E+00	-0.284987E+00	-0.219841E+00
9	0.138845E+00	0.786267E-01	-0.100887E+00	-0.136816E-01
10	0.341364E+00	0.340894E+00	0.147643E+00	0.244535E+00
11	0.559359E+00	0.589762E+00	0.415128E+00	0.488339E+00

1	XD(1)	XD(1)	XD(1)	XD(1)
2	-0.363160E-01	-0.136185E+00	-0.547484E+00	-0.618556E+00
3	-0.879517E-01	-0.209522E+00	-0.105218E+01	-0.105547E+01
4	-0.960236E-01	-0.208906E+00	-0.117245E+01	-0.111170E+01
5	-0.627967E-01	-0.123358E+00	-0.916858E+00	-0.762976E+00
6	0.182155E-01	0.100267E-01	-0.289472E+00	-0.106971E+00
7	0.114511E+00	0.191935E+00	0.482526E+00	0.746090E+00
8	0.238516E+00	0.375367E+00	0.138800E+01	0.167033E+01
9	0.358701E+00	0.499280E+00	0.220328E+01	0.239624E+01
10	0.429269E+00	0.522310E+00	0.265575E+01	0.268039E+01
11	0.420343E+00	0.448691E+00	0.258326E+01	0.195527E+01

4.53

The problem-dependent data to be used in Program 5 and results are given.

C FOLLOWING 11 LINES CONTAIN PROBLEM-DEPENDENT DATA

```

      DIMENSION F(51),FF(51),DELF(51),T(51),X(51),XD(51),X1(51),
2     XD1(51),X2(51),XD2(51),X3(51),XD3(51),X4(51),XD4(51)
      DATA F/0.,2.,4.,6.,8.,10.,12.,14.,16.,18.,20.,20.,20.,20.,
2     20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,0.,0.,0.,0.,
3     0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,
4     0./
      DO 5 I=1,51
5     FF(I)=F(I)
      XAT=0.0
      OMU=27.3861
      DELT=0.01
      XK=1500.0
      NP=51

```

C END OF PROBLEM-DEPENDENT DATA

VALUE OF	METHOD #1 (FIG.4.18)	METHOD #1 (FIG.4.19)	METHOD #2 (FIG.4.20)	METHOD #3 (FIG.4.21)
1	X(1)	X(1)	X(1)	X(1)
2	0.496882E-04	0.493764E-04	0.496882E-04	0.662924E-04
3	0.244738E-03	0.439787E-03	0.244738E-03	0.326397E-03
4	0.669987E-03	0.109524E-02	0.669987E-03	0.860303E-03
5	0.139312E-02	0.211625E-02	0.139312E-02	0.172759E-02
6	0.245961E-02	0.352610E-02	0.245961E-02	0.296301E-02
7	0.388935E-02	0.531910E-02	0.388935E-02	0.457384E-02
8	0.567516E-02	0.746097E-02	0.567516E-02	0.653940E-02
9	0.778331E-02	0.989145E-02	0.778330E-02	0.881258E-02
10	0.101560E-01	0.125288E-01	0.101560E-01	0.113233E-01
:				

45	0.798152E-02	0.845167E-02	0.798155E-02	0.827565E-02
46	0.831680E-02	0.807062E-02	0.831682E-02	0.825346E-02
47	0.803221E-02	0.708804E-02	0.803222E-02	0.761612E-02
48	0.714897E-02	0.557718E-02	0.714896E-02	0.641114E-02
49	0.573290E-02	0.365064E-02	0.573288E-02	0.472831E-02
50	0.388954E-02	0.145202E-02	0.388951E-02	0.269308E-02
51	0.175629E-02	-0.854840E-03	0.175624E-02	0.457121E-03

I	XD(I)	XD(I)	XD(I)	XD(I)
2	0.360601E-03	0.721201E-03	0.987545E-02	0.148443E-01
3	0.105493E-02	0.174925E-02	0.288903E-01	0.385198E-01
4	0.203123E-02	0.300753E-02	0.556274E-01	0.692620E-01
5	0.321673E-02	0.440224E-02	0.880938E-01	0.104780E+00
6	0.452309E-02	0.582945E-02	0.123870E+00	0.142425E+00
7	0.585294E-02	0.718278E-02	0.160289E+00	0.179393E+00
8	0.710715E-02	0.836136E-02	0.194637E+00	0.212929E+00
9	0.819225E-02	0.927735E-02	0.224354E+00	0.240531E+00
10	0.902736E-02	0.986248E-02	0.247224E+00	0.260144E+00
:				
45	0.233952E-02	-0.244374E-03	0.640694E-01	0.289820E-01
46	0.937302E-04	-0.252103E-02	0.256574E-02	-0.333924E-01
47	-0.215904E-02	-0.460977E-02	-0.591291E-01	-0.932780E-01
48	-0.425090E-02	-0.635495E-02	-0.116417E+00	-0.146211E+00
49	-0.602593E-02	-0.762648E-02	-0.165028E+00	-0.188247E+00
50	-0.735183E-02	-0.832959E-02	-0.201339E+00	-0.216253E+00
51	-0.812978E-02	-0.841188E-02	-0.222644E+00	-0.228140E+00

4.54

The problem-dependent data to be used in Program 5 and output are given below.

```

C FOLLOWING 11 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION F(51),FF(51),DELF(51),T(51),X(51),XD(51),X1(51);
  2  XD1(51),X2(51),XD2(51),X3(51),XD3(51),X4(51),XD4(51)
  DATA F/0.,2.,4.,6.,8.,10.,12.,14.,16.,18.,20.,20.,20.,20.,
  2  20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,0.,0.,0.,
  3  0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,
  4  0./
  DO 5 I=1,51
  5  FF(I)=F(I)
    XA1=0.0912872
    OMN=27.3861
    DELT=0.01
    XK=1500.0
    NP=51

```

C END OF PROBLEM-DEPENDENT DATA

VALUE OF	METHOD #1 (FIG.4.18)	METHOD #1 (FIG.4.19)	METHOD #2 (FIG.4.20)	METHOD #3 (FIG.4.21)
1	X(I)	X(I)	X(I)	X(I)
2	0.488713E-04	0.977126E-04	0.488713E-04	0.652702E-04
3	0.237610E-03	0.426350E-03	0.237610E-03	0.316883E-03
4	0.642607E-03	0.104760E-02	0.642607E-03	0.824391E-03
5	0.132081E-02	0.199900E-02	0.132080E-02	0.163535E-02
6	0.230651E-02	0.329221E-02	0.230651E-02	0.277294E-02
7	0.360999E-02	0.491347E-02	0.360999E-02	0.423546E-02
8	0.521792E-02	0.682586E-02	0.521792E-02	0.599767E-02
9	0.709550E-02	0.897307E-02	0.709550E-02	0.801378E-02
10	0.919000E-02	0.112845E-01	0.919000E-02	0.102218E-01
:				

45	0.440657E-02	0.534203E-02	0.440660E-02	0.491147E-02
46	0.529242E-02	0.576879E-02	0.529244E-02	0.556998E-02
47	0.575031E-02	0.575478E-02	0.575032E-02	0.579144E-02
48	0.576783E-02	0.532398E-02	0.576783E-02	0.558107E-02
49	0.536517E-02	0.452545E-02	0.536516E-02	0.497521E-02
50	0.459210E-02	0.343772E-02	0.459208E-02	0.403720E-02
51	0.352290E-02	0.215312E-02	0.352286E-02	0.285144E-02

I	XD(I)	XD(I)	XD(I)	XD(I)
2	0.350266E-03	0.700531E-03	0.963263E-02	0.145198E-01
3	0.100825E-02	0.166623E-02	0.277278E-01	0.369691E-01
4	0.191111E-02	0.281397E-02	0.525573E-01	0.653292E-01
5	0.298127E-02	0.405142E-02	0.819876E-01	0.972501E-01
6	0.413276E-02	0.528425E-02	0.113655E+00	0.130237E+00
7	0.527791E-02	0.642306E-02	0.145147E+00	0.161841E+00
8	0.633377E-02	0.738963E-02	0.174185E+00	0.189831E+00
9	0.722794E-02	0.812211E-02	0.198775E+00	0.212348E+00
10	0.790330E-02	0.857866E-02	0.217348E+00	0.228023E+00
...				
45	0.395372E-02	0.234794E-02	0.108730E+00	0.870512E-01
46	0.246078E-02	0.745291E-03	0.676726E-01	0.441850E-01
47	0.861748E-03	-0.833375E-03	0.236977E-01	0.197140E-03
48	-0.721946E-03	-0.227447E-02	-0.198553E-01	-0.416597E-01
49	-0.217591E-02	-0.347992E-02	-0.598408E-01	-0.784465E-01
50	-0.340079E-02	-0.137360E-02	-0.935257E-01	-0.107736E+00
51	-0.431863E-02	-0.190573E-02	-0.118769E+00	-0.127765E+00

4.55
$$x(t) = \frac{1}{m \omega_d} \int_0^t F(\tau) e^{-\gamma \omega_n(t-\tau)} \sin \omega_d(t-\tau) d\tau$$

But $\sin \omega_d(t-\tau) = \sin \omega_d t \cos \omega_d \tau - \cos \omega_d t \sin \omega_d \tau$

$$x(t) = \{A(t) \cdot \sin \omega_d t - B(t) \cdot \cos \omega_d t\} \frac{e^{-\gamma \omega_n t}}{m \omega_d}$$

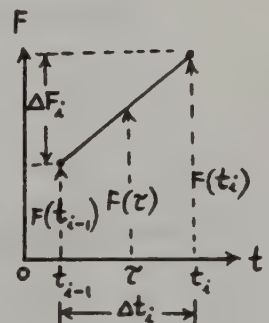
where

$$A(t) = \int_0^t F(\tau) e^{\gamma \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$B(t) = \int_0^t F(\tau) e^{\gamma \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

Let $F(\tau)$ be taken as a piecewise linear function during (t_{i-1}, t_i) as

$$F(\tau) = F_{i-1} + \left(\frac{\tau - t_{i-1}}{t_i - t_{i-1}} \right) (F_i - F_{i-1})$$



We can write

$$A(t_i) = A(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left(\frac{\tau}{\Delta t_i} \Delta F_i \right) e^{\gamma \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$+ \int_{t_{i-1}}^{t_i} \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) e^{\gamma \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$= A(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} P_1 + \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) P_2$$

where

$$P_1 = \int_{t_{i-1}}^{t_i} \tau e^{\gamma \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$P_2 = \int_{t_{i-1}}^{t_i} e^{\gamma \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$B(t_i) = B(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left(\frac{\tau}{\Delta t_i} \Delta F_i \right) e^{\gamma \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$+ \int_{t_{i-1}}^{t_i} \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) e^{\gamma \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$= B(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} P_3 + \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) P_4$$

where

$$P_3 = \int_{t_{i-1}}^{t_i} e^{\gamma \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$P_4 = \int_{t_{i-1}}^{t_i} \tau e^{\gamma \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

Assuming $A(t_1=0) = B(t_1=0) = 0$,

$$x(t_i) = \frac{e^{-\gamma \omega_n t_i}}{m \omega_d} \left[A(t_i) \sin \omega_d t_i - B(t_i) \cos \omega_d t_i \right]$$

The integrals in P_1, P_2, P_3 and P_4 can be evaluated in closed form.

The computer program and output are given.

```

C =====
C
C PROBLEM 4.55
C NUMERICAL INTEGRATION OF DUHAMEL INTEGRAL
C
C =====
C PROBLEM-DEPENDENT DATA
C   DIMENSION F(21), X(21), DELT(21), T(21), A(21), B(21)
C   NP=21
C   XAI=0.1
C   OMN=1.0

```

```

XM=1.0
DATA F/1.0,.8436,.6910,.5460,.4122,.2929,.1910,.1090,
2 .04894,.01231,.0,.0,.0,.0,.0,.0,.0,.0,.0/
DO 10 I=1,21
10 DELT(I)=0.31416
C END OF PROBLEM-DEPENDENT DATA
A(1)=0.0
B(1)=0.0
OMD=OMN*SQRT(1.0-XAI**2)
T(1)=0.0
DO 20 I=2,NP
T(I)=T(I-1)+DELT(I)
TIME=T(I)
CALL PI1(TIME,XAI,OMN,OMD,PP1)
CALL PI2(TIME,XAI,OMN,OMD,PP2)
CALL PI3(TIME,XAI,OMN,OMD,PP3)
CALL PI4(TIME,XAI,OMN,OMD,PP4)
TIME=T(I-1)
CALL PI1(TIME,XAI,OMN,OMD,PM1)
CALL PI2(TIME,XAI,OMN,OMD,PM2)
CALL PI3(TIME,XAI,OMN,OMD,PM3)
CALL PI4(TIME,XAI,OMN,OMD,PM4)
P1=PP1-PM1
P2=PP2-PM2
P3=PP3-PM3
P4=PP4-PM4
DELF=F(I)-F(I-1)
A(I)=A(I-1)+(DELF/DELT(I))*P1+(F(I-1)-T(I-1)*DELF/DELT(I))*P2
B(I)=B(I-1)+(DELF/DELT(I))*P4+(F(I-1)-T(I-1)*DELF/DELT(I))*P3
X(I)=(EXP(-XAI*OMN*T(I))/(XM*OMD))*(A(I)*SIN(OMD*T(I))-
2 B(I)*COS(OMD*T(I)))
20 CONTINUE
PRINT 30
30 FORMAT (//,2X,41H NUMERICAL EVALUATION OF DUHAMEL INTEGRAL,
2 //,5X,2H I,6X,5H T(I),10X,5H F(I),10X,5H X(I),/)
DO 40 I=2,NP
40 PRINT 50, I,T(I),F(I),X(I)
50 FORMAT (2X,I5,3E15.8)
STOP
END

C =====
C
C SUBROUTINE PI1
C
C =====
SUBROUTINE PI1 (T,XAI,OMN,OMD,P)
DEN=(XAI*OMN)**2+OMD**2
P=(T*EXP(XAI*OMN*T)/DEN)*(XAI*OMN*COS(OMD*T)+OMD*SIN(OMD*T))
2 -(EXP(XAI*OMN*T)/(DEN**2))*(((XAI*OMN)**2-OMD**2)*COS(OMD*T)
3 +2.0*XAI*OMN*OMD*SIN(OMD*T))
RETURN
END

```

```

C =====
C
C SUBROUTINE PI2
C
C =====
      SUBROUTINE PI2 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(EXP(XAI*OMN*T)/DEN)*(XAI*OMN*COS(OMD*T)+OMD*SIN(OMD*T))
      RETURN
      END
C =====
C
C SUBROUTINE PI3
C
C =====
      SUBROUTINE PI3 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(EXP(XAI*OMN*T)/DEN)*(XAI*OMN*SIN(OMD*T)-OMD*COS(OMD*T))
      RETURN
      END
C =====
C
C SUBROUTINE PI4
C
C =====
      SUBROUTINE PI4 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(T*EXP(XAI*OMN*T)/DEN)*(XAI*OMN*SIN(OMD*T)-OMD*COS(OMD*T))
      2 -(EXP(XAI*OMN*T)/(DEN**2))*(((XAI*OMN)**2-OMD**2)*SIN(OMD*T))
      3 -2.0*XAI*OMN*OMD*COS(OMD*T))
      RETURN
      END

```

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.31415999E+00	0.84359998E+00	0.45415938E-01
3	0.62831998E+00	0.69099998E+00	0.16377741E+00
4	0.94247997E+00	0.54600000E+00	0.32499743E+00
5	0.12566404E+01	0.41219997E+00	0.49747676E+00
6	0.15708008E+01	0.29290003E+00	0.65152758E+00
7	0.18849611E+01	0.19099998E+00	0.76239210E+00
8	0.21991215E+01	0.10900003E+00	0.81256396E+00
9	0.25132818E+01	0.48939999E-01	0.79324198E+00
10	0.28274422E+01	0.12309998E-01	0.70482814E+00
11	0.31416025E+01	0.00000000E+00	0.55646867E+00
12	0.34557629E+01	0.00000000E+00	0.36454141E+00
13	0.37699232E+01	0.00000000E+00	0.14971566E+00
14	0.40840836E+01	0.00000000E+00	-0.66231847E-01
15	0.43982439E+01	0.00000000E+00	-0.26274461E+00
16	0.47124043E+01	0.00000000E+00	-0.42236131E+00
17	0.50265646E+01	0.00000000E+00	-0.53218621E+00
18	0.53407249E+01	0.00000000E+00	-0.58483124E+00
19	0.56548853E+01	0.00000000E+00	-0.57879375E+00
20	0.59690456E+01	0.00000000E+00	-0.51819199E+00
21	0.62832060E+01	0.00000000E+00	-0.41212624E+00

4.56

The computer program of problem 4.55 can be used to find the relative displacement $z(t)$ of the water tank provided $-m\ddot{y}(\tau)$ is used in place of $F(\tau)$.

Here $\zeta = 0$, $\omega_n = 22.3607$ rad/sec and

$$F(\tau) = -10000 \times 9.81 \times \ddot{y}(\tau) \text{ if } \ddot{y} \text{ is in g's.}$$

The problem-dependent data for the program of problem 4.55 and the output are given below.

```

C =====
C
C PROBLEM 4.56
C
C =====
C PROBLEM-DEPENDENT DATA
      DIMENSION F(15), X(15), DELT(15), T(15), A(15), B(15)
      NP=15
      XAI=0.0
      OMN=22.3607
      XM=10000.0
      DATA F/.0, .45, -.8, -.9, -.6, -.75, -.7, .55, 1.75, 1.65, .25,
2 -1.1, -1.4, -1.05, .0/
      DO 10 I=1, 15
10  DELT(I)=0.025
      DO 11 I=1, 15
11  F(I)=-XM*9.81*F(I)
C END OF PROBLEM-DEPENDENT DATA

```

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.24999999E-01	-0.44144996E+05	-0.45271125E-03
3	0.49999997E-01	0.78480000E+05	-0.17453337E-02
4	0.74999988E-01	0.88290000E+05	0.11150776E-02
5	0.99999964E-01	0.58860004E+05	0.86095147E-02
6	0.12499994E+00	0.73575000E+05	0.17519452E-01
7	0.14999992E+00	0.68670000E+05	0.25374368E-01
8	0.17499989E+00	-0.53955000E+05	0.28478131E-01
9	0.19999987E+00	-0.17167500E+06	0.19676924E-01
10	0.22499985E+00	-0.16186494E+06	-0.42601228E-02
11	0.24999982E+00	-0.24525000E+05	-0.35448216E-01
12	0.27499980E+00	0.10791006E+06	-0.57387743E-01
13	0.29999977E+00	0.13734000E+06	-0.56341648E-01
14	0.32499975E+00	0.10300506E+06	-0.30433871E-01
15	0.34999973E+00	0.00000000E+00	0.10307007E-01

4.57

The problem-dependent data (to be used in the program of Problem 4.55) and output are given.

C PROBLEM-DEPENDENT DATA

DIMENSION F(30),X(30),DELT(30),T(30),A(30),B(30)

NP=30

XA1=0.0

QMN=8.660254

XN=2.0

DATA F /60.0,60.0,60.0,60.0,60.0,60.0,100.0,100.0,100.0,100.0,100.0,
2 30.0,30.0,30.0,30.0,30.0,30.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
3 0.0,0.0,0.0,0.0,0.0,0.0,0.0/

DO 10 I=1,30

10 DELT(I)=0.01

C END OF PROBLEM-DEPENDENT DATA

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.99999998E-02	0.60000000E+02	0.14990796E-02
3	0.20000000E-01	0.60000000E+02	0.59849964E-02
4	0.29999999E-01	0.60000000E+02	0.13424230E-01
5	0.39999999E-01	0.60000000E+02	0.23760952E-01
6	0.50000001E-01	0.10000000E+03	0.37250735E-01
7	0.60000002E-01	0.10000000E+03	0.55125091E-01
8	0.70000000E-01	0.10000000E+03	0.77583194E-01
9	0.79999998E-01	0.10000000E+03	0.10445669E+00
10	0.89999996E-01	0.10000000E+03	0.13554408E+00
11	0.99999994E-01	0.30000000E+02	0.17002963E+00
12	0.10999999E+00	0.30000000E+02	0.20532255E+00
13	0.11999999E+00	0.30000000E+02	0.24057558E+00
14	0.13000000E+00	0.30000000E+02	0.27552453E+00
15	0.14000000E+00	0.30000000E+02	0.30990744E+00
16	0.15000001E+00	0.30000000E+02	0.34346649E+00
17	0.16000001E+00	0.00000000E+00	0.37570029E+00
18	0.17000002E+00	0.00000000E+00	0.40536803E+00
19	0.18000002E+00	0.00000000E+00	0.43199742E+00
20	0.19000003E+00	0.00000000E+00	0.45538884E+00
21	0.20000003E+00	0.00000000E+00	0.47536695E+00
22	0.21000004E+00	0.00000000E+00	0.49178207E+00
23	0.22000004E+00	0.00000000E+00	0.50451112E+00
24	0.23000005E+00	0.00000000E+00	0.51345861E+00
25	0.24000005E+00	0.00000000E+00	0.51855767E+00
26	0.25000006E+00	0.00000000E+00	0.51976985E+00
27	0.26000005E+00	0.00000000E+00	0.51708633E+00
28	0.27000004E+00	0.00000000E+00	0.51052707E+00
29	0.28000003E+00	0.00000000E+00	0.50014120E+00
30	0.29000002E+00	0.00000000E+00	0.48600671E+00

4.58

(i) Find natural frequency:

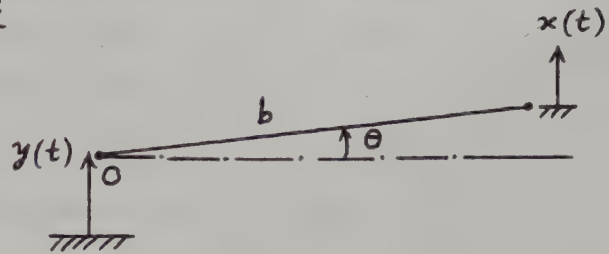
$$J_0 = m b^2 = 9 m a^2$$

$$\frac{1}{2} k_t \theta^2 = \frac{1}{2} k (a\theta)^2 = \frac{1}{2} k a^2 \theta^2 \Rightarrow k_t = k a^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{k a^2}{9 m a^2}} = \frac{1}{3} \sqrt{\frac{k}{m}} = 2\pi(10) = 20\pi \frac{\text{rad}}{\text{sec}} \quad (E_1)$$

(ii) Find equation of motion:

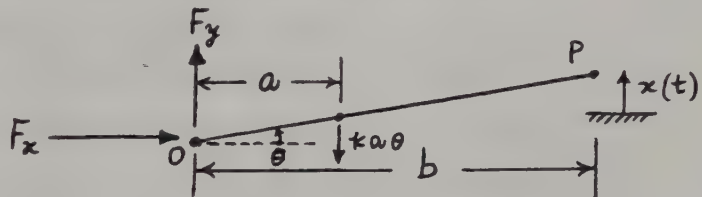
When the base and hence the pivot O is displaced by $y(t)$, the displacement of mass m , $x(t)$, is given by



$$x(t) = y(t) + b \theta(t) \quad (E_2)$$

Relative displacement of mass is

$$z(t) = x(t) - y(t) = b \theta(t) \quad (E_3)$$



Equation of motion:

$$\sum \text{Forces along } y(\text{vertical}) \text{ direction} = 0 \Rightarrow F_y - k a \theta = m \ddot{x} \quad (E_4)$$

$$\sum \text{Moments about P} = 0 \Rightarrow F_y \cdot b - k a \theta (b - a) = 0 \quad (E_5)$$

Solve (E₄) for F_y and substitute the result in (E₅):

$$(k a \theta + m \ddot{x}) b - k a \theta (b - a) = 0$$

$$\text{i.e., } m b \ddot{x} + \theta k a^2 = 0 \quad (E_6)$$

$$\text{But } x = y + z \text{ and } \theta(t) = \frac{1}{b} z(t)$$

Hence (E₆) becomes

$$m b \ddot{z} + \frac{k a^2}{b} z = -m b \ddot{y}$$

$$\text{or } m \ddot{z} + \frac{k a^2}{b^2} z = -m \ddot{y} \quad (E_7)$$

E₇ (E₇) can be compared with a standard forced vibration equation for an undamped system:

$$\ddot{x} + \frac{k}{m} x = \frac{F(t)}{m} \quad (E_8)$$

Comparison of (E₇) and (E₈) shows that

$$x = z, \quad m = m, \quad k = \frac{k a^2}{b^2} \quad \text{and} \quad F = -m \ddot{y} \quad (E_9)$$

(iii) Find solution:

Solution of Eq. (E₈) under a rectangular pulse is given in problem 4.19. Hence the solution of problem 4.19 can be used to find the relative displacement.

(iv) Design:

$$\text{Eq. (E}_1\text{) gives } \sqrt{k} = 60 \pi \sqrt{m} \quad \text{or} \quad k = 3600 \pi^2 m \quad (E_{10})$$

Following procedure can be used to solve the problem:

- Assume a value of a .
- Assume a small value of m .
- Find k from Eq. (E₁₀).
- Evaluate the solution numerically as outlined in part (iii).
- If the maximum relative displacement is larger than or equal to 0.02 m, the design is complete.
- Otherwise, increase the value of m and go to step c.
- If necessary change the value of a in step a.

4.59 Let k and m denote the equivalent stiffness and mass of the cutting head.

Equation of motion is:
$$m \ddot{x} + k x = F(t) \quad (E_1)$$
 where $F(t)$ is given by Figs. 4.34 (a) and (b).

Solution of Eq. (E₁) under the force given by Fig. 4.56 (a) can be obtained as in problem 4.15. Solution of (E₁) under the force of Fig. 4.56 (b) can be determined using a procedure similar to that of problem 4.24.

The values of k and m can be determined as follows:

- Assume a small value for m .
- Assume a small value for k .
- Evaluate the response under $F(t)$ given by Fig. 4.56 (a).

- d. Evaluate the response under $F(t)$ given by Fig. 4.56(b).
- e. If the responses in steps c and d are approximately equal to 0.1 mm and 0.05 mm, the current values of m and k are the desired values.
- f. Otherwise, increment the value of m and/or k and go to step c.

4.60

Model the system as a single d.o.f. torsional system with:

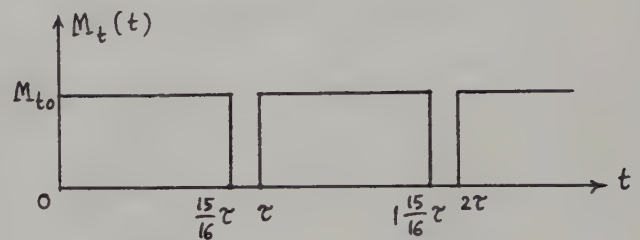
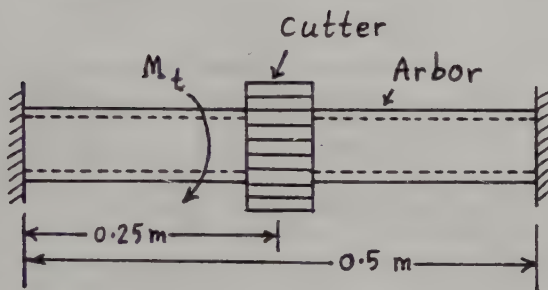
$J_0 = 0.1 \text{ N-m}^2$; $c_t = 0$ (no damping assumed) ;

$$k_t = \frac{G J}{\ell} = \frac{1}{0.25} (80 (10^9)) \left(\frac{\pi}{32} (d_o^4 - d_i^4) \right)$$

where d_o = outer diameter of shaft, and d_i = inner diameter.

$$k_t = 62.832 (10^9) (d_o^4 - d_i^4) \quad (1)$$

Torque acting on the arbor due to breakage of one tooth can be modeled as shown in figure.



$M_{t0} = 500 \text{ N-m}$, $\tau = \frac{2\pi}{\omega} = \frac{2\pi}{N(2\pi)/60} = 60/N = 60/1000 = 0.06 \text{ sec}$ where N = speed of cutter = 1000 rpm. Express $M_t(t)$ in Fourier series:

$$M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n \omega t + b_n \sin n \omega t \right) \quad (2)$$

(see solution of Problem 4.6 for procedure). where $\omega = \frac{2\pi N}{60} = 104.72 \text{ rad/sec}$.

Equation of motion:

$$J_0 \ddot{\theta} + k_t \theta = M_t(t) \quad (3)$$

where $M_t(t)$ is given by Eq. (2). Find solution of Eq. (3) and determine the maximum value of θ , $\theta_{\max}$. This value must be less than 1° .

Note:

The solution requires an iterative procedure. Assume values for d_i and d_o . Compute k_t , find solution of $\theta(t)$, and the value of $\theta_{\max}$. If $\theta_{\max}$ exceeds 1° , choose a different set of values for d_i and d_o and continue the process until $\theta_{\max}$ comes out to be less than 1° .

Chapter 5

Two Degree of Freedom Systems

5.1 Equations of motion

$$\begin{aligned} m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 &= 0 \\ m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 &= 0 \end{aligned} \quad (E_1)$$

With $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2$, Eqs. (E₁) give the frequency equation

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{vmatrix} = 0$$

$$\text{or } \omega^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right) \omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \quad (E_2)$$

Roots of Eq. (E₂) are

$$\omega_1^2, \omega_2^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2}} \quad (E_3)$$

$$\text{If } \vec{x}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} = r_1 X_1^{(1)} \end{Bmatrix} \quad \text{and} \quad \vec{x}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} = r_2 X_1^{(2)} \end{Bmatrix},$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1 \omega_1^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_1^2 + k_2} \quad (E_4)$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1 \omega_2^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_2^2 + k_2} \quad (E_5)$$

General solution of (E₁) is

$$x_1(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) \quad (E_6)$$

$$x_2(t) = r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 and ϕ_2 can be found using Eqs. (5.18).

For $m_1 = m$, $m_2 = 2m$, $k_1 = k$ and $k_2 = 2k$, (E₃) gives

$$\omega_1^2 = (2 - \sqrt{3}) \frac{k}{m}, \quad \omega_2^2 = (2 + \sqrt{3}) \frac{k}{m} \quad (E_7)$$

When $k = 1000 \text{ N/m}$ and $m = 20 \text{ kg}$,

$$\omega_1 = 3.6603 \text{ rad/sec} \quad \text{and} \quad \omega_2 = 13.6603 \text{ rad/sec}$$

$$r_1 = \frac{k_2}{-m_2 \omega_1^2 + k_2} = 1.36604, \quad r_2 = \frac{k_2}{-m_2 \omega_2^2 + k_2} = -0.36602$$

With $x_1(0) = 1$, $\dot{x}_1(0) = 0$, $x_2(0) = -1$ and $\dot{x}_2(0) = 0$, Eqs. (5.18)

give $x_1^{(1)} = -0.36602$, $x_1^{(2)} = -1.36603$, $\phi_1 = 0$, $\phi_2 = 0$

Response of the system is

$$x_1(t) = -0.36602 \cos 3.6603 t - 1.36603 \cos 13.6603 t$$

$$x_2(t) = -0.5 \cos 3.6603 t + 0.5 \cos 13.6603 t$$

5.2

Taking moments about O and mass m_1 ,

$$\begin{aligned} m_1 l_1^2 \ddot{\theta}_1 &= -W_1 (l_1 \sin \theta_1) + Q \sin \theta_2 (l_1 \cos \theta_1) \\ &\quad - Q \cos \theta_2 (l_1 \sin \theta_1) \\ &= -W_1 l_1 \theta_1 + W_2 l_1 (\theta_2 - \theta_1) \quad (E_1) \end{aligned}$$

assuming $Q \approx W_2$.

$$\begin{aligned} m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 (l_1 \ddot{\theta}_1) &= -W_2 (l_2 \sin \theta_2) \\ &= -W_2 l_2 \theta_2 \quad (E_2) \end{aligned}$$

Using the relations $\theta_1 = \frac{x_1}{l_1}$ and $\theta_2 = \frac{x_2 - x_1}{l_2}$,
Eqs. (E₁) and (E₂) become

$$m_1 l_1 \ddot{x}_1 + [W_1 + W_2 \left(\frac{l_1 + l_2}{l_2} \right)] x_1 - \frac{W_2 l_1}{l_2} x_2 = 0 \quad (E_3)$$

$$m_2 l_2 \ddot{x}_2 - W_2 x_1 + W_2 x_2 = 0 \quad (E_4)$$

When $m_1 = m_2 = m$, $l_1 = l_2 = l$ and $W_1 = W_2 = mg$, Eqs. (E₃) and (E₄) give

$$m l \ddot{x}_1 + 3 m g x_1 - m g x_2 = 0 \quad (E_5)$$

$$m l \ddot{x}_2 - m g x_1 + m g x_2 = 0$$

For harmonic motion $x_i(t) = X_i \cos \omega t$; $i = 1, 2$, Eqs. (E₅) become

$$-\omega^2 m l X_1 + 3 m g X_1 - m g X_2 = 0 \quad (E_6)$$

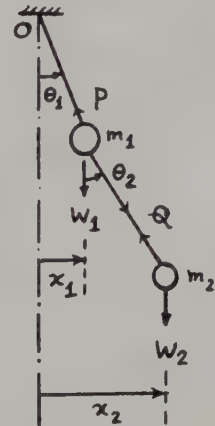
$$-\omega^2 m l X_2 - m g X_1 + m g X_2 = 0$$

from which the frequency equation can be obtained as

$$\omega^4 m^2 l^2 - (4 m^2 l g) \omega^2 + 2 m^2 g^2 = 0$$

i.e. $\omega_1^2, \omega_2^2 = (2 \mp \sqrt{2}) \frac{g}{l}$

$$\therefore \omega_1 = 0.7654 \sqrt{\frac{g}{l}}, \quad \omega_2 = 1.8478 \sqrt{\frac{g}{l}}$$



Ratio of amplitudes is given by Eq. (E6) as

$$\frac{x_1}{x_2} = \frac{mg}{-\omega^2 ml + 3mg} = \frac{1}{(-\omega^2 \frac{l}{g} + 3)}$$

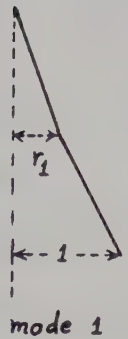
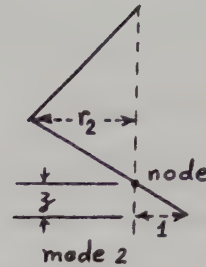
In mode 1, $\omega_1 = 0.7654 \sqrt{\frac{g}{l}}$, $r_1 = \left(\frac{x_1}{x_2}\right)^{(1)} = 0.4142$
No node.

In mode 2, $\omega_2 = 1.8478 \sqrt{\frac{g}{l}}$,

$$r_2 = \left(\frac{x_1}{x_2}\right)^{(2)} = -2.4133$$

one node located at z :

$$\frac{z}{l} = \frac{1-z}{2.4133} \quad \text{or} \quad z = 0.2930$$



5.3

Let R_1, R_2 and R_3 be the restoring forces in springs. Equations of motion of mass m in x and y directions are

$$m\ddot{x} = \sum_{i=1}^3 R_i \cos \alpha_i \quad (E_1)$$

$$m\ddot{y} = \sum_{i=1}^3 R_i \sin \alpha_i \quad (E_2)$$

$$\text{where } R_i = -k_i (x \cos \alpha_i + y \sin \alpha_i) \quad (E_3)$$

Eqs. (E1) to (E3) give

$$m\ddot{x} + \sum_{i=1}^3 k_i (x \cos^2 \alpha_i + y \sin \alpha_i \cos \alpha_i) = 0 \quad (E_4)$$

$$m\ddot{y} + \sum_{i=1}^3 k_i (x \sin \alpha_i \cos \alpha_i + y \sin^2 \alpha_i) = 0 \quad (E_5)$$

For $\alpha_1 = 45^\circ$, $\alpha_2 = 135^\circ$, $\alpha_3 = 270^\circ$ and $k_1 = k_2 = k_3 = k$, Eqs. (E4) and (E5) reduce to

$$m\ddot{x} + kx = 0 \quad (E_6)$$

$$m\ddot{y} + 2ky = 0 \quad (E_7)$$

These equations are uncoupled. For harmonic motion,

$x(t) = X \cos(\omega t + \phi)$, $y(t) = Y \cos(\omega t + \phi)$, and hence

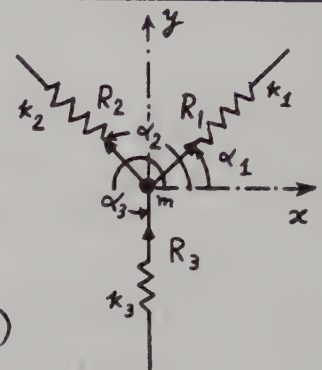
$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{for motion in } x \text{ direction}$$

$$\omega_2 = \sqrt{\frac{2k}{m}} \quad \text{for motion in } y \text{ direction}$$

Natural modes are given by $x(t) = X \cos\left(\sqrt{\frac{k}{m}} t + \phi_1\right)$

$$y(t) = Y \cos\left(\sqrt{\frac{2k}{m}} t + \phi_2\right)$$

where X, ϕ_1, Y and ϕ_2 can be determined from initial conditions.



5.4

Equations of motion in terms of x and θ :

$$m\ddot{x} + k_1(x - l_1\theta) + k_2(x + l_2\theta) = 0 \quad (E_1)$$

$$J_0\ddot{\theta} - k_1l_1(x - l_1\theta) + k_2l_2(x + l_2\theta) = 0 \quad (E_2)$$

For free vibration,

$$x(t) = X \cos(\omega t + \phi) \quad (E_3)$$

$$\theta(t) = \Theta \cos(\omega t + \phi) \quad (E_4)$$

and Eqs. (E₁) and (E₂) become

$$\begin{bmatrix} -m\omega^2 + k_1 + k_2 & -(k_1l_1 - k_2l_2) \\ -(k_1l_1 - k_2l_2) & -J_0\omega^2 + k_1l_1^2 + k_2l_2^2 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_5)$$

Frequency equation is

$$\begin{vmatrix} -m\omega^2 + k_1 + k_2 & -(k_1l_1 - k_2l_2) \\ -(k_1l_1 - k_2l_2) & -J_0\omega^2 + k_1l_1^2 + k_2l_2^2 \end{vmatrix} = 0 \quad (E_6)$$

i.e.,

$$\begin{vmatrix} -\omega^2 + 5000 & 100 \\ 100 & -0.3\omega^2 + 2030 \end{vmatrix} = 0$$

i.e.,

$$0.3\omega^4 - 3530\omega^2 + 10.14 \times 10^6 = 0$$

i.e.,

$$\omega^2 = 6785.3373, \quad 4981.3293$$

$$\therefore \omega_1 = 70.5785 \text{ rad/sec}, \quad \omega_2 = 82.3732 \text{ rad/sec}$$

Mode shapes:

$$(-1000\omega_1^2 + 5 \times 10^6)X + 0.1 \times 10^6\Theta = 0$$

$$\text{or} \quad \left. \frac{X}{\Theta} \right|_{\omega_1} = \frac{-0.1 \times 10^6}{-1000\omega_1^2 + 5 \times 10^6} = -5.3476$$

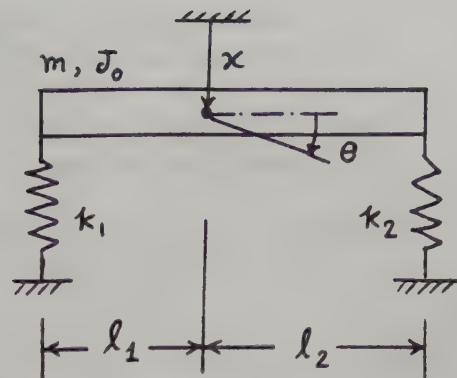
and

$$\left. \frac{X}{\Theta} \right|_{\omega_2} = \frac{-0.1 \times 10^6}{-1000\omega_2^2 + 5 \times 10^6} = 0.05601$$

5.5

$$k_1 = \text{stiffness of girder} = \frac{48EI}{\ell^3} = \frac{48(6(10^{12}))}{(30(12))^3} = 6.1728(10^8) \text{ lb/in}$$

$$k_2 = \text{stiffness of rope} = \frac{AE}{\ell} = \frac{A(30(10^6))}{(20)(12)} = 12.5(10^4) A \text{ lb/in}$$



$$m_1 = \text{mass of trolley} = 8000/386.4 = 20.7039 \text{ lb-sec}^2/\text{in}$$

$$m_2 = \text{mass of load} = 2000/386.4 = 5.1760 \text{ lb-sec}^2/\text{in}$$

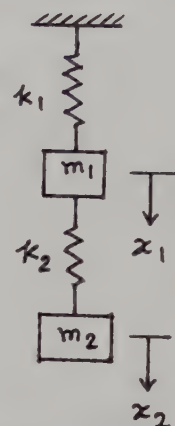
Desired frequency value: $\omega_1 > 20 \text{ Hz}$. Let $\omega_1 = 25 \text{ Hz} = 157.08 \text{ rad/sec}$ or $\omega_1^2 = 24674.1264 \text{ (rad/sec)}^2$.

Fundamental natural frequency is given by (see Eq. (E3) in solution of Problem 5.1):

$$\omega_1^2 = \frac{k_1 + k_2}{2 m_1} + \frac{k_2}{2 m_2} - \sqrt{\frac{1}{4} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2}} \quad (1)$$

Using the known values of k_1 , m_1 , and m_2 , a series of trial values of A are given and Eq. (1) is evaluated to find the corresponding values of ω_1^2 . The results are given in the table below. It can be seen that $A = 1.1 \text{ in}^2$ yields a frequency that satisfies the specification.

$A \text{ (in}^2\text{)}$	$k_2 \text{ (lb/in)}$	$\omega_1^2 \text{ (rad/sec)}^2$	$\omega_1 \text{ (rad/sec)}$
0.6	$0.7500 (10^5)$	$0.1434 (10^5)$	119.7
0.7	$0.8750 (10^5)$	$0.1715 (10^5)$	131.0
0.8	$0.1000 (10^6)$	$0.1946 (10^5)$	139.5
0.9	$0.1125 (10^6)$	$0.2176 (10^5)$	147.5
1.0	$0.1250 (10^6)$	$0.2406 (10^5)$	155.1
1.1	$0.1375 (10^6)$	$0.2662 (10^5)$	163.2
1.2	$0.1500 (10^6)$	$0.2918 (10^5)$	170.8
1.3	$0.1625 (10^6)$	$0.3123 (10^5)$	176.7
1.4	$0.1750 (10^6)$	$0.3379 (10^5)$	183.8
1.5	$0.1875 (10^6)$	$0.3610 (10^5)$	190.0



5.6

$$k_1 = \frac{48 EI}{l^3} = \frac{48 (2.06 \times 10^{11}) (0.02)}{(40)^3} = 3.09 \times 10^6 \text{ N/m}$$

$$k_2 = 0.3 \times 10^6 \text{ N/m}, \quad m_1 = 1000 \text{ kg}, \quad m_2 = 5000 \text{ kg}$$

Eq. (E3) in the solution of problem 5.1 gives

$$\omega_1^2, \omega_2^2 = \frac{3.39 \times 10^6}{2000} + \frac{0.3 \times 10^6}{10000} \mp \sqrt{\frac{1}{4} \left(\frac{3.39 \times 10^6}{1000} + \frac{0.3 \times 10^6}{5000} \right)^2 - \frac{3.09 \times 0.3 \times 10^{12}}{5 \times 10^6}}$$

$$= (1.725 \mp 1.6704) \times 10^3$$

$$\omega_1 = 7.3892 \text{ rad/s}, \quad \omega_2 = 58.2701 \text{ rad/s}$$

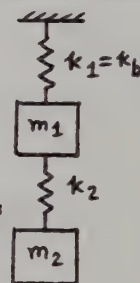
From Eqs. (E4) and (E5) of solution of problem 5.1,

$$r_1 = \frac{k_2}{-m_2 \omega_1^2 + k_2} = \frac{0.3 \times 10^6}{-5000 (54.6003) + 0.3 \times 10^6} = 11.1117$$

$$r_2 = \frac{k_2}{-m_2 \omega_2^2 + k_2} = \frac{0.3 \times 10^6}{-5000 (3395.4046) + 0.3 \times 10^6} = -0.01799$$

Mode shapes are

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 11.1117 \end{Bmatrix} X_1^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -0.01799 \end{Bmatrix} X_1^{(2)}$$



5.7

Frequency equation:

$$\left| \left[-\omega^2 [m] + [k] \right] \right| = 0$$

or

$$\begin{vmatrix} (k_{11} - \omega^2 m_1) & k_{12} \\ k_{21} & (k_{22} - \omega^2 m_2) \end{vmatrix} = 0 \quad (1)$$

Expansion of the determinantal equation (1) gives:

$$(m_1 m_2) \omega^4 - (m_1 k_{22} + m_2 k_{11}) \omega^2 + (k_{11} k_{22} - k_{12}^2) = 0 \quad (2)$$

Roots of Eq. (2):

$$\omega_2^2, \omega_1^2 = \frac{(m_1 k_{22} + m_2 k_{11}) \pm \sqrt{(m_1 k_{22} - m_2 k_{11})^2 + 4 m_1 m_2 k_{12}^2}}{2 m_1 m_2} \quad (3)$$

Substitution of known expressions for k_{11} , k_{12} , and k_{22} into Eq. (3) yields:

$$\omega_2^2, \omega_1^2 = \frac{48}{7} \frac{EI}{m_1 m_2} \left[(m_1 + 8 m_2) \pm \sqrt{(m_1 - 8 m_2)^2 + 25 m_1 m_2} \right] \quad (4)$$

5.8

Equations of motion :

$$\left. \begin{aligned} m_1 \ddot{x}_1 + k_1 x_1 - k_1 x_2 &= 0 \\ m_2 \ddot{x}_2 + (k_1 + k_2) x_2 - k_1 x_1 &= 0 \end{aligned} \right\} (E_1)$$

$$\text{Let } x_i(t) = X_i \cos(\omega t + \phi); \quad i = 1, 2 \quad (E_2)$$

Eq. (E₁) becomes

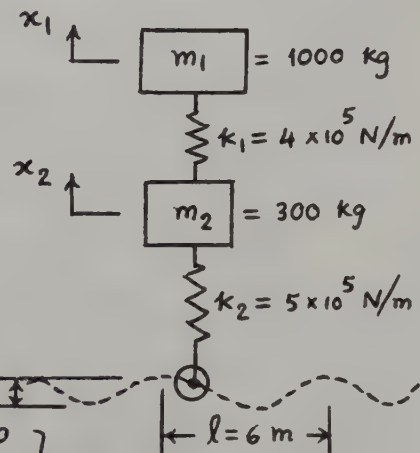
$$\begin{bmatrix} -m_1 \omega^2 + k_1 & -k_1 \\ -k_1 & -m_2 \omega^2 + k_1 + k_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\begin{vmatrix} -m_1 \omega^2 + k_1 & -k_1 \\ -k_1 & -m_2 \omega^2 + k_1 + k_2 \end{vmatrix} = 0$$

i.e.,

$$m_1 m_2 \omega^4 - (m_1 k_1 + m_1 k_2) \omega^2 - k_1 m_2 \omega^2 + k_1 k_2 = 0$$



i.e.,

$$\omega^2 = \left[(m_1 k_1 + m_1 k_2 + m_2 k_1) \pm (m_1^2 k_1^2 + m_1^2 k_2^2 + m_2^2 k_1^2 + 2 m_1^2 k_1 k_2 - 2 m_1 m_2 k_1 k_2 + 2 m_1 m_2 k_1^2)^{\frac{1}{2}} \right] / 2 m_1 m_2 \quad (E_3)$$

Since $m_1 = 1000 \text{ kg}$, $m_2 = 300 \text{ kg}$, $k_1 = 4 \times 10^5 \text{ N/m}$ and $k_2 = 5 \times 10^5 \text{ N/m}$,
Eq. (E₃) gives

$$\omega_1 = 14.4539 \text{ rad/sec}, \quad \omega_2 = 56.4897 \text{ rad/sec}$$

$$f_1 = \frac{14.4539}{2\pi} \text{ Hz} = \frac{s_1 (1000)}{3600} \left(\frac{1}{l} \right) = \frac{s_1}{21.6}$$

where $l = 6 \text{ m}$ and s_1 is in km/hr .

$$\therefore s_1 = \text{critical velocity \# 1} = \frac{14.4539 (21.6)}{2\pi} = 49.6887 \text{ km/hr}$$

$$f_2 = \frac{56.4897}{2\pi} \text{ Hz} = \frac{s_2 (1000)}{3600} \left(\frac{1}{l} \right) = \frac{s_2}{21.6}$$

$$\therefore s_2 = \text{critical velocity \# 2} = \frac{56.4897 (21.6)}{2\pi} = 194.1968 \text{ km/hr.}$$

5.9

Equations of motion for rotation about O and A give

$$\begin{aligned} m_1 l_1^2 \ddot{\theta}_1 &= -W_1 l_1 \sin \theta_1 + Q \sin \theta_2 (l_1 \cos \theta_1) - Q \cos \theta_2 (l_1 \sin \theta_1) \\ &= -W_1 l_1 \theta_1 + Q l_1 (\theta_2 - \theta_1) \\ &= -W_1 l_1 \theta_1 + W_2 l_1 (\theta_2 - \theta_1) \quad \text{---- (E}_1\text{)} \\ &\text{since } Q \approx W_2. \end{aligned}$$

$$\begin{aligned} m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 (l_1 \ddot{\theta}_1) &= -W_2 l_2 \sin \theta_2 \\ &= -W_2 l_2 \theta_2 \quad \text{---- (E}_2\text{)} \end{aligned}$$

For $m_1 = m_2 = m$ and $l_1 = l_2 = l$, Eqs. (E₁) and (E₂) become

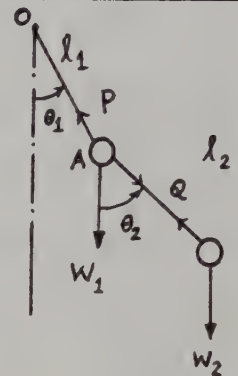
$$m l \ddot{\theta}_1 + 2 m g \theta_1 - m g \theta_2 = 0$$

$$m l \ddot{\theta}_1 + m l \ddot{\theta}_2 + m g \theta_2 = 0$$

Assuming $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i = 1, 2$, we get

$$\begin{bmatrix} -\omega^2 m l + 2 m g & -m g \\ -\omega^2 m l & -\omega^2 m l + m g \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{---- (E}_3\text{)}$$

Defining $\lambda = \frac{\omega^2 m l}{m g} = \frac{\omega^2 l}{g}$, frequency equation can be obtained as

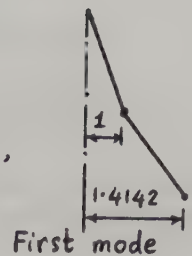
$$\begin{vmatrix} -\lambda + 2 & -1 \\ -\lambda & -\lambda + 1 \end{vmatrix} = \lambda^2 - 4\lambda + 2 = 0$$


$$\lambda_1 = 2 - \sqrt{2} = 0.5858, \quad \lambda_2 = 2 + \sqrt{2} = 3.4142$$

$$\omega_1 = 0.7654 \sqrt{\frac{g}{l}}, \quad \omega_2 = 1.8478 \sqrt{\frac{g}{l}}$$

For ω_1 , first equation in (E₃) gives for $\Theta_1 = 1$,

$$\Theta_2 = -\lambda_1 + 2 = \sqrt{2} = 1.4142$$

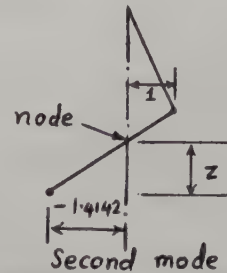


For ω_2 , first equation in (E₃) gives for $\Theta_1 = 1$,

$$\Theta_2 = -\lambda_2 + 2 = -\sqrt{2} = -1.4142$$

Location of node:

$$\frac{z}{1.4142} = \frac{1-z}{1} ; \quad z = 0.4142$$



5.10

Eg. (E₃) in the solution of problem 5.1 gives

$$\omega_1^2, \omega_2^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

For $m_1 = m_2 = m$ and $k_1 = k_2 = k$, we get

$$\omega_1^2, \omega_2^2 = \frac{k}{m} + \frac{k}{2m} \mp \sqrt{\frac{1}{4} \left(\frac{3k}{m} \right)^2 - \frac{k^2}{m^2}} = \frac{3k}{2m} \mp \frac{k}{m} \sqrt{\frac{5}{4}}$$

$$\omega_1^2 = 0.382 \frac{k}{m}, \quad \omega_2^2 = 2.618 \frac{k}{m}$$

$$\omega_1 = 0.6181 \sqrt{\frac{k}{m}}, \quad \omega_2 = 1.6180 \sqrt{\frac{k}{m}}$$

Mode shapes are given by (see Egs. (E₄) and (E₅) of problem 5.1)

$$r_1 = \frac{k_2}{-m_2 \omega_1^2 + k_2} = 1.6181 ; \quad \vec{x}^{(1)} = \begin{Bmatrix} 1 \\ 1.6181 \end{Bmatrix} x_1^{(1)}$$

$$r_2 = \frac{k_2}{-m_2 \omega_2^2 + k_2} = -0.6180 ; \quad \vec{x}^{(2)} = \begin{Bmatrix} 1 \\ -0.6180 \end{Bmatrix} x_1^{(2)}$$

5.11

For the system of Fig. 5.3(a),

$$\begin{aligned} \vec{x}^{(1)T} [m] \vec{x}^{(2)} &= \begin{Bmatrix} x_1^{(1)} & x_2^{(1)} \end{Bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{Bmatrix} = x_1^{(1)} x_1^{(2)} \{1 \quad r_1\} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} 1 \\ r_2 \end{Bmatrix} \\ &= x_1^{(1)} x_1^{(2)} (m_1 + m_2 r_1 r_2) \end{aligned}$$

But

$$m_1 + m_2 r_1 r_2 = m_1 + m_2 \left(\frac{k_2}{-m_2 \omega_1^2 + k_2 + k_3} \right) \left(\frac{k_2}{-m_2 \omega_2^2 + k_2 + k_3} \right) \equiv \frac{N}{D}$$

where

$$N = m_1 (-m_2 \omega_1^2 + k_2 + k_3) (-m_2 \omega_2^2 + k_2 + k_3) + m_2 k_2^2, \text{ and}$$

$$D = (-m_2 \omega_1^2 + k_2 + k_3) (-m_2 \omega_2^2 + k_2 + k_3)$$

By substituting

$$\omega_1^2, \omega_2^2 = \frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{2m_1m_2} \mp \frac{1}{2} \left\{ \left[\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1m_2} \right]^2 - 4 \frac{(k_1 + k_2)(k_2 + k_3) - k_2^2}{m_1m_2} \right\}^{1/2}, \text{ we get}$$

$$\begin{aligned} N &= m_1 m_2^2 \omega_1^2 \omega_2^2 - m_1 m_2 k_2 \omega_2^2 - m_1 m_2 k_3 \omega_2^2 - m_1 m_2 k_2 \omega_1^2 \\ &\quad + m_1 k_2^2 + m_1 k_2 k_3 - m_1 m_2 k_3 \omega_1^2 + m_1 k_2 k_3 + m_1 k_3^2 + m_2 k_2^2 \\ &= m_1 m_2^2 \left\{ \left[\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{2m_1m_2} \right]^2 - \frac{1}{4} \left[\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1m_2} \right]^2 \right. \\ &\quad \left. - 4 \left(\frac{(k_1 + k_2)(k_2 + k_3) - k_2^2}{m_1m_2} \right) \right\} - m_1 m_2 k_2 \left\{ \frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1m_2} \right\} \\ &\quad - m_1 m_2 k_3 \left\{ \frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1m_2} \right\} + m_1 k_2^2 + m_2 k_2^2 + 2m_1 k_2 k_3 + m_1 k_3^2 \\ &= 0 \end{aligned}$$

5.12 Eq. (5.10) gives

$$\begin{aligned} \omega_1^2, \omega_2^2 &= \frac{1}{2} \left\{ \frac{(800)1 + (700)2}{2} \right\} \mp \frac{1}{2} \left[\left\{ \frac{(800)1 + (700)2}{2} \right\}^2 - 4 \left\{ \frac{(800)(700) - (500)^2}{2} \right\} \right]^{1/2} \\ &= 550 \mp 384.0573 = 165.9427, 934.0573 \end{aligned}$$

$$\omega_1 = 12.8819 \text{ rad/s}, \quad \omega_2 = 30.5624 \text{ rad/s}$$

5.13 Eq. (E₃) in the solution of problem 5.1 gives

$$\omega_1^2, \omega_2^2 = \frac{8000}{2} + \frac{6000}{2} \mp \sqrt{\frac{1}{4} \left(\frac{8000}{1} + \frac{6000}{1} \right)^2 - \frac{12 \times 10^6}{1}} = 917.2, 13082.8$$

$$\omega_1 = 30.2853 \text{ rad/sec}, \quad \omega_2 = 114.3801 \text{ rad/s}$$

Eqs. (E₄) and (E₅) of solution of problem 5.1 give

$$r_1 = \frac{k_2}{-m_2 \omega_1^2 + k_2} = \frac{6000}{-917.2 + 6000} = 1.1805; \quad \vec{x}^{(1)} = \begin{Bmatrix} 1 \\ 1.1805 \end{Bmatrix} x_1^{(1)}$$

$$r_2 = \frac{k_2}{-m_2 \omega_2^2 + k_2} = \frac{6000}{-13082.8 + 6000} = -0.8471; \quad \vec{x}^{(2)} = \begin{Bmatrix} 1 \\ -0.8471 \end{Bmatrix} x_1^{(2)}$$

5.14 Same as example 5.1 with $m_1 = m_2 = 25 \text{ kg}$ and $k_1 = k_2 = k_3 = 50000 \text{ N/m}$

$$\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{50000}{25}} = 44.7214 \text{ rad/s}$$

$$\omega_2 = \sqrt{\frac{3k}{m}} = \sqrt{\frac{150000}{25}} = 77.4597 \text{ rad/s}$$

General motion is given by Eq. (E₁₀) of Example 5.1:

$$x_1(t) = x_1^{(1)} \cos(44.7214t + \phi_1) + x_1^{(2)} \cos(77.4597t + \phi_2)$$

$$x_2(t) = x_1^{(1)} \cos(44.7214t + \phi_1) - x_1^{(2)} \cos(77.4597t + \phi_2)$$

5.15

Eg. (5.10) gives

$$\omega_1^2, \omega_2^2 = \frac{1}{2} \left\{ \frac{3000(2) + 4000(1)}{2} \right\} \pm \frac{1}{2} \left[\left\{ \frac{3000(2) + 4000(1)}{2} \right\}^2 - 4 \left\{ \frac{3000(4000) - 1000^2}{2} \right\} \right]^{\frac{1}{2}}$$

$$= 1633.9746, 3366.0254$$

$$\omega_1 = 40.4225 \text{ rad/s}, \quad \omega_2 = 58.0175 \text{ rad/s}$$

$$r_1 = \frac{k_2}{-m_2 \omega_1^2 + k_2 + k_3} = \frac{1000}{-2(1633.9746) + 4000} = 1.3660$$

$$r_2 = \frac{k_2}{-m_2 \omega_2^2 + k_2 + k_3} = \frac{1000}{-2(3366.0254) + 4000} = -0.3660$$

When $x_1(0) = x_2(0) = \dot{x}_2(0) = 0$ and $\dot{x}_1(0) = 20 \text{ m/s}$,

$$x_1^{(1)} = \frac{1}{(-0.366 - 1.366)} \left[\frac{+0.366(20)}{40.4225} \right] = -0.1046$$

$$x_1^{(2)} = \frac{1}{-1.732} \left[\frac{1.366(20)}{58.0175} \right] = -0.2719$$

$$\phi_1 = \phi_2 = \tan^{-1}(\infty) = \frac{\pi}{2}$$

Motion of the two masses is given by Eg. (5.15):

$$x_1(t) = 0.1046 \sin 40.4225t + 0.2719 \sin 58.0175t$$

$$x_2(t) = +(1.3660)(0.1046) \sin 40.4225t - (-0.3660)(-0.2719) \sin 58.0175t$$

$$= 0.1429 \sin 40.4225t - 0.09952 \sin 58.0175t$$

5.16

$$(a) \quad \omega_1^2 = 917.2, \quad \omega_2^2 = 13082.8, \quad r_1 = 1.1805, \quad r_2 = -0.8471$$

$$x_1(0) = 0.2, \quad x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$$

Eg. (5.18) gives

$$x_1^{(1)} = \frac{1}{(-0.8471 - 1.1805)} [(-0.8471)(0.2)] = 0.08356$$

$$x_1^{(2)} = \frac{1}{(-0.8471 - 1.1805)} [(-1.1805)(0.2)] = 0.11644$$

$$\phi_1 = \phi_2 = \tan^{-1}(0) = 0$$

$$x_1(t) = 0.08356 \cos 30.2853t + 0.11644 \cos 114.3801t$$

$$x_2(t) = (1.1805)(0.08356) \cos 30.2853t + (-0.8471)(0.11644) \cos 114.3801t$$

$$= 0.09864 \cos 30.2853t - 0.09864 \cos 114.3801t$$

$$(b) \quad x_1(0) = 0.2, \quad x_2(0) = \dot{x}_1(0) = 0, \quad \dot{x}_2(0) = 5.0$$

Eg. (5.18) gives

$$x_1^{(1)} = \frac{1}{-2.0276} \left[\{(-0.8471)(0.2)\}^2 + \frac{(5)^2}{917.2} \right]^{\frac{1}{2}} = -0.1167$$

$$x_1^{(2)} = \frac{1}{-2.0276} \left[\{-1.1805(0.2)\}^2 + \frac{(-5)^2}{13082.8} \right]^{1/2} = -0.1184$$

$$\phi_1 = \tan^{-1} \left\{ \frac{5.0}{30.2853 (-0.8471)(0.2)} \right\} = \tan^{-1}(-0.9745) = 135.7399^\circ$$

$$\phi_2 = \tan^{-1} \left\{ \frac{-5.0}{114.3801 (-1.1805)(0.2)} \right\} = \tan^{-1}(0.1851) = 10.4895^\circ$$

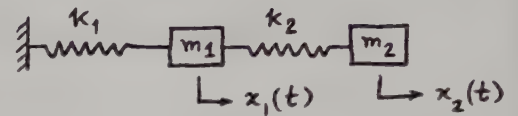
$$x_1(t) = -0.1167 \cos(30.2853t + 135.7399^\circ) - 0.1184 \cos(114.3801t + 10.4895^\circ)$$

$$\begin{aligned} x_2(t) &= (1.1805)(-0.1167) \cos(30.2853t + 135.7399^\circ) \\ &\quad - 0.8471(-0.1184) \cos(114.3801t + 10.4895^\circ) \\ &= -0.1378 \cos(30.2853t + 135.7399^\circ) \\ &\quad + 0.1003 \cos(114.3801t + 10.4895^\circ) \end{aligned}$$

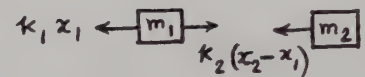
5.17

Equivalent system is shown in figure:

$$k_i = 2 \left(\frac{24 EI_i}{h_i^3} \right); \quad i = 1, 2$$



$$k_1 = k_2 = k = \frac{48 EI}{h^3}; \quad m_1 = 2m, \quad m_2 = m$$



Equations of motion:

$$\begin{aligned} m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 &= 0 \\ m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 &= 0 \end{aligned}$$

For harmonic motion $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2$, we get

$$\begin{bmatrix} (-\omega^2 m_1 + k_1 + k_2) & -k_2 \\ -k_2 & (-\omega^2 m_2 + k_2) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{--- (E}_1\text{)}$$

Frequency equation is

$$\begin{vmatrix} (-\omega^2 m_1 + k_1 + k_2) & -k_2 \\ -k_2 & (-\omega^2 m_2 + k_2) \end{vmatrix} = 0$$

$$\text{or} \quad \omega^4 m_1 m_2 - \omega^2 (m_2 k_1 + m_2 k_2 + m_1 k_2) + k_1 k_2 = 0$$

$$\omega^2 = \frac{(m_2 k_1 + m_2 k_2 + m_1 k_2) \pm \sqrt{(m_2 k_1 + m_2 k_2 + m_1 k_2)^2 - 4 m_1 m_2 k_1 k_2}}{2 m_1 m_2} \quad \text{--- (E}_2\text{)}$$

For given data,

$$\omega^2 = \frac{(mk + mk + 2mk) \pm \sqrt{(mk + mk + 2mk)^2 - 8m^2 k^2}}{4m^2} = \frac{k}{m} \left(1 \pm \frac{1}{\sqrt{2}} \right)$$

$$\omega_1 = 0.5412 \sqrt{\frac{k}{m}} = 3.7495 \sqrt{\frac{EI}{mh^3}}; \quad \omega_2 = 1.3066 \sqrt{\frac{k}{m}} = 9.0524 \sqrt{\frac{EI}{mh^3}}$$

From Eq. (E₁), we get

$$r_1 = \frac{x_2^{(1)}}{x_1^{(1)}} = \frac{-\omega_1^2 m_1 + k_1 + k_2}{k_2} = \frac{-2m\omega_1^2 + 2k}{k} = \frac{-2(0.2929k) + 2k}{k} = 1.4142$$

$$r_2 = \frac{x_2^{(2)}}{x_1^{(2)}} = \frac{-\omega_2^2 m_1 + k_1 + k_2}{k_2} = \frac{-2m\omega_2^2 + 2k}{k} = \frac{-2(1.7071k) + 2k}{k} = -1.4142$$

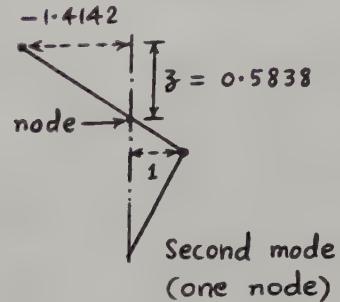
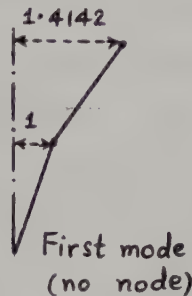
Mode shapes are:

$$\vec{x}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.4142 \end{Bmatrix} x_1^{(1)}$$

$$\vec{x}^{(2)} = \begin{Bmatrix} 1.0 \\ -1.4142 \end{Bmatrix} x_1^{(2)}$$

Location of node:

$$\frac{z}{1.4142} = \frac{1-z}{1} ; z = 0.5838$$



5.18

Let P be the tension in the string. Horizontal components of tension (along x_1 direction) in the string lying above and below m_1 are $-\frac{x_1 P}{l_1}$ and $-\frac{(x_1 - x_2) P}{l_2}$, respectively.

Newton's second law gives

$$m_1 \ddot{x}_1 = -\frac{x_1 P}{l_1} - \frac{(x_1 - x_2) P}{l_2} \quad \text{or} \quad m_1 \ddot{x}_1 + \frac{x_1 P}{l_1} + \left(\frac{x_1 - x_2}{l_2}\right) P = 0$$

similarly

$$m_2 \ddot{x}_2 + \frac{x_2 P}{l_3} - \left(\frac{x_1 - x_2}{l_2}\right) P = 0$$

With $x_i(t) = X_i \cos(\omega t + \phi)$; $i=1,2$, and $l_1 = l_2 = l_3 = l$, $m_1 = m_2 = m$,

$$\begin{bmatrix} \left(-m\omega^2 + \frac{2P}{l}\right) & -\frac{P}{l} \\ -\frac{P}{l} & \left(-m\omega^2 + \frac{2P}{l}\right) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{--- (E}_1\text{)}$$

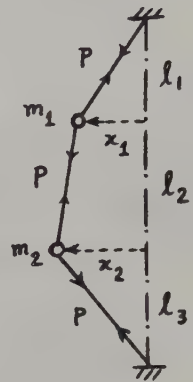
This gives the frequency equation

$$\left(-m\omega^2 + \frac{2P}{l}\right)^2 - \left(\frac{P}{l}\right)^2 = \left(-m\omega^2 + \frac{3P}{l}\right)\left(-m\omega^2 + \frac{P}{l}\right) = 0$$

$$\therefore \omega_1 = \sqrt{\frac{P}{ml}}, \quad \omega_2 = \sqrt{\frac{3P}{ml}}$$

From first of Eqs. (E₁),

$$r_1 = \frac{x_2^{(1)}}{x_1^{(1)}} = \frac{-m\omega_1^2 + \frac{2P}{l}}{\left(\frac{P}{l}\right)} = 1 ; \quad r_2 = \frac{x_2^{(2)}}{x_1^{(2)}} = \frac{-m\omega_2^2 + \frac{2P}{l}}{\left(\frac{P}{l}\right)} = -1$$



mode shapes are:

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} X_1^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(2)}$$

No node

one node at middle of the two masses

5.19

For $m_1 = 3m$, $m_2 = m$, $k_1 = 3k$ and $k_2 = k$, Eq. (E₂) in solution of problem 5.17 gives

$$\omega^2 = \frac{(3mk + mk + 3mk) \pm \sqrt{(3mk + mk + 3mk)^2 - 36k^2m^2}}{6m^2} = \frac{k}{6m} (7 \pm \sqrt{13})$$

$$\omega^2 = 0.5657 \frac{k}{m}, \quad 1.7676 \frac{k}{m}$$

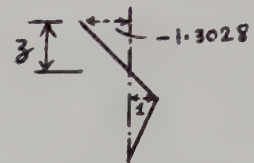
$$\omega_1 = 0.7521 \sqrt{\frac{k}{m}}, \quad \omega_2 = 1.3295 \sqrt{\frac{k}{m}}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-\omega_1^2 m_1 + k_1 + k_2}{k_2} = \frac{-0.5657(3) + 3 + 1}{1} = 2.3029$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-\omega_2^2 m_1 + k_1 + k_2}{k_2} = \frac{-1.7676(3) + 3 + 1}{1} = -1.3028$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 2.3029 \end{Bmatrix} X_1^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -1.3028 \end{Bmatrix} X_1^{(2)}$$

No node one node



$$\frac{z}{1.3028} = \frac{1-z}{1}; \quad z = 0.5657$$

5.20

$$k_1 = \frac{3EI}{b^3} = \frac{3E \left(\frac{1}{12} at^3 \right)}{b^3} = \frac{E at^3}{4b^3}$$

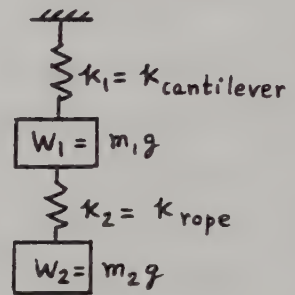
$$k_2 = \frac{AE}{l} = \frac{\pi d^2 E}{4l}$$

From problem 5.1,

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

$$= \left(\frac{E at^3}{4b^3} + \frac{\pi d^2 E}{4l} \right) \frac{g}{2W_1} + \frac{\pi d^2 E g}{8l W_2}$$

$$\mp \sqrt{\left[\frac{1}{4} \left\{ \left(\frac{E at^3}{4b^3} + \frac{\pi d^2 E}{4l} \right) \frac{g}{W_1} + \frac{\pi d^2 E g}{4l W_2} \right\}^2 - \frac{E^2 at^3 \pi d^2 g^2}{16 l b^3 W_1 W_2} \right]}$$



5.21

From solution of Problem 5.1, we find that for $m_1 = m$, $m_2 = 2m$, $k_1 = k$ and $k_2 = 2k$,

$$\omega_1^2 = (2 - \sqrt{3}) \frac{k}{m}; \quad \omega_2^2 = (2 + \sqrt{3}) \frac{k}{m}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-m_2 \omega_1^2 + k_2} = \frac{1}{-1 + \sqrt{3}}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{k_2}{-m_2 \omega_2^2 + k_2} = \frac{1}{-1 - \sqrt{3}}$$

First mode shape:

$$\begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ \left(\frac{X_1^{(1)}}{\sqrt{3} - 1} \right) \cos(\omega_1 t + \phi_1) \end{Bmatrix}$$

For the motion to be identical with the first normal mode, we need to have $X_1^{(2)} = 0$. This requires that (from Eq. (5.18)):

$$\frac{1}{r_2 - r_1} \left[\left\{ -r_1 x_1(0) + x_2(0) \right\}^2 + \frac{1}{\omega_2^2} \left\{ r_1 \dot{x}_1(0) - \dot{x}_2(0) \right\}^2 \right]^{\frac{1}{2}} = 0$$

or

$$\begin{aligned} x_2(0) &= r_1 x_1(0) = \frac{x_1(0)}{\sqrt{3} - 1} \\ \dot{x}_2(0) &= r_1 \dot{x}_1(0) = \frac{\dot{x}_1(0)}{\sqrt{3} - 1} \end{aligned}$$

5.22

Let $m_1 = m$, $m_2 = 2m$, $k_1 = k$, $k_2 = 2k$.

Initial conditions: $x_1(0) = 0$, $x_2(0) = 0.1$ m, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$

Eqs. (5.18) yield:

$$\begin{aligned} X_1^{(1)} &= \frac{1}{r_2 - r_1} \left[(0 - 0.1)^2 \right]^{\frac{1}{2}} = \frac{0.1}{r_2 - r_1} = \frac{0.1}{\left(\frac{-1}{\sqrt{3} + 1} - \frac{1}{\sqrt{3} - 1} \right)} = -\frac{0.1}{\sqrt{3}} \\ X_1^{(2)} &= \frac{1}{r_2 - r_1} \left[(0.1)^2 \right]^{\frac{1}{2}} = \frac{0.1}{r_2 - r_1} = -\frac{0.1}{\sqrt{3}} \\ \phi_1 &= \tan^{-1}(0) = 0 \\ \phi_2 &= \tan^{-1}(0) = 0 \end{aligned}$$

where ω_1 and ω_2 are given by Eq. (E3) of solution of Problem 5.1.

Resulting motion:

$$\begin{aligned} x_1(t) &= X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) = -\frac{0.1}{\sqrt{3}} \left\{ \cos \omega_1 t + \cos \omega_2 t \right\} \\ x_2(t) &= r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ &= \left(\frac{1}{\sqrt{3} - 1} \right) \left(-\frac{0.1}{\sqrt{3}} \right) \cos \omega_1 t + \left(-\frac{1}{\sqrt{3} + 1} \right) \left(-\frac{0.1}{\sqrt{3}} \right) \cos \omega_2 t \end{aligned}$$

$$= -\frac{0.1}{\sqrt{3}} \left[\left(\frac{1}{\sqrt{3}-1} \right) \cos \omega_1 t - \left(\frac{1}{\sqrt{3}+1} \right) \cos \omega_2 t \right]$$

5.23 $\omega_1^2 \geq (2\pi \times 10)^2 = 3947.8602 \text{ (rad/sec)}^2$

From solution of problem 5.20, this inequality can be expressed as:

$$\omega_1^2 = \left(\frac{E a t^3}{4 b^3} + \frac{\pi d^2 E}{4 l} \right) \frac{g}{2 W_1} + \frac{\pi d^2 E g}{8 l W_2} - \sqrt{\left[\frac{1}{4} \left\{ \left(\frac{E a t^3}{4 b^3} + \frac{\pi d^2 E}{4 l} \right) \frac{g}{W_1} + \frac{\pi d^2 E g}{4 l W_2} \right\}^2 - \frac{E^2 a t^3 \pi d^2 g^2}{16 l b^3 W_1 W_2} \right]}$$

$$\geq 3947.8602 \quad (E_1)$$

Data: $E = 30 \times 10^6 \text{ psi}$, $W_1 = 1000 \text{ lb}$, $W_2 = 500 \text{ lb}$, $g = 386.4 \text{ in/s}^2$,
 $b = 30 \text{ in}$, $l = 60 \text{ in}$.

Unknowns: a , t , d .

Let $a = 10 t$ and $d = t$.

For this data, t is incremented from 0.1 in in increments of 0.01 in and the left hand side of the inequality (E_1) is evaluated. This gives a value of $t = 1.54 \text{ in}$ for satisfying (E_1) .

$\therefore$ Design is $t = 1.54''$, $d = t = 1.54''$, $a = 10 t = 15.4''$

5.24 With $k_{t1} = k_t$, $k_{t2} = 2 k_t$, $J_1 = J_0$, $J_2 = 2 J_0$, $k_{t3} = 0$ and $M_{t1} = M_{t2} = 0$, Eqs. (5.20) give

$$J_0 \ddot{\theta}_1 + 3 k_t \theta_1 - 2 k_t \theta_2 = 0$$

$$2 J_0 \ddot{\theta}_2 - 2 k_t \theta_1 + 2 k_t \theta_2 = 0$$

For harmonic solution, $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$, $i = 1, 2$,

$$\begin{bmatrix} (-\omega^2 J_0 + 3 k_t) & -2 k_t \\ -2 k_t & (-2 \omega^2 J_0 + 2 k_t) \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

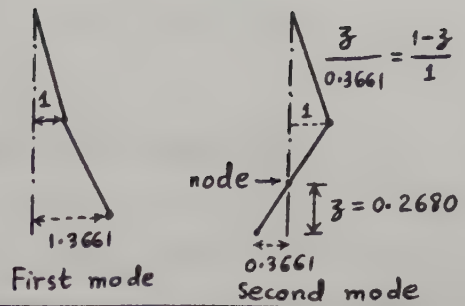
Frequency equation is

$$\begin{vmatrix} -\omega^2 J_0 + 3 k_t & -2 k_t \\ -2 k_t & -2 \omega^2 J_0 + 2 k_t \end{vmatrix} = 2 J_0^2 \omega^4 - 8 J_0 k_t \omega^2 + 2 k_t^2 = 0$$

$$\omega^2 = (2 \mp \sqrt{3}) \frac{k_t}{J_0} ; \quad \omega_1 = 0.5176 \sqrt{\frac{k_t}{J_0}}, \quad \omega_2 = 1.9319 \sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-J_0 \omega_1^2 + 3k_t}{2k_t} = 1.3661$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-J_0 \omega_2^2 + 3k_t}{2k_t} = -0.3661$$



5.25 Equation of motion of mass m : $m \ddot{x} = -k_2 (x - r\theta) \dots (E_1)$

Equation of motion of cylinder of mass m_0 and mass moment of inertia $J_0 = \frac{1}{2} m_0 r^2$:

$$J_0 \ddot{\theta} = -k_1 r^2 \theta - k_2 (r\theta - x)r \dots (E_2)$$

For $x(t) = X \cos(\omega t + \phi)$ and $\theta(t) = \Theta \cos(\omega t + \phi)$, Eqs. (E1) and (E2) give the frequency equation

$$\begin{vmatrix} -m\omega^2 + k_2 & -k_2 r \\ -k_2 r & -\frac{1}{2} m_0 r^2 \omega^2 + k_1 r^2 + k_2 r^2 \end{vmatrix} = 0$$

$$\text{i.e. } \omega^4 - \omega^2 \left(\frac{k_2}{m} + \frac{2\{k_1 + k_2\}}{m_0} \right) + \frac{2k_1 k_2}{m_0 m} = 0$$

$$\omega_1^2, \omega_2^2 = \frac{k_2}{2m} + \frac{(k_1 + k_2)}{m_0} \mp \sqrt{\frac{1}{4} \left(\frac{k_2}{m} + \frac{2k_1}{m_0} + \frac{2k_2}{m_0} \right)^2 - \frac{2k_1 k_2}{m m_0}}$$

5.26 For $J_1 = J_0$, $J_2 = 2J_0$, $k_{t1} = k_{t2} = k_{t3} = k_t$, and $M_{t1} = M_{t2} = 0$, Eqs. (5.20) give

$$J_0 \ddot{\theta}_1 + 2k_t \theta_1 - k_t \theta_2 = 0$$

$$2J_0 \ddot{\theta}_2 - k_t \theta_1 + 2k_t \theta_2 = 0$$

For harmonic motion, these equations give

$$\begin{bmatrix} -\omega^2 J_0 + 2k_t & -k_t \\ -k_t & -2\omega^2 J_0 + 2k_t \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

from which the frequency equation can be obtained as

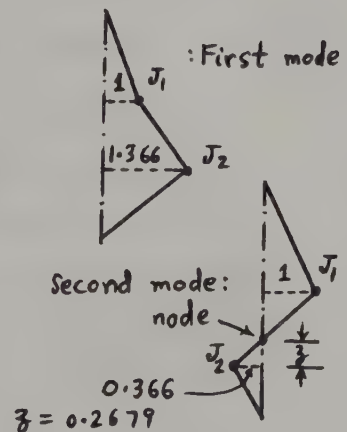
$$2J_0^2 \omega^4 - 6J_0 k_t \omega^2 + 3k_t^2 = 0$$

$$\omega_1^2, \omega_2^2 = \frac{1}{2} (3 \mp \sqrt{3}) \frac{k_t}{J_0}$$

$$\therefore \omega_1 = 0.79623 \sqrt{\frac{k_t}{J_0}}; \quad \omega_2 = 1.53819 \sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\theta_2^{(1)}}{\theta_1^{(1)}} = \frac{-\omega_1^2 J_0 + 2k_t}{k_t} = 1.36603$$

$$r_2 = \frac{\theta_2^{(2)}}{\theta_1^{(2)}} = \frac{-\omega_2^2 J_0 + 2k_t}{k_t} = -0.36603$$



5.27

Eqs. (5.20) give

$$J_0 \ddot{\theta}_1 + 6k_t \theta_1 - 5k_t \theta_2 = 0$$

$$5J_0 \ddot{\theta}_2 - 5k_t \theta_1 + 5k_t \theta_2 = 0$$

These equations can be expressed as, for harmonic motion,

$$\begin{bmatrix} -\omega^2 J_0 + 6k_t & -5k_t \\ -5k_t & -5\omega^2 J_0 + 5k_t \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$J_0^2 \omega^4 - 7k_t J_0 \omega^2 + k_t^2 = 0$$

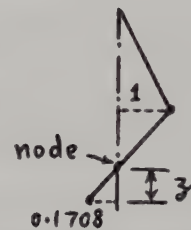
$$\omega_1^2, \omega_2^2 = \frac{k_t}{J_0} \left(\frac{7}{2} \mp \frac{1}{2} \sqrt{45} \right) = 0.1459 \frac{k_t}{J_0}, 6.8541 \frac{k_t}{J_0}$$

$$\omega_1 = 0.38197 \sqrt{\frac{k_t}{J_0}}, \quad \omega_2 = 2.61803 \sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\theta_2^{(1)}}{\theta_1^{(1)}} = \frac{-\omega_1^2 J_0 + 6k_t}{5k_t} = 1.1708$$

$$r_2 = \frac{\theta_2^{(2)}}{\theta_1^{(2)}} = \frac{-\omega_2^2 J_0 + 6k_t}{5k_t} = -0.1708$$

First mode



Second mode

$$\frac{z}{0.1708} = \frac{1-z}{1}$$

$$z = 0.1459$$

5.28

(i) Using $x(t)$ and $\theta(t)$:

For translatory motion:

$$m \ddot{x} = -k_1(x-l_1\theta) - c_1(\dot{x}-l_1\dot{\theta}) - k_2(x+l_2\theta) - c_2(\dot{x}+l_2\dot{\theta}) \quad \text{--- (E}_1\text{)}$$

For rotational motion about C.G.:

$$J_0 \ddot{\theta} = k_1(x-l_1\theta)l_1 + c_1(\dot{x}-l_1\dot{\theta})l_1 - k_2(x+l_2\theta)l_2 - c_2(\dot{x}+l_2\dot{\theta})l_2 \quad \text{--- (E}_2\text{)}$$

Eqs. (E₁) and (E₂) can be rewritten as

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_1l_1+c_2l_2 \\ -c_1l_1+c_2l_2 & c_1l_1^2+c_2l_2^2 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_1l_1+k_2l_2 \\ -k_1l_1+k_2l_2 & k_1l_1^2+k_2l_2^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(ii) Using $y(t)$ and $\theta(t)$:

For translatory motion:

$$m\ddot{y} = -\kappa_1(y - l'_1\theta) - c_1(\dot{y} - l'_1\dot{\theta}) - \kappa_2(y + l'_2\theta) - c_2(\dot{y} + l'_2\dot{\theta}) - me\ddot{\theta} \quad \text{---- (E}_3\text{)}$$

For rotational motion:

$$J_p\ddot{\theta} = \kappa_1(y - l'_1\theta)l'_1 + c_1(\dot{y} - l'_1\dot{\theta})l'_1 - \kappa_2(y + l'_2\theta)l'_2 - c_2(\dot{y} + l'_2\dot{\theta})l'_2 - me\ddot{y} \quad \text{---- (E}_4\text{)}$$

Eqs. (E₃) and (E₄) can be rewritten as

$$\begin{bmatrix} m & me \\ me & J_p \end{bmatrix} \begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_1 l'_1 + c_2 l'_2 \\ -c_1 l'_1 + c_2 l'_2 & c_1 l'^2_1 + c_2 l'^2_2 \end{bmatrix} \begin{Bmatrix} \dot{y} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} \kappa_1 + \kappa_2 & -\kappa_1 l'_1 + \kappa_2 l'_2 \\ -\kappa_1 l'_1 + \kappa_2 l'_2 & \kappa_1 l'^2_1 + \kappa_2 l'^2_2 \end{bmatrix} \begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

5.29

For small angular deflections, equations of motion are

$$m_1 l_1^2 \ddot{\theta}_1 = -W_1 l_1 \sin \theta_1 + \kappa (l_2 \theta_2 - l_1 \theta_1) l_1 \cos \theta_1$$

$$m_2 l_2^2 \ddot{\theta}_2 = -W_2 l_2 \sin \theta_2 - \kappa (l_2 \theta_2 - l_1 \theta_1) l_2 \cos \theta_2$$

or

$$m_1 l_1^2 \ddot{\theta}_1 + \theta_1 (W_1 l_1 + \kappa l_1^2) - \kappa l_1 l_2 \theta_2 = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + \theta_2 (W_2 l_2 + \kappa l_2^2) - \kappa l_1 l_2 \theta_1 = 0$$

For harmonic motion,

$\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i = 1, 2$, we get

$$\begin{bmatrix} -\omega^2 m_1 l_1^2 + W_1 l_1 + \kappa l_1^2 & -\kappa l_1 l_2 \\ -\kappa l_1 l_2 & -\omega^2 m_2 l_2^2 + W_2 l_2 + \kappa l_2^2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\omega^4 (m_1 m_2 l_1^2 l_2^2) - \omega^2 [m_2 l_2^2 (W_1 l_1 + \kappa l_1^2) + m_1 l_1^2 (W_2 l_2 + \kappa l_2^2)] + [W_1 l_1 W_2 l_2 + W_2 l_2 \kappa l_1^2 + W_1 l_1 \kappa l_2^2] = 0 \quad \text{---- (E}_1\text{)}$$

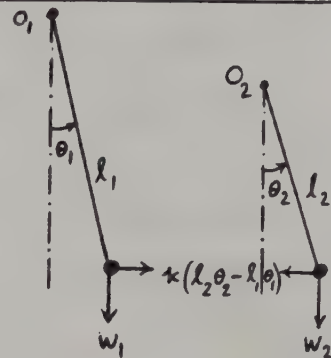
Roots of this equation give the natural frequencies ω_1 and ω_2 .

Amplitude ratios are given by

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 m_1 l_1^2 + W_1 l_1 + \kappa l_1^2}{\kappa l_1 l_2}$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 m_1 l_1^2 + W_1 l_1 + \kappa l_1^2}{\kappa l_1 l_2}$$

where $W_1 = m_1 g$, $W_2 = m_2 g$.



5.30

Equations of motion:

$$4ml^2 \ddot{\theta} = -kl\theta \cdot l - k(l\theta + x)l$$

$$m \ddot{x} = -kx - k(l\theta + x)$$

i.e. $4ml^2 \ddot{\theta} + 2kl^2 \theta + klx = 0$

$$m \ddot{x} + 2kx + kl\theta = 0$$

For harmonic motion, these equations give

$$\begin{bmatrix} -4ml^2\omega^2 + 2kl^2 & kl \\ kl & -m\omega^2 + 2k \end{bmatrix} \begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$4m^2\omega^4 - 10km\omega^2 + 3k^2 = 0$$

$$\omega^2 = \frac{k}{m} \left(\frac{5}{4} \mp \frac{\sqrt{13}}{4} \right) = 0.3486 \frac{k}{m}, 2.1514 \frac{k}{m}$$

$$\omega_1 = 0.5904 \sqrt{\frac{k}{m}}, \quad \omega_2 = 1.4668 \sqrt{\frac{k}{m}}$$

Amplitude ratios are

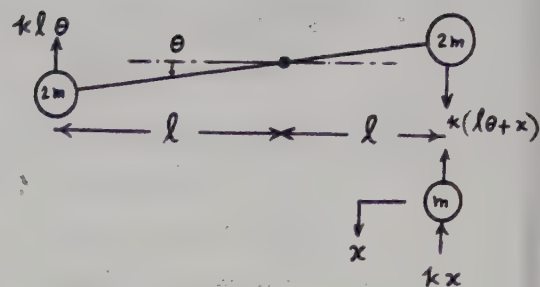
$$r_1 = \frac{x^{(1)}}{\theta^{(1)}} = \frac{-4ml^2\omega_1^2 + 2kl^2}{-kl} = -0.6056l$$

$$r_2 = \frac{x^{(2)}}{\theta^{(2)}} = \frac{-4ml^2\omega_2^2 + 2kl^2}{-kl} = 6.6056l$$

Mode shapes are

$$\vec{X}^{(1)} = \begin{Bmatrix} \theta^{(1)} \\ x^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.6056l \end{Bmatrix} \theta^{(1)}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} \theta^{(2)} \\ x^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 6.6056l \end{Bmatrix} \theta^{(2)}$$



5.31

Equations of motion:

$$m(\ddot{x} - e\ddot{\theta}) = -kx$$

$$J_{CG} \ddot{\theta} = -k_t \theta - kxe$$

i.e. $m \ddot{x} + kx - me \ddot{\theta} = 0$

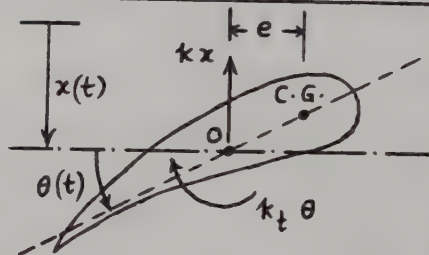
$$(J_0 - me^2) \ddot{\theta} + k_t \theta + kex = 0$$

For harmonic motion, we get the frequency equation as

$$\begin{vmatrix} -m\omega^2 + k & me\omega^2 \\ ke & -(J_0 - me^2)\omega^2 + k_t \end{vmatrix} = 0$$

$$\text{or } (J_0 - me^2)m\omega^4 - (J_0k + m k_t)\omega^2 + k k_t = 0$$

Roots of this equation give the natural frequencies of the system.



5.32 Speed becomes unfavorable when it is related to l as

$$v \tau_n = l$$

$$\text{i.e., } v = \frac{l}{\tau_n} = l f_n = \frac{l \omega_n}{2\pi}$$

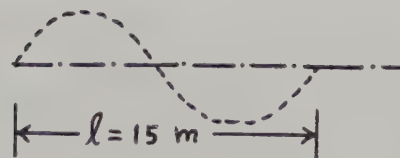
Example 5.6 gives

$$\omega_1 = 5.8593 \text{ rad/sec (bouncing)}$$

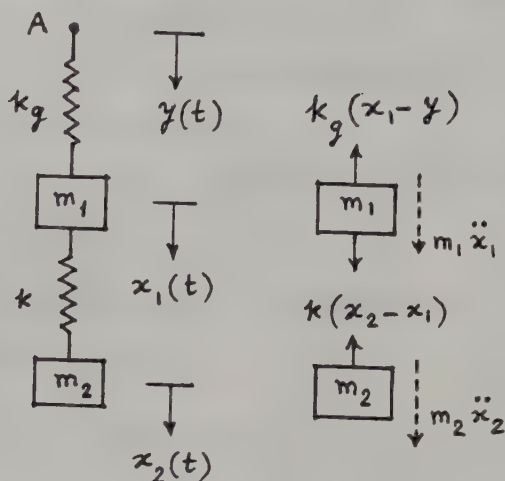
$$\omega_2 = 9.4341 \text{ rad/sec (pitching)}$$

$$\therefore v_1 = \frac{l \omega_1}{2\pi} = \frac{15 (5.8593)}{2\pi} = 13.9880 \text{ m/s (bouncing)}$$

$$v_2 = \frac{l \omega_2}{2\pi} = \frac{15 (9.4341)}{2\pi} = 22.5222 \text{ m/s (pitching)}$$



5.33



Equations of motion:

$$\begin{aligned} m_1 \ddot{x}_1 + (k_g + k) x_1 - k x_2 &= k_g y \\ m_2 \ddot{x}_2 - k x_1 + k x_2 &= 0 \end{aligned}$$

Free body diagrams of masses

Since velocity of crane in z -direction = $30 \text{ ft/min} = 0.5 \text{ ft/sec}$, τ = time to complete one cycle = $10/0.5 = 20 \text{ sec}$, and $\omega = \frac{2\pi}{\tau} = \frac{2\pi}{20} = 0.31416 \text{ rad/sec}$.

Base motion for m_1 (girder motion due to unevenness of rails):

$$y(t) = Y \sin \omega t$$

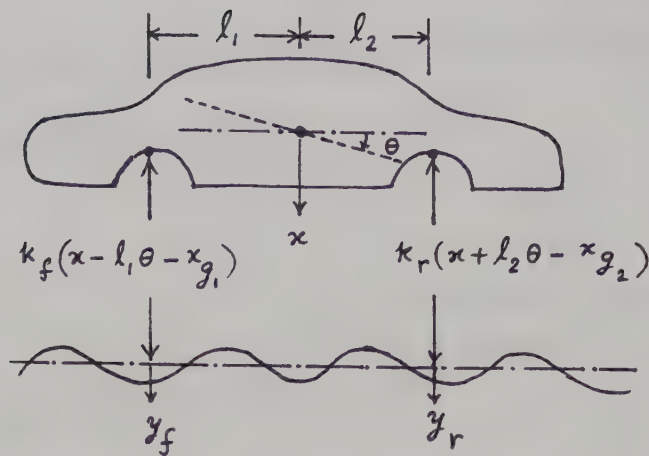
where $Y = 2 \text{ in}$ and $\omega = 0.31416 \text{ rad/sec}$.

5.34 Road surface varies sinusoidally with amplitude, $Y = 0.05 \text{ m}$ and wavelength, $d = 10 \text{ m}$. If v = velocity of automobile (m/sec), time to travel one wave length = $\tau = d/v$ sec. $\tau = 10/v \text{ sec}$, $\omega = \frac{2\pi}{\tau} = \frac{2\pi v}{10} \text{ rad/sec}$.

$$v = 50 \text{ km/hr} = (50 (10^3)) / (60 (60)) = 13.8889 \text{ m/sec,}$$

$$J_0 = m r_g^2 = 1000 (0.9)^2 = 810 \text{ kg-m}^2.$$

Equations of motion:



y_f (y_r) = ground or base displacement
of front (rear) wheels, downwards

For motion along x :

$$m \ddot{x} + x(k_f + k_r) + \theta(k_r \ell_2 - k_f \ell_1) = k_f y_f + k_r y_r \quad (1)$$

For motion along θ :

$$J_0 \ddot{\theta} + x(\ell_2 k_r - \ell_1 k_f) + \theta(k_r \ell_2^2 + k_f \ell_1^2) = k_r \ell_2 y_r - k_f \ell_1 y_f \quad (2)$$

where the ground (base) motions can be expressed as

$$y_f(t) = Y \sin \omega t = 0.05 \sin \frac{2\pi v}{10} t \text{ m} \quad (3)$$

$$y_r(t) = Y \sin(\omega t - \phi) = 0.05 \sin \left(\frac{2\pi v}{10} t - \frac{2\pi(\ell_1 + \ell_2)}{d} \right) \text{ m} \quad (4)$$

For given data, Eqs. (1) and (2) take the form:

$$1000 \ddot{x} + 40(10^3)x + 15000\theta = 900 \sin 8.7267 t + 1100 \sin(8.7267 t - 1.5708) \quad (5)$$

$$810 \ddot{\theta} + 15000x + 67500\theta = 1650 \sin(8.7267 t - 1.5708) - 900 \sin 8.7267 t \quad (6)$$

5.35 Natural frequencies are given by:

$$\left[-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \right] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (1)$$

where m_1 = mass of pulley = $\frac{200}{386.4}$ lb-sec²/in ; m_2 = mass of motor = $\frac{500}{386.4}$ lb-sec²/in

X_1 = amplitude of pulley, X_2 = amplitude of motor,

$$\frac{EI}{\ell^3} = \frac{(30(10^8)) \left(\frac{\pi}{64} (2^4) \right)}{(90^3)} = 32.3210 \text{ lb/in}$$

Frequency equation becomes:

$$\begin{vmatrix} (-\omega^2 m_1 + k_{11}) & k_{12} \\ k_{12} & (-\omega^2 m_2 + k_{22}) \end{vmatrix} = 0$$

or

$$(m_1 m_2) \omega^4 - (k_{11} m_2 + k_{22} m_1) \omega^2 + (k_{11} k_{22} - k_{12}^2) = 0 \quad (2)$$

From known data, Eq. (2) can be expressed as:

$$0.6698 \omega^4 - 11563.2894 \omega^2 + 7.3108 (10^6) = 0 \quad (3)$$

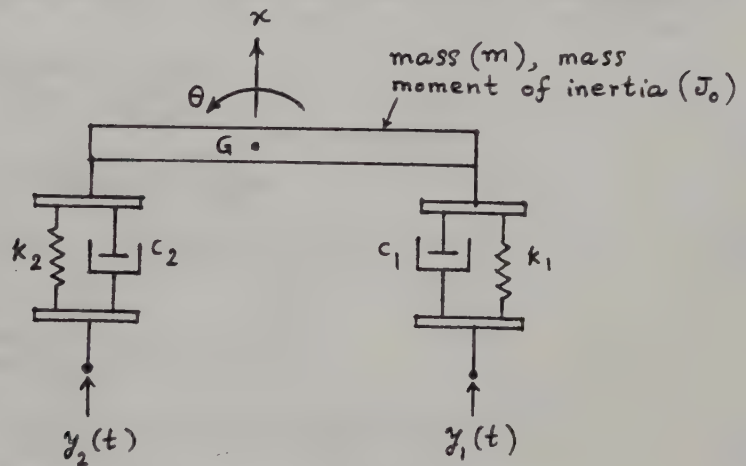
Roots of Eq. (3):

$$\omega^2 = 657.26642, 16606.5300 \quad (4)$$

or

$$\omega_1 = 25.6372 \text{ rad/sec}, \omega_2 = 128.8663 \text{ rad/sec}$$

5.36



1. Model the bicycle and the rider as a two d.o.f system as shown in figure.
2. Find the equivalent stiffness (k_1) and damping coefficient (c_1) of the front wheel in the vertical direction.
3. Find the equivalent stiffness (k_2) and damping coefficient (c_2), if applicable, of the rear wheel in the vertical direction.
4. Describe the road roughness under the wheels as $y_1(t)$ and $y_2(t)$.
5. Derive the equations of motion of the system subjected to base excitation.
6. Solve the resulting system of equations to find the steady state response.

5.37

(a)

Choose unknown coordinates as $x(t)$ and $\theta(t)$. Equations of motion:

$$m \ddot{x} = -k(x - \ell \theta/2) - 2k(x + \ell \theta/3) + F(t)$$

$$J_0 \ddot{\theta} = k(x - \ell \theta/2)(\ell/2) - 2k(x + \ell \theta/3)(\ell/3) + F(t)(\ell/3)$$

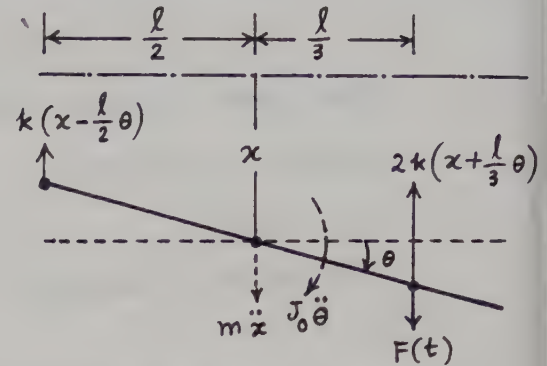
or

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 3k & k\ell/6 \\ k\ell/6 & 17k\ell^2/36 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} F(t) \\ \ell F(t)/3 \end{Bmatrix}$$

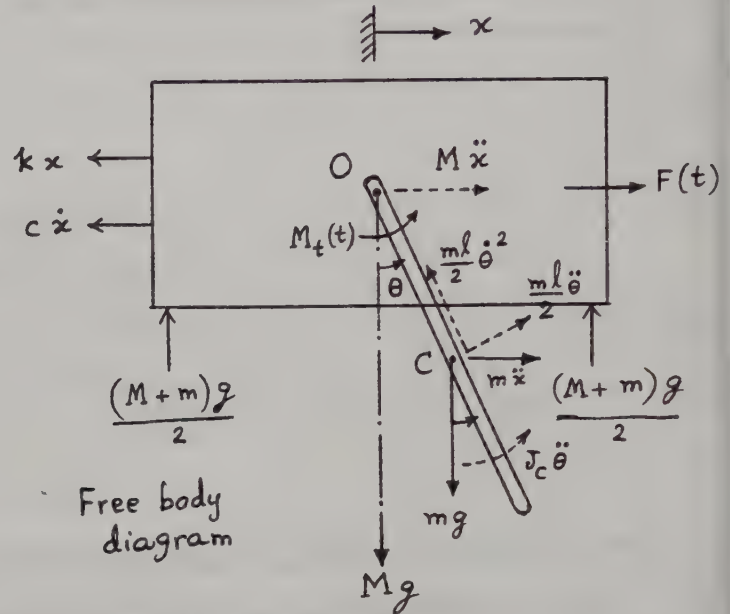
where $J_0 = \frac{m\ell^2}{12}$ and $F(t) = F_0 \sin \omega t$.

(b)

Elastic or static coupling.



5.38



Equations of motion with coordinates $x(t)$ and $\theta(t)$:

For motion along x :

$$M \ddot{x} = -kx - c\dot{x} - (m\ell/2) \ddot{\theta} \cos \theta - m\ddot{x} + (m\ell/2) \dot{\theta}^2 \sin \theta + F(t) \quad (1)$$

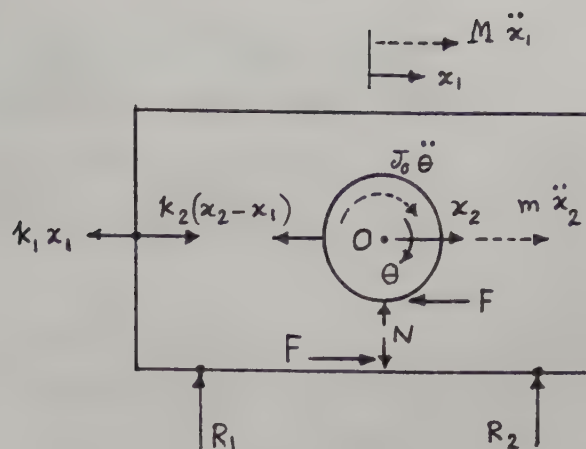
For rotation about O:

$$J_c \ddot{\theta} + (m\ell/2) \ddot{\theta} (\ell/2) + m\ddot{x} (\ell \cos \theta/2) = -mg(\ell/2) \sin \theta + M_t(t) \quad (2)$$

Using $J_c = \frac{1}{12} m \ell^2$, $\cos \theta \approx 1$ and $\sin \theta \approx \theta$ and neglecting the nonlinear term involving $\dot{\theta}^2$ in Eq. (1), Eqs. (1) and (2) can be rewritten in matrix form as:

$$\begin{bmatrix} (M+m) & m\ell/2 \\ m\ell/2 & (J_c + m\ell^2/4) \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & mg\ell/2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} F(t) \\ M_t(t) \end{Bmatrix}$$

5.39



Free body diagram

N = normal reaction between cylinder and trailer, F = friction force, R_1, R_2 = reactions between trailer and ground.

Equation of motion for linear motion of cylinder:

$$\sum F = m \ddot{x}_2 \quad \text{or} \quad m \ddot{x}_2 = -F - k_2 (x_2 - x_1) \quad (1)$$

Equation of motion for rotational motion of cylinder:

$$\sum M_O = J_O \ddot{\theta} \quad \text{or} \quad J_O \ddot{\theta} = F r \quad (2)$$

where $J_O = \frac{1}{2} m r^2$ and $\theta = \frac{x_2 - x_1}{r}$.

Equation of motion for linear motion of trailer:

$$\sum F = M \ddot{x}_1 \quad \text{or} \quad M \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1) + F \quad (3)$$

Eq. (2) gives

$$F = \frac{J_O \ddot{\theta}}{r} = \frac{1}{r} \left(\frac{1}{2} m r^2 \right) \left(\frac{\ddot{x}_2 - \ddot{x}_1}{r} \right) = \frac{m}{2} (\ddot{x}_2 - \ddot{x}_1) \quad (4)$$

Substitution of Eq. (4) into Eqs. (1) and (3) yields the equations of motion as:

$$\frac{3m}{2} \ddot{x}_2 - \frac{1}{2} m \ddot{x}_1 - k_2 x_1 + k_2 x_2 = 0 \quad (5)$$

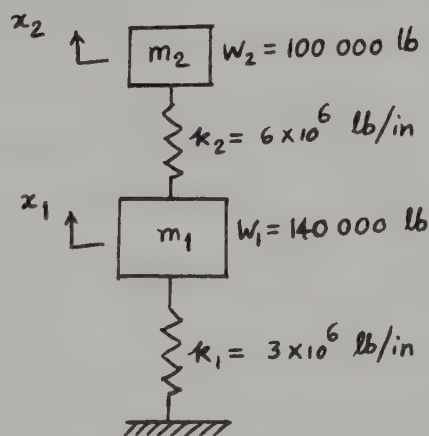
$$\left(M + \frac{m}{2} \right) \ddot{x}_1 - \frac{m}{2} \ddot{x}_2 + x_1 (k_1 + k_2) - k_2 x_2 = 0 \quad (6)$$

5.40 (a) Natural frequencies of the system:

From problem 5.1,

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2}$$

$$\mp \sqrt{\frac{1}{4} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$



$$\frac{\omega_{1,2}^2}{g} = \frac{9 \times 10^6}{28 \times 10^4} + \frac{6 \times 10^6}{20 \times 10^4}$$

$$\mp \sqrt{\frac{1}{4} \left(\frac{9 \times 10^6}{14 \times 10^4} + \frac{6 \times 10^6}{10 \times 10^4} \right)^2 - \frac{18 \times 10^{12}}{140 \times 10^8}}$$

or

$$\omega_1 = 66.3408 \text{ rad/sec}$$

$$\omega_2 = 208.8557 \text{ rad/sec}$$

(b) Initial conditions of the system:

Let v_2 = initial velocity of anvil and frame just after the impact of tup

From conservation of momentum principle,

momentum of tup plus momentum of anvil just before impact
= momentum of tup plus momentum of anvil just after impact

$$\text{i.e., } m_{\text{tup}} v_{\text{tup}} + m_{\text{anvil}} (0) = m_{\text{tup}} v_0 + m_{\text{anvil}} v_2 \quad (E_1)$$

Where v_0 = velocity of rebound of tup after impact

Also,

$$\text{coefficient of restitution } (e) = \frac{\text{relative velocity after impact}}{\text{relative velocity before impact}}$$

$$\text{i.e., } e = \frac{v_2 - v_0}{v_{\text{tup}}} \quad \text{or} \quad v_0 = v_2 - e v_{\text{tup}} \quad (E_2)$$

From Eqs. (E₁) and (E₂),

$$v_2 = \frac{m_{\text{tup}} v_{\text{tup}} (1+e)}{m_{\text{tup}} + m_{\text{anvil}}} \quad (E_3)$$

For given data,

$$v_2 = \frac{5000 (180) (1+0.5)}{105000} = 12.8571 \text{ in/sec}$$

$\therefore$ Initial conditions are:

$$x_1(0) = 0, \quad \dot{x}_1(0) = 0$$

$$x_2(0) = 0, \quad \dot{x}_2(0) = 12.8571 \text{ in/sec}$$

(C) Displacements of anvil and foundation block:

We can use results of section 5.3 with $k_3 = 0$.

$$r_1 = \frac{x_2^{(1)}}{x_1^{(1)}} = \frac{-m_1 \omega_1^2 + k_1 + k_2}{k_2} = \frac{-\frac{140000}{386.4} (4401.096) + 9 \times 10^6}{6 \times 10^6} \\ = 1.2342$$

$$r_2 = \frac{x_2^{(2)}}{x_1^{(2)}} = \frac{-m_1 \omega_2^2 + k_1 + k_2}{k_2} = \frac{-\frac{140000}{386.4} (43620.696) + 9 \times 10^6}{6 \times 10^6} \\ = -1.1341$$

Response of the system can be computed using Eqs. (5.18):

$$x_1^{(1)} = \frac{1}{r_2 - r_1} \left(\frac{\dot{x}_2}{\omega_1} \right) = \frac{1}{-2.3683} \left(\frac{12.8571}{66.3408} \right) = -0.08183 \text{ in}$$

$$x_1^{(2)} = \frac{1}{r_2 - r_1} \left(\frac{\dot{x}_2}{\omega_2} \right) = \frac{1}{-2.3683} \left(\frac{12.8571}{208.8557} \right) = -0.02599 \text{ in}$$

$$\phi_1 = \tan^{-1} \left\{ \frac{\dot{x}_2(0)}{0} \right\} = \frac{\pi}{2}$$

$$\phi_2 = \tan^{-1} \left\{ -\frac{\dot{x}_2(0)}{0} \right\} = -\frac{\pi}{2}$$

Response is given by Eqs. (5.15):

$$x_1(t) = x_1^{(1)} \cos(\omega_1 t + \phi_1) + x_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$= -0.08183 \cos(66.3408 t + \frac{\pi}{2}) - 0.02599 \cos(208.8557 t - \frac{\pi}{2}) \text{ in.}$$

$$x_2(t) = r_1 x_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 x_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$= -0.1010 \cos(66.3408 t + \frac{\pi}{2}) + 0.02948 \cos(208.8557 t - \frac{\pi}{2}) \text{ in.}$$

5.41

(a) Natural frequencies:

$$\text{Equations of motion: } m_1 \ddot{x}_1 + k_1 x_1 - k_1 x_2 = F_1(t) \quad (E_1)$$

$$m_2 \ddot{x}_2 + (k_1 + k_2) x_2 - k_1 x_1 = 0 \quad (E_2)$$

Frequency equation:

$$\begin{vmatrix} -\omega^2 m_1 + k_1 & -k_1 \\ -k_1 & -m_2 \omega^2 + k_1 + k_2 \end{vmatrix} = 0$$

or

$$\omega^4 - \left(\frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2} \right) \omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0$$

$$\therefore \omega_{1,2}^2 = \frac{k}{2m_1} + \frac{k_1 + k_2}{2m_2} \mp \sqrt{\frac{1}{4} \left(\frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

Here $m_1 = 2 \times 10^5 \text{ kg}$, $m_2 = 2.5 \times 10^5 \text{ kg}$, $k_1 = 150 \times 10^6 \text{ N/m}$
and $k_2 = 75 \times 10^6 \text{ N/m}$.

$$\begin{aligned} \omega_{1,2}^2 &= \frac{150 \times 10^6}{4 \times 10^5} + \frac{225 \times 10^6}{5 \times 10^5} \mp \sqrt{\frac{1}{4} \left(\frac{150 \times 10^6}{2 \times 10^5} + \frac{225 \times 10^6}{2.5 \times 10^5} \right)^2 - \frac{150 \times 75 \times 10^{12}}{5 \times 10^{10}}} \\ &= 150, 1500 \text{ (rad/sec)}^2 \end{aligned}$$

$$\therefore \omega_1 = 12.2474 \text{ rad/sec}, \quad \omega_2 = 38.7298 \text{ rad/sec}$$

(b) Response:

Assuming zero initial conditions, the Laplace transforms of (E_1) and (E_2) can be written as

$$m_1 s^2 \bar{x}_1(s) + k_1 \bar{x}_1(s) - k_1 \bar{x}_2(s) = \bar{F}_1(s)$$

$$m_2 s^2 \bar{x}_2(s) + (k_1 + k_2) \bar{x}_2(s) - k_1 \bar{x}_1(s) = 0$$

$$\text{i.e. } (m_1 s^2 + k_1) \bar{x}_1(s) - k_1 \bar{x}_2(s) = \bar{F}_1(s)$$

$$-k_1 \bar{x}_1(s) + (m_2 s^2 + k_1 + k_2) \bar{x}_2(s) = 0$$

Solution of these equations gives

$$\bar{x}_1(s) = \left\{ \frac{(m_2 s^2 + k_1 + k_2)}{m_1 m_2 s^4 + s^2 (m_1 k_1 + m_1 k_2 + m_2 k_1) + k_1 k_2} \right\} \bar{F}_1(s) \quad \text{---- (E}_3\text{)}$$

$$\bar{x}_2(s) = \left\{ \frac{k_1}{m_1 m_2 s^4 + s^2 (m_1 k_1 + m_1 k_2 + m_2 k_1) + k_1 k_2} \right\} \bar{F}_1(s) \quad \text{---- (E}_4\text{)}$$

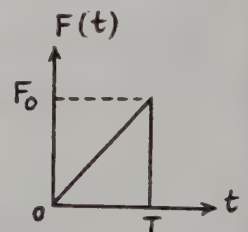
For the forcing function given,

$$\bar{F}_1(s) = \bar{F}(s) = \frac{F_0}{T} \left\{ \frac{1}{s^2} - e^{-Ts} \left(\frac{T}{s} - \frac{1}{s^2} \right) \right\} \quad \text{---- (E}_5\text{)}$$

For the given data, Eqs. (E₃) to (E₅) become

$$\bar{x}_1(s) = \frac{2.5 \times 10^5 s^2 + 225 \times 10^6}{(5 \times 10^{10} s^4 + 825 \times 10^{11} s^2 + 11250 \times 10^{12})} \bar{F}_1(s)$$

$$= \frac{s^2 + 900}{2 \times 10^5 s^4 + 330 \times 10^6 s^2 + 45 \times 10^9} \bar{F}_1(s) \quad \text{---- (E}_6\text{)}$$



$$\bar{x}_2(s) = \frac{150 \times 10^6}{5 \times 10^{10} s^4 + 825 \times 10^{11} s^2 + 11250 \times 10^{12}} \bar{F}_1(s)$$

$$= \frac{30}{10^4 s^4 + 165 \times 10^5 s^2 + 225 \times 10^7} \bar{F}_1(s) \quad \text{---- (E7)}$$

where $\bar{F}_1(s) = 2 \times 10^5 \left[\frac{1}{s^2} - e^{-0.5s} \left(\frac{1}{2s} - \frac{1}{s^2} \right) \right]$ ---- (E8)

The inverse transforms of (E6) and (E7) yield $x_1(t)$ and $x_2(t)$.

5.42 Equations of motion for free vibration are (from Eqs. (5.1) and (5.2)):

$$\left. \begin{aligned} m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 + (k_1 + k_2) x_1 - c_2 \dot{x}_2 - k_2 x_2 &= 0 \\ m_2 \ddot{x}_2 + (c_2 + c_3) \dot{x}_2 + (k_2 + k_3) x_2 - c_2 \dot{x}_1 - k_2 x_1 &= 0 \end{aligned} \right\} \quad (E1)$$

Assuming the solution as

$$x_i(t) = \bar{c}_i e^{s_i t} \quad ; \quad i = 1, 2$$

Eqs. (E1) can be rewritten as:

$$\begin{bmatrix} m_1 s^2 + (c_1 + c_2) s + (k_1 + k_2) & -(c_2 s + k_2) \\ -(c_2 s + k_2) & m_2 s^2 + (c_2 + c_3) s + (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} \bar{c}_1 \\ \bar{c}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E2)$$

For a non-trivial solution of Eqs. (E2),

$$\begin{vmatrix} m_1 s^2 + (c_1 + c_2) s + k_1 + k_2 & -(c_2 s + k_2) \\ -(c_2 s + k_2) & m_2 s^2 + (c_2 + c_3) s + k_2 + k_3 \end{vmatrix} = 0$$

i.e.,

$$\begin{aligned} & s^4 (m_1 m_2) + s^3 [m_1 (c_2 + c_3) + m_2 (c_1 + c_2)] + s^2 [m_1 (k_2 + k_3) \\ & + (c_1 + c_2)(c_2 + c_3) + m_2 (k_1 + k_2) - c_2^2] + s [(c_1 + c_2)(k_2 + k_3) \\ & + (c_2 + c_3)(k_1 + k_2) - 2c_2 k_2] + [(k_1 + k_2)(k_2 + k_3) - k_2^2] = 0 \end{aligned} \quad (E3)$$

This equation can be expressed as

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0 \quad (E4)$$

where $a_0, a_1, \dots, a_4$ can be identified by comparing Eqs. (E4) and (E3).

Nature of possible solutions:

If s_1, s_2, s_3 and s_4 are the roots of Eq. (E₄), the general solution of the system can be expressed as

$$\left. \begin{aligned} x_1(t) &= c_1^{(1)} e^{s_1 t} + c_1^{(2)} e^{s_2 t} + c_1^{(3)} e^{s_3 t} + c_1^{(4)} e^{s_4 t} \\ x_2(t) &= c_2^{(1)} e^{s_1 t} + c_2^{(2)} e^{s_2 t} + c_2^{(3)} e^{s_3 t} + c_2^{(4)} e^{s_4 t} \end{aligned} \right\} \quad (E_5)$$

where the constants $c_i^{(i)}$, $i=1$ to 4, can be found from the four initial conditions of the system, namely, $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$ and $\dot{x}_2(0)$. The ratios of amplitudes $c_1^{(i)} / c_2^{(i)}$ can be determined from Eqs. (E₂) as

$$\frac{c_1^{(i)}}{c_2^{(i)}} = \frac{c_2 s_i + k_2}{m_1 s_i^2 + (c_1 + c_2) s_i + k_1 + k_2} = \frac{m_2 s_i^2 + (c_2 + c_3) s_i + k_2 + k_3}{c_2 s_i + k_2};$$
$$i = 1, 2, 3, 4 \quad (E_6)$$

If any root s_i has a positive real part, $x_1(t)$ and $x_2(t)$ will increase with time. If all s_i have negative real parts as

$$s_j = -r_j + i \omega_j$$

then the solution, $x_1(t)$, can be expressed as

$$x_1(t) = \sum_{j=1}^4 c_1^{(j)} e^{-r_j t} e^{i \omega_j t} = \sum_{j=1}^4 c_1^{(j)} e^{-r_j t} (\cos \omega_j t + i \sin \omega_j t)$$

If two roots s_1 and s_2 are complex conjugates as

$$s_1 = -(r_1 + i \omega_1) \text{ and } s_2 = -(r_1 - i \omega_1),$$

$x_1(t)$ can be expressed as

$$\begin{aligned} x_1(t) &= e^{-r_1 t} \left\{ c_1^{(1)} \cos \omega_1 t - i c_1^{(1)} \sin \omega_1 t \right\} \\ &\quad + e^{-r_1 t} \left\{ c_1^{(2)} \cos \omega_1 t + i c_1^{(2)} \sin \omega_1 t \right\} \\ &\quad + c_1^{(3)} e^{s_3 t} + c_1^{(4)} e^{s_4 t} \end{aligned}$$

Similar expressions can be derived for $x_2(t)$.

5.43

$$m_1 = m_2 = 10 \text{ kg}, \quad k_1 = k_2 = 2000 \text{ N/m}, \quad k_3 = 2 \text{ N/m}, \\ c_1 = 100 \text{ N-s/m}, \quad c_2 = c_3 = 1 \text{ N-s/m}.$$

Equations (E_3) and (E_4) in the solution of problem 5.42 give

$$a_0 = m_1 m_2 = 100$$

$$a_1 = 10(2) + 10(101) = 1030$$

$$a_2 = 10(2002) + 101(2) + 10(4000) - 1 = 60221$$

$$a_3 = 101(2002) + 2(4000) - 2(1)(2000) = 206202$$

$$a_4 = 4000(2002) - 4 \times 10^6 = 4008000$$

and

$$100 s^4 + 1030 s^3 + 60221 s^2 + 206202 s + 4008000 = 0 \quad (E_1)$$

Using PROGRAM 10, the roots of E_1 can be found as

$$s_{1,2} = -1.4714 \pm i 8.8272 \quad (E_2)$$

$$s_{3,4} = -3.6786 \pm i 22.0668 \quad (E_3)$$

Thus the solution is given by

$$x_1(t) = \sum_{j=1}^4 \zeta_1^{(j)} e^{s_j t}, \quad x_2(t) = \sum_{j=1}^4 \zeta_2^{(j)} e^{s_j t} \quad (E_4, E_5)$$

where $\zeta_1^{(j)}$, $j=1,2,3,4$, can be found from the initial conditions, and the ratios of amplitudes $\left\{ \frac{\zeta_1^{(j)}}{\zeta_2^{(j)}} \right\}$ can be obtained from E_6 in problem 5.42:

$$\frac{\zeta_1^{(j)}}{\zeta_2^{(j)}} = \frac{c_2 s_j + k_2}{m_1 s_j^2 + (c_1 + c_2) s_j + k_1 + k_2} = \frac{s_j + 2000}{10 s_j^2 + 101 s_j + 4000}; \quad j=1,2,3,4 \quad (E_6)$$

For $j=1$, $s_1 = -1.4714 + i 8.8272$ and (E_6) gives†

$$\alpha_1 = \zeta_1^{(1)} / \zeta_2^{(1)} = 0.6207 - i 0.1239 \quad (E_7)$$

For $j=2$, $s_2 = -1.4714 - i 8.8272$ and (E_6) gives

$$\alpha_2 = \zeta_1^{(2)} / \zeta_2^{(2)} = 0.6207 + i 0.1239 \quad (E_8)$$

For $j=3$, $s_3 = -3.6786 + i 22.0668$ and (E_6) yields

$$\alpha_3 = \zeta_1^{(3)} / \zeta_2^{(3)} = -1.3808 - i 0.7758 \quad (E_9)$$

For $j=4$, $s_4 = -3.6786 - i 22.0668$ and (E_6) yields

$$\alpha_4 = \zeta_1^{(4)} / \zeta_2^{(4)} = -1.3808 + i 0.7758 \quad (E_{10})$$

Thus the solution of Eqs. (E_4) and (E_5) can be rewritten as

$$x_1(t) = \sum_{j=1}^4 \alpha_j \zeta_2^{(j)} e^{s_j t}, \quad x_2(t) = \sum_{j=1}^4 \alpha_j e^{s_j t} \quad (E_{11}, E_{12})$$

Since the pairs (α_1, α_2) , (α_3, α_4) , (s_1, s_2) and (s_3, s_4) are complex conjugates, we can express them as

$$\left. \begin{aligned} \alpha_1, \alpha_2 &= p_1 \pm i v_1; & \alpha_3, \alpha_4 &= p_2 \pm i v_2 \\ s_1, s_2 &= u_1 \pm i v_1; & s_3, s_4 &= u_2 \pm i v_2 \end{aligned} \right\} \quad (E_{13})$$

and (E_{11}) and (E_{12}) can be simplified further. However, we proceed directly with (E_{11}) and (E_{12}) and use the initial conditions to evaluate the constants $\zeta_2^{(j)}$; $j=1,2,3,4$:

$$\left. \begin{aligned} x_1(0) &= \alpha_1 \zeta_2^{(1)} + \alpha_2 \zeta_2^{(2)} + \alpha_3 \zeta_2^{(3)} + \alpha_4 \zeta_2^{(4)} = 0.2 \\ x_2(0) &= \zeta_2^{(1)} + \zeta_2^{(2)} + \zeta_2^{(3)} + \zeta_2^{(4)} = 0.1 \\ \dot{x}_1(0) &= s_1 \alpha_1 \zeta_2^{(1)} + s_2 \alpha_2 \zeta_2^{(2)} + s_3 \alpha_3 \zeta_2^{(3)} + s_4 \alpha_4 \zeta_2^{(4)} = 0 \\ \dot{x}_2(0) &= s_1 \zeta_2^{(1)} + s_2 \zeta_2^{(2)} + s_3 \zeta_2^{(3)} + s_4 \zeta_2^{(4)} = 0 \end{aligned} \right\} \quad (E_{14})$$

Once $\zeta_2^{(j)}$, $j=1,2,3,4$ are determined from Eqs. (E_{14}) , the displacements of masses $x_1(t)$ and $x_2(t)$ can be obtained using Eqs. (E_{11}) and (E_{12}) .

† If $s = a + ib$, $s^2 = (a^2 - b^2) + i(2ab)$

If $x = \frac{a+bi}{c+di}$, it can be rewritten as

$$x = \frac{(a+bi)(c-di)}{(c^2+d^2)} = \left(\frac{ac+bd}{c^2+d^2} \right) + i \left(\frac{bc-ad}{c^2+d^2} \right)$$

5.44

$$\omega = \frac{2\pi(1200)}{60} = 125.664 \text{ rad/sec}$$

$$F_1(t) = m e \omega^2 \cos \omega t$$

$$= \left(\frac{0.5}{386.4} \right) (6) (125.664)^2 \cos 125.664 t$$

$$= 122.6044 \cos 125.664 t \text{ lb}$$

$$m_1 = 800/386.4 = 2.0704 \text{ lb-s}^2/\text{in}$$

$$m_2 = 2000/386.4 = 5.1760 \text{ lb-s}^2/\text{in}$$

$$k_1 = 2000 \text{ lb/in}, \quad k_2 = 1000 \text{ lb/in}, \quad c_2 = 200 \text{ lb-s/in}, \quad F_{10} = 122.6044,$$

$$F_{20} = 0.$$

Equations of motion are [substitute $k_1 = k_2$, $c_1 = c_2$, $m_1 = m_2$, $F_1 = 0$, $k_2 = k_1$, $c_2 = 0$, $m_2 = m_1$, $F_2 = F_1$, $k_3 = 0$, $c_3 = 0$ in Eq. (5.3)]:

$$\begin{bmatrix} m_2 & 0 \\ 0 & m_1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_1 \end{Bmatrix} + \begin{bmatrix} c_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{Bmatrix} + \begin{bmatrix} k_2 + k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} x_2 \\ x_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_1(t) \end{Bmatrix} \dots (E_1)$$

or

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix} \dots (E_2)$$

Comparing (E₂) with Eq. (5.27), we find that

$$m_{11} = m_1, \quad m_{12} = 0, \quad m_{22} = m_2, \quad c_{11} = 0, \quad c_{12} = 0, \quad c_{22} = c_2, \quad k_{11} = k_1, \quad k_{12} = 0 \text{ and}$$

$$k_{22} = k_1 + k_2.$$

Application of Eq. (5.31) leads to

$$Z_{11}(i\omega) = -\omega^2 m_{11} + i\omega c_{11} + k_{11} = -m_1 \omega^2 + k_1$$

$$Z_{12}(i\omega) = -\omega^2 m_{12} + i\omega c_{12} + k_{12} = -k_1$$

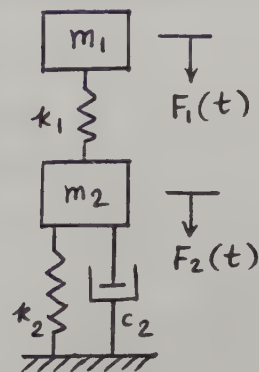
$$Z_{22}(i\omega) = -\omega^2 m_{22} + i\omega c_{22} + k_{22} = -m_2 \omega^2 + i\omega c_2 + k_1 + k_2$$

Response of the system can be expressed as

$$x_j(t) = X_j e^{i\omega t} = X_j \cos \omega t \quad (\text{real part})$$

with X_j given by Eq. (5.35):

$$X_1(i\omega) = \frac{Z_{22}(i\omega) \cdot F_{10} - Z_{12}(i\omega) \cdot F_{20}}{Z_{11}(i\omega) \cdot Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$



$$\begin{aligned}
&= \frac{(-m_2 \omega^2 + i\omega c_2 + k_1 + k_2) F_{10}}{(-m_1 \omega^2 + k_1)(-m_2 \omega^2 + i\omega c_2 + k_1 + k_2) - k_1^2} \\
&= \frac{\{-5.176 (125.664)^2 + i(125.664)(200) + 3000\} 122.6044}{\{-2.0704 (125.664)^2 + 2000\} [-5.176 (125.664)^2 + i(125.664)(200) + 3000] - 4 \times 10^6} \\
&= (-40.0042 - 0.01919 i) \times 10^{-4} \text{ in} \\
X_2(i\omega) &= \frac{-Z_{12}(i\omega) F_{10} + Z_{11}(i\omega) F_{20}}{Z_{11}(i\omega) Z_{22}(i\omega) - Z_{12}^2(i\omega)} = \frac{k F_{10}}{Z_{11}(i\omega) Z_{22}(i\omega) - Z_{12}^2(i\omega)} \\
&= \frac{2000 (122.6044)}{(24.1272 - 7.7143 i) 10^8} = (0.9221 + 0.2948 i) \times 10^{-4} \text{ in}
\end{aligned}$$

5.45 $k_1 = k_{\text{beam}} = \frac{192 E (\frac{1}{12} a t^3)}{l^3} = \frac{16 E a t^3}{l^3}$

Equations of motion:

$$\begin{cases} m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = F_1(t) = F_0 \cos \omega t \\ m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0 \end{cases} \dots (E_1)$$

Assuming harmonic response

$$x_j(t) = X_j \cos \omega t ; j=1,2$$

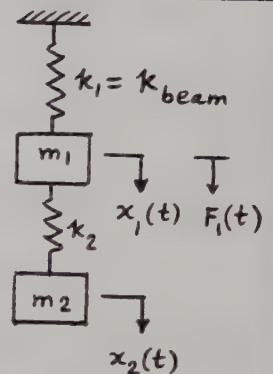
Eqs. (E₁) yield

$$X_1 = \frac{(k_2 - m_2 \omega^2) F_0}{(k_1 + k_2 - m_1 \omega^2)(k_2 - m_2 \omega^2) - k_2^2}$$

$$X_2 = \frac{k_2 F_0}{(k_1 + k_2 - m_1 \omega^2)(k_2 - m_2 \omega^2) - k_2^2}$$

For no steady state vibration of the beam, $X_1 = 0$ and hence the condition to be satisfied is

$$\frac{k_2}{m_2} = \omega^2$$



5.46 Equations of motion:

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = F_1(t) = F_0 \sin \omega t \quad \dots (E_1)$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0 \quad \dots (E_2)$$

We use $F_0 e^{i\omega t}$ (with $i = \sqrt{-1}$) for $F_1(t)$ and consider only the imaginary part at the end.

Let $x_j(t) = X_j e^{i\omega t}$; $j = 1, 2$

Eqs. (E₁) and (E₂) become

$$-m_1 \omega^2 X_1 e^{i\omega t} + (k_1 + k_2) X_1 e^{i\omega t} - k_2 X_2 e^{i\omega t} = F_0 e^{i\omega t}$$

$$-m_2 \omega^2 X_2 e^{i\omega t} + k_2 X_2 e^{i\omega t} - k_2 X_1 e^{i\omega t} = 0$$

i.e. $[Z(i\omega)] \vec{X} = \vec{F}_0$ ----- (E₃)

where $\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, $\vec{F}_0 = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$,

$$[Z(i\omega)] = \begin{bmatrix} Z_{11}(i\omega) & Z_{12}(i\omega) \\ Z_{21}(i\omega) & Z_{22}(i\omega) \end{bmatrix},$$

$$Z_{11}(i\omega) = -m_1 \omega^2 + k_1 + k_2, \quad Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2,$$

$$Z_{22}(i\omega) = -m_2 \omega^2 + k_2.$$

Eqs. (5.35) give

$$X_1(i\omega) = \frac{(-m_2 \omega^2 + k_2) F_0}{(-m_1 \omega^2 + k_1 + k_2)(-m_2 \omega^2 + k_2) - k_2^2}$$

$$X_2(i\omega) = \frac{k_2 F_0}{(-m_1 \omega^2 + k_1 + k_2)(-m_2 \omega^2 + k_2) - k_2^2}$$

Since $F_0 \sin \omega t = \text{Im} (F_0 e^{i\omega t})$, $x_j(t) = \text{Im} (X_j e^{i\omega t}) = X_j \sin \omega t$

$$\therefore x_1(t) = \frac{(-m_2 \omega^2 + k_2) F_0}{(-m_1 \omega^2 + k_1 + k_2)(-m_2 \omega^2 + k_2) - k_2^2} \sin \omega t$$

$$x_2(t) = \frac{k_2 F_0}{(-m_1 \omega^2 + k_1 + k_2)(-m_2 \omega^2 + k_2) - k_2^2} \sin \omega t$$

5.47

Equations of motion:

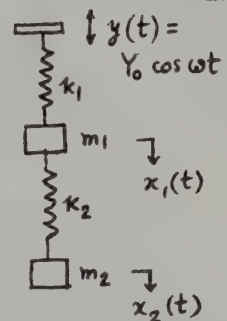
$$m_1 \ddot{x}_1 = -k_1(x_1 - y) - k_2(x_1 - x_2)$$

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$$

or $m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = k_1 Y_0 \cos \omega t$ --- (E₁)

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$
 --- (E₂)

As there is no damping, the masses vibrate either in phase or 180° out of phase with respect to the base motion. Hence the response can be taken as



$$x_j(t) = X_j \cos \omega t \quad ; \quad j = 1, 2 \quad \text{--- (E}_3\text{)}$$

Eqs. (E₁) and (E₂) reduce to

$$(-\omega^2 m_1 + k_1 + k_2) X_1 - k_2 X_2 = k_1 Y_0$$

$$-k_2 X_1 + (-\omega^2 m_2 + k_2) X_2 = 0$$

i.e. $[Z(i\omega)] \vec{X} = \vec{F}$ where $\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, $\vec{F} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{Bmatrix} k_1 Y_0 \\ 0 \end{Bmatrix}$,

$$Z_{11}(i\omega) = -\omega^2 m_1 + k_1 + k_2, \quad Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2,$$

$$Z_{22}(i\omega) = -\omega^2 m_2 + k_2.$$

Eqs. (5.35) and (E₃) give

$$x_1(t) = \frac{(-\omega^2 m_2 + k_2) k_1 Y_0}{(-m_1 \omega^2 + k_1 + k_2)(-\omega^2 m_2 + k_2) - k_2^2} \cos \omega t$$

$$x_2(t) = \frac{k_1 k_2 Y_0}{(-m_1 \omega^2 + k_1 + k_2)(-\omega^2 m_2 + k_2) - k_2^2} \cos \omega t$$

5.48

Equations of motion:

$$m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 + (k_1 + k_2) x_1 - c_2 \dot{x}_2 - k_2 x_2 = F_1(t) = F_0 e^{i\omega t}$$

$$m_2 \ddot{x}_2 + c_2 \dot{x}_2 + k_2 x_2 - c_2 \dot{x}_1 - k_2 x_1 = 0$$

Let $x_j(t) = X_j e^{i\omega t}$, $j = 1, 2$

Equations of motion become

$$[-\omega^2 m_1 + i\omega(c_1 + c_2) + k_1 + k_2] X_1 - (i\omega c_2 + k_2) X_2 = F_0 \quad \text{--- (E}_1\text{)}$$

$$-(i\omega c_2 + k_2) X_1 + [-\omega^2 m_2 + i\omega c_2 + k_2] X_2 = 0 \quad \text{--- (E}_2\text{)}$$

For given data, (E₁) and (E₂) become

$$[Z(i\omega)] \vec{X} = \vec{F}_0 \quad \text{--- (E}_3\text{)}$$

where $Z_{11}(i\omega) = 400i + 999$

$$Z_{12}(i\omega) = Z_{21}(i\omega) = -200i - 500$$

$$Z_{22}(i\omega) = 200i + 499$$

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}, \quad \vec{F}_0 = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

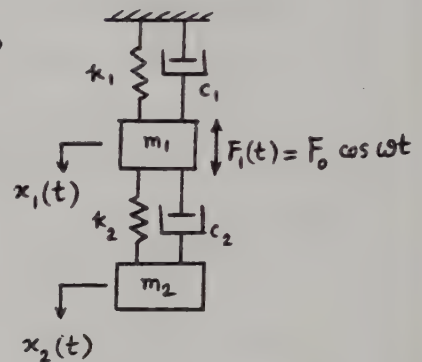
Solution of (E₃) is, using Eqs. (5.35),

$$k_1 = k_2 = 500 \text{ N/m}$$

$$m_1 = m_2 = 1 \text{ kg}$$

$$c_1 = c_2 = c = 200 \text{ N.s/m}$$

$$\omega = 1 \text{ rad/s}$$



$$\begin{aligned}
 x_1 &= \frac{(200i + 499) F_0}{(400i + 999)(200i + 499) - (-200i - 500)^2} = \frac{(200i + 499) F_0}{(199400i + 208501)} \\
 &= \frac{(200i + 499)(-199400i + 208501) F_0}{(199400i + 208501)(-199400i + 208501)} \\
 &= (17.2915 \times 10^{-4} - 6.9444 \times 10^{-4} i) F_0 \quad \text{---- (E}_4\text{)}
 \end{aligned}$$

$$\begin{aligned}
 x_2 &= \frac{(200i + 500) F_0}{(400i + 999)(200i + 499) - (-200i - 500)^2} = \frac{(200i + 500) F_0}{(199400i + 208501)} \\
 &= \frac{(200i + 500)(-199400i + 208501) F_0}{(199400i + 208501)(-199400i + 208501)} \\
 &= (17.3165 \times 10^{-4} - 6.9684 \times 10^{-4} i) F_0 \quad \text{---- (E}_5\text{)}
 \end{aligned}$$

Final solution is given by the real parts as

$$\begin{aligned}
 x_1(t) &= \operatorname{Re}(x_1 e^{i\omega t}) = \operatorname{Re}(x_1 \cos \omega t + i x_1 \sin \omega t) \\
 &= \operatorname{Re}[(17.2915 \times 10^{-4} - 6.9444 \times 10^{-4} i) F_0 \cos \omega t \\
 &\quad + (17.2915 \times 10^{-4} i + 6.9444 \times 10^{-4}) F_0 \sin \omega t] \\
 &= 17.2915 \times 10^{-4} F_0 \cos \omega t + 6.9444 \times 10^{-4} F_0 \sin \omega t \quad \text{--- (E}_6\text{)}
 \end{aligned}$$

$$\begin{aligned}
 x_2(t) &= \operatorname{Re}(x_2 e^{i\omega t}) = \operatorname{Re}(x_2 \cos \omega t + i x_2 \sin \omega t) \\
 &= \operatorname{Re}[(17.3165 \times 10^{-4} - 6.9684 \times 10^{-4} i) F_0 \cos \omega t \\
 &\quad + (17.3165 \times 10^{-4} i + 6.9684 \times 10^{-4}) F_0 \sin \omega t] \\
 &= 17.3165 \times 10^{-4} F_0 \cos \omega t + 6.9684 \times 10^{-4} F_0 \sin \omega t \quad \text{--- (E}_7\text{)}
 \end{aligned}$$

5.49

Equations of motion:

$$\begin{aligned}
 m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 &= F_{10} \cos \omega t = \operatorname{Re}(F_{10} e^{i\omega t}) \\
 m_2 \ddot{x}_2 + (k_2 + k_3) x_2 - k_2 x_1 &= F_{20} \cos \omega t = \operatorname{Re}(F_{20} e^{i\omega t})
 \end{aligned}$$

Assuming $x_j(t) = X_j e^{i\omega t}$; $j = 1, 2$, along with $F_j(t) = F_{j0} e^{i\omega t}$; $j = 1, 2$, the equations of motion can be expressed as

$$(-\omega^2 m_1 + k_1 + k_2) X_1 - k_2 X_2 = F_{10}$$

$$-k_2 X_1 + (-\omega^2 m_2 + k_2 + k_3) X_2 = F_{20}$$

i.e. $[Z(i\omega)] \vec{X} = \vec{F}_0 \quad \text{---- (E}_1\text{)}$

where $Z_{11}(i\omega) = -\omega^2 m_1 + k_1 + k_2$, $Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2$,

$$Z_{22}(i\omega) = -\omega^2 m_2 + k_2 + k_3,$$

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}, \quad \vec{F}_0 = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

Solution of (E_1) can be expressed, using Eqs. (5.35), as

$$X_1 = \frac{(-\omega^2 m_2 + k_2 + k_3) F_{10} + k_2 F_{20}}{(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2 + k_3) - k_2^2} \quad \text{---- (E}_2\text{)}$$

$$X_2 = \frac{k_2 F_{10} + (-\omega^2 m_1 + k_1 + k_2) F_{20}}{(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2 + k_3) - k_2^2} \quad \text{---- (E}_3\text{)}$$

Since X_1 and X_2 are real (since there is no damping), the final solution is given by

$$x_1(t) = X_1 \cos \omega t$$

$$x_2(t) = X_2 \cos \omega t$$

where X_1 and X_2 are given by (E_2) and (E_3) .

5.50

From the solution of problem 5.46, we have

$$x_1(t) = \frac{(-m_2 \omega^2 + k_2) F_0}{(-m_1 \omega^2 + k_1 + k_2)(-m_2 \omega^2 + k_2) - k_2^2} \sin \omega t$$

$$x_2(t) = \frac{k_2 F_0}{(-m_1 \omega^2 + k_1 + k_2)(-m_2 \omega^2 + k_2) - k_2^2} \sin \omega t$$

For the data $F_1(t) = 50 \sin 4\pi t$, $F_0 = 50$ N, $\omega = 4\pi$ rad/s,

$m_1 = 10$ kg, $m_2 = 5$ kg, $k_1 = 8000$ N/m and $k_2 = 2000$ N/m,

$$x_1(t) = \frac{(-5 \times 16 \pi^2 + 2000) 50}{(-10 \times 16 \pi^2 + 8000 + 2000)(-5 \times 16 \pi^2 + 2000) - (2000)^2} \sin 4\pi t$$

$$= 0.009773 \sin 4\pi t \quad \text{meters}$$

$$x_2(t) = \frac{2000(50)}{(-10 \times 16 \pi^2 + 8000 + 2000)(-5 \times 16 \pi^2 + 2000) - (2000)^2} \sin 4\pi t$$

$$= 0.016148 \sin 4\pi t \quad \text{meters}$$

5.51

k_1 = total stiffness = 800 N/m

k_2 = total stiffness = 600 N/m

$m_1 = 50$ kg, $m_2 = 50$ kg

$Y = 0.2$ m, $\omega = \pi$ rad/s

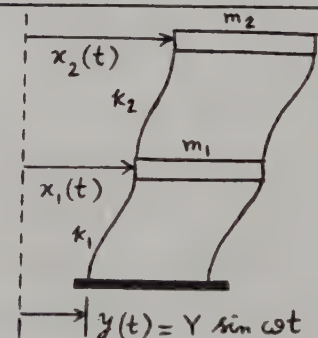
Equations of motion:

$$m_1 \ddot{x}_1 = -k_1(x_1 - y) - k_2(x_1 - x_2)$$

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$$

i.e. $m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = k_1 y = k_1 Y \sin \omega t$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$



Assuming $x_i(t) = X_i \sin \omega t$; $i = 1, 2$, we get

$$(-m_1 \omega^2 + k_1 + k_2) X_1 - k_2 X_2 = k_1 Y$$

$$-k_2 X_1 + (-m_2 \omega^2 + k_2) X_2 = 0$$

For given data, these equations take the form

$$(-50\pi^2 + 1400) X_1 - 600 X_2 = (800)(0.2)$$

$$-600 X_1 + (-50\pi^2 + 600) X_2 = 0$$

Solution of these equations gives $X_1 = -0.06469 \text{ m}$, $X_2 = -0.36439 \text{ m}$

$$\therefore x_1(t) = -0.06469 \sin \pi t \text{ m}; \quad x_2(t) = -0.36439 \sin \pi t \text{ m}.$$

5.52

Equations of motion:

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = F_1(t) \quad \text{--- (E}_1\text{)}$$

$$m_2 \ddot{x}_2 + (k_2 + k_3) x_2 - k_2 x_1 = 0 \quad \text{--- (E}_2\text{)}$$

Laplace transforms of (E₁) and (E₂) are

$$m_1 [\delta^2 \bar{x}_1(s) - \delta x_1(0) - \dot{x}_1(0)] + (k_1 + k_2) \bar{x}_1(s) - k_2 \bar{x}_2(s) = \bar{F}_1(s)$$

$$m_2 [\delta^2 \bar{x}_2(s) - \delta x_2(0) - \dot{x}_2(0)] + (k_2 + k_3) \bar{x}_2(s) - k_2 \bar{x}_1(s) = 0$$

Rearranging these equations, we get

$$(m_1 \delta^2 + k_1 + k_2) \bar{x}_1(s) - k_2 \bar{x}_2(s) = \bar{F}_1(s) + \delta m_1 x_1(0) + m_1 \dot{x}_1(0) \quad \text{--- (E}_3\text{)}$$

$$-k_2 \bar{x}_1(s) + (m_2 \delta^2 + k_2 + k_3) \bar{x}_2(s) = \delta m_2 x_2(0) + m_2 \dot{x}_2(0) \quad \text{--- (E}_4\text{)}$$

When $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$, (E₃) and (E₄) give

$$(m \delta^2 + 2k) \bar{x}_1(s) - k \bar{x}_2(s) = \bar{F}_1(s) + \delta m x_1(0) + m \dot{x}_1(0) \quad \text{--- (E}_5\text{)}$$

$$-k \bar{x}_1(s) + (m \delta^2 + 2k) \bar{x}_2(s) = \delta m x_2(0) + m \dot{x}_2(0) \quad \text{--- (E}_6\text{)}$$

Solution of Eqs. (E₅) and (E₆) gives

$$\bar{x}_1(s) = \frac{(m \delta^2 + 2k) \{ \bar{F}_1(s) + m x_1(0) \cdot \delta + m \dot{x}_1(0) \} + k \{ m x_2(0) \cdot \delta + m \dot{x}_2(0) \}}{(m \delta^2 + 2k)(m \delta^2 + 2k) - k^2}$$

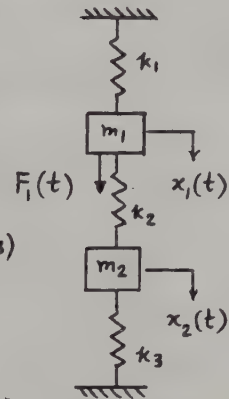
$$\bar{x}_2(s) = \frac{k \{ \bar{F}_1(s) + m x_1(0) \cdot \delta + \dot{x}_1(0) m \} + (m \delta^2 + 2k) \{ m x_2(0) \cdot \delta + m \dot{x}_2(0) \}}{(m \delta^2 + 2k)(m \delta^2 + 2k) - k^2}$$

These equations become, for $x_1(0) = x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$,

$$F_1(t) = 5 u(t), \quad \bar{F}_1(s) = \frac{5}{s}, \quad m = 1 \text{ and } k = 100,$$

$$\bar{x}_1(s) = \frac{5(\delta^2 + 200)}{s[(\delta^2 + 200)^2 - 10000]} = \frac{5(\delta^2 + 200)}{s(\delta^2 + 300)(\delta^2 + 100)} \quad \text{--- (E}_7\text{)}$$

$$\bar{x}_2(s) = 100 \left(\frac{5}{s} \right) \frac{1}{[(\delta^2 + 200)^2 - 10000]} = \frac{500}{s(\delta^2 + 300)(\delta^2 + 100)} \quad \text{--- (E}_8\text{)}$$



By expressing

$$\bar{x}_1(s) = \frac{5(s^2 + 200)}{s(s^2 + 300)(s^2 + 100)} = \frac{A_1}{s} + \frac{A_2 s + A_3}{s^2 + 100} + \frac{A_4 s + A_5}{s^2 + 300}$$

$$\bar{x}_2(s) = \frac{500}{s(s^2 + 300)(s^2 + 100)} = \frac{B_1}{s} + \frac{B_2 s + B_3}{s^2 + 100} + \frac{B_4 s + B_5}{s^2 + 300}$$

we can find $A_1, A_2, \dots, B_1, B_2, \dots$ (partial fractions method).

This leads to

$$\bar{x}_1(s) = \frac{1}{30s} - \frac{s}{40(s^2 + 100)} - \frac{s}{120(s^2 + 300)} \quad \dots (E_9)$$

$$\bar{x}_2(s) = \frac{1}{60s} - \frac{s}{40(s^2 + 100)} + \frac{s}{120(s^2 + 300)} \quad \dots (E_{10})$$

Inverse Laplace transforms of (E_9) and (E_{10}) give

$$x_1(t) = \left(\frac{1}{30} - \frac{1}{40} \cos 10t - \frac{1}{120} \cos 10\sqrt{3}t \right) u(t)$$

$$x_2(t) = \left(\frac{1}{60} - \frac{1}{40} \cos 10t + \frac{1}{120} \cos 10\sqrt{3}t \right) u(t)$$

It is to be noted that $x_1(t) = \frac{1}{30}$ meter and $x_2(t) = \frac{1}{60}$ meter are the static deflections associated with 5 N static force applied to mass m_1 . $\omega_1 = 10$ rad/s and $\omega_2 = 10\sqrt{3}$ rad/s are the two resonant frequencies associated with the two degree of freedom system.

5.53

Equivalent mass of cylinder with respect to $x_2 = (m_2)_{eq} = m_2 + \frac{J_0}{r^2}$

Equations of motion: $m_1 \ddot{x}_1 = -k(x_1 - x_2)$

$$(m_2)_{eq} \ddot{x}_2 = -k(x_2 - x_1)$$

$$\text{i.e. } m_1 \ddot{x}_1 + kx_1 - kx_2 = 0 \quad \dots (E_1)$$

$$\left(m_2 + \frac{J_0}{r^2}\right) \ddot{x}_2 + kx_2 - kx_1 = 0 \quad \dots (E_2)$$

Assuming $x_i(t) = X_i \cos(\omega t + \phi)$; $i=1, 2$, Eqs. (E_1) and (E_2) can be written as

$$(-m_1 \omega^2 + k) X_1 - k X_2 = 0$$

$$-k X_1 + (-\omega^2 \{m_2 + \frac{J_0}{r^2}\} + k) X_2 = 0$$

Frequency equation is:

$$(-m_1 \omega^2 + k) (-\omega^2 m_2 - \omega^2 \frac{J_0}{r^2} + k) - k^2 = 0$$

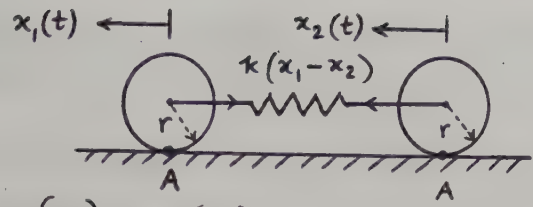
$$\text{or } \omega^4 \left(m_1 m_2 + \frac{m_1 J_0}{r^2}\right) - \omega^2 \left(m_1 k + m_2 k + \frac{k J_0}{r^2}\right) = 0$$

$$\omega_1 = 0, \quad \omega_2 = \left\{ \left(m_1 k + m_2 k + \frac{k J_0}{r^2}\right) / \left(m_1 m_2 + \frac{m_1 J_0}{r^2}\right) \right\}^{1/2}$$

5.54

Equivalent mass of each cylinder for translatory motion, referred to point A, is

$$m_{eq} = \frac{J_A}{r^2} = \frac{\frac{mr^2}{2} + mr^2}{r^2} = \frac{3m}{2} = (m_1)_{eq} = (m_2)_{eq}$$



Equations of motion:

$$(m_1)_{eq} \ddot{x}_1 + k(x_1 - x_2) = 0$$

$$(m_2)_{eq} \ddot{x}_2 - k(x_1 - x_2) = 0$$

Frequency equation:

$$\begin{vmatrix} -(m_1)_{eq} \omega^2 + k & -k \\ -k & -(m_2)_{eq} \omega^2 + k \end{vmatrix} = 0$$

$$\text{or } (m_1)_{eq} (m_2)_{eq} \omega^4 - \omega^2 [(m_1)_{eq} k + (m_2)_{eq} k] = 0$$

$$\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{(m_1)_{eq} k + (m_2)_{eq} k}{(m_1)_{eq} (m_2)_{eq}}} = \sqrt{\frac{4k}{3m}}$$

5.55

For harmonic motion $x_i(t) = X_i \cos(\omega t + \phi)$; $i=1,2$

given equations lead to

$$(-\omega^2 a_1 + b_1) X_1 + c_1 X_2 = 0$$

$$b_2 X_1 + (-\omega^2 a_2 + c_2) X_2 = 0$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 a_1 + b_1 & c_1 \\ b_2 & -\omega^2 a_2 + c_2 \end{vmatrix} = 0$$

$$\text{or } \omega^4 (a_1 a_2) - \omega^2 (a_1 c_2 + b_1 a_2) + (b_1 c_2 - c_1 b_2) = 0$$

Condition for degeneracy is: $b_1 c_2 - c_1 b_2 = 0$

5.56

Equations of motion:

$$J_1 \ddot{\theta}_1 + k_t \theta_1 - k_t \theta_2 = 0$$

$$J_2 \ddot{\theta}_2 + k_t \theta_2 - k_t \theta_1 = 0$$

For $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i=1,2$,

$$\begin{bmatrix} -\omega^2 J_1 + k_t & -k_t \\ -k_t & -\omega^2 J_2 + k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\omega^4 (J_1 J_2) - \omega^2 (J_1 k_t + J_2 k_t) = 0$$

$$\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{k_t (J_1 + J_2)}{J_1 J_2}}$$

Amplitude ratios:

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 J_1 + k_t}{k_t} = 1$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 J_1 + k_t}{k_t} = -\frac{J_1}{J_2}$$

General solution is given by equations similar to Eqs. (5.15). With $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$, we obtain from equations similar to Eqs. (5.18):

$$\begin{aligned}\Theta_1^{(1)} &= \frac{1}{r_2 - r_1} \{ r_2 \theta_1(0) - \theta_2(0) \} = - \left(\frac{J_2}{J_1 + J_2} \right) \left[- \frac{J_1}{J_2} \theta_1(0) - \theta_2(0) \right] \\ &= \left\{ \frac{J_1 \theta_1(0) + J_2 \theta_2(0)}{J_1 + J_2} \right\}\end{aligned}$$

$$\begin{aligned}\Theta_1^{(2)} &= \frac{1}{r_2 - r_1} \{ -r_1 \theta_1(0) + \theta_2(0) \} = - \left(\frac{J_2}{J_1 + J_2} \right) \left[\frac{J_1}{J_2} \theta_1(0) + \theta_2(0) \right] \\ &= - \left\{ \frac{J_1 \theta_1(0) + J_2 \theta_2(0)}{J_1 + J_2} \right\}\end{aligned}$$

$$\phi_1 = \phi_2 = 0$$

$$\theta_1(t) = \Theta_1^{(1)} \cos \omega t + \Theta_1^{(2)} \cos \omega_2 t = \Theta_1^{(1)} + \Theta_1^{(2)} \cos \omega_2 t$$

$$\theta_2(t) = \Theta_1^{(1)} - \frac{J_1}{J_2} \Theta_1^{(2)} \cos \omega_2 t$$

5.57

When $k_{t1} = 0$, the system becomes identical to the system of problem 5.56 with $k_t = k_{t1}$. Normal modes are given by

$$\vec{\Theta}^{(1)} = \begin{Bmatrix} \Theta_1^{(1)} \\ \Theta_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \Theta_1^{(1)} ; \quad \vec{\Theta}^{(2)} = \begin{Bmatrix} \Theta_1^{(2)} \\ \Theta_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -(\frac{J_1}{J_2}) \end{Bmatrix} \Theta_1^{(2)}$$

Equations of motion can be rewritten as

$$\ddot{\Theta}_1 + \frac{k_t}{J_1} (\Theta_1 - \Theta_2) = 0 \quad \text{---- (E}_1\text{)}$$

$$\ddot{\Theta}_2 - \frac{k_t}{J_2} (\Theta_1 - \Theta_2) = 0 \quad \text{---- (E}_2\text{)}$$

Subtracting (E₂) from (E₁) gives

$$(\ddot{\Theta}_1 - \ddot{\Theta}_2) + (\Theta_1 - \Theta_2) \left(\frac{k_t}{J_1} + \frac{k_t}{J_2} \right) = 0 \quad \text{---- (E}_3\text{)}$$

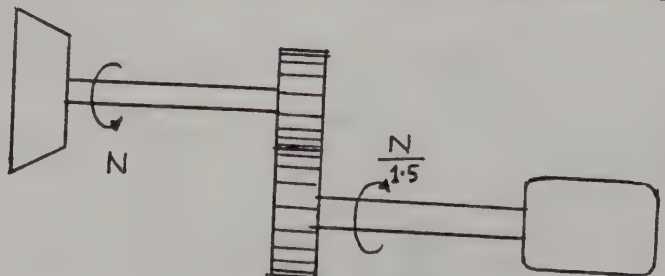
Defining $\alpha = \Theta_1 - \Theta_2$, (E₃) can be written as

$$\ddot{\alpha} + \left(\frac{k_t}{J_1} + \frac{k_t}{J_2} \right) \alpha = 0 \quad \text{---- (E}_4\text{)}$$

This is a single equation for which the natural frequency is

$$\omega = \sqrt{\left(\frac{k_t}{J_1} + \frac{k_t}{J_2} \right)} = \sqrt{\frac{k_t (J_1 + J_2)}{J_1 J_2}} \equiv \omega_2 \text{ of problem 5.45.}$$

5.58



Since the length of shaft 1 is small and its diameter large, it will be very rigid and hence the turbine and gear 1 are assumed to be rigidly connected. This helps in modeling the system as a two d.o.f. system.

$$J_{01} = J_{\text{turbine}} + J_{\text{gear1}} + \frac{J_{\text{gear2}}}{1.5^2} = 3000 + 500 + (1000/2.25) = 3944.4444 \text{ kg-m}^2$$

$$k_{t2} = \left(\frac{GJ}{\ell} \right)_{\text{shaft2}} = \frac{(80 (10^9)) \left(\frac{\pi}{32} (0.1^4) \right)}{1} = 7.854 (10^5) \text{ N/m}$$

$$J_{02} = J_{\text{generator}} = 2000 \text{ kg-m}^2$$

System is a semi-definite system. Its natural frequencies are given by (see Eq. (5.40)):

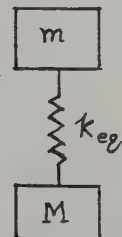
$$\omega_1 = 0$$

$$\omega_2 = \sqrt{\frac{k_{t2} (J_{01} + J_{02})}{J_{01} J_{02}}} = \sqrt{\frac{(78.54 (10^4)) (5944.4444)}{(3944.4444) (2000)}} = 24.3273 \text{ rad/sec}$$

5.59 Natural frequencies are given by Eq. (5.40):

$$\omega_1 = 0 ; \quad \omega_2 = \sqrt{\frac{k (m_1 + m_2)}{m_1 m_2}} = \sqrt{\frac{6 k (m + M)}{m M}}$$

Assumption: Balloon is a point mass.



$$k_{e2} = 12 k \cos^2 45^\circ = 6 k$$

5.60 Equations of motion:

$$J_1 \ddot{\theta}_1 + c_{t1} \dot{\theta}_1 - c_{t2} (\dot{\theta}_2 - \dot{\theta}_1) + k_{t1} \theta_1 - k_{t2} (\theta_2 - \theta_1) = 0$$

$$J_2 \ddot{\theta}_2 + c_{t2} (\dot{\theta}_2 - \dot{\theta}_1) + k_{t2} (\theta_2 - \theta_1) = 0$$

i.e.,

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} c_{t1} + c_{t2} & -c_{t2} \\ -c_{t2} & c_{t2} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_{t1} + k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \theta_j(t) = X_j e^{st} \quad \text{----- (E}_2\text{)} \quad \text{--- (E}_1\text{)}$$

Eqs. (E₁) yield:

$$\begin{aligned} & \left(\begin{bmatrix} J_1 s^2 & 0 \\ 0 & J_2 s^2 \end{bmatrix} + \begin{bmatrix} (c_{t1} + c_{t2})s & -c_{t2}s \\ -c_{t2}s & c_{t2}s \end{bmatrix} + \begin{bmatrix} (k_{t1} + k_{t2}) & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{st} \\ & = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{----- (E}_3\text{)} \end{aligned}$$

The characteristic equation becomes:

$$\begin{vmatrix} J_1 \delta^2 + (c_{t1} + c_{t2}) \delta + k_{t1} + k_{t2} & -(c_{t2} \delta + k_{t2}) \\ -(c_{t2} \delta + k_{t2}) & J_2 \delta^2 + c_{t2} \delta + k_{t2} \end{vmatrix} = 0$$

i.e.,

$$a_0 \delta^4 + a_1 \delta^3 + a_2 \delta^2 + a_3 \delta + a_4 = 0 \quad \dots (E_4)$$

where

$$a_0 = J_1 J_2$$

$$a_1 = J_1 c_{t2} + J_2 (c_{t1} + c_{t2})$$

$$a_2 = J_1 k_{t2} + c_{t2} (c_{t1} + c_{t2}) + J_2 (k_{t1} + k_{t2}) - c_{t2}^2$$

$$a_3 = k_{t2} (c_{t1} + c_{t2}) + c_{t2} (k_{t1} + k_{t2}) - 2 c_{t2} k_{t2}$$

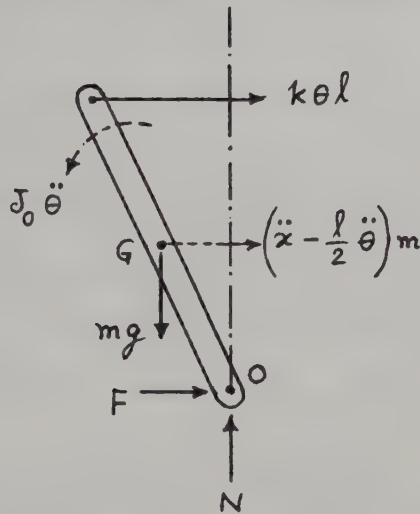
$$a_4 = k_{t2} (k_{t1} + k_{t2}) - k_{t2}^2$$

For the stability of the system, the conditions derived in section 5.8 are applicable:

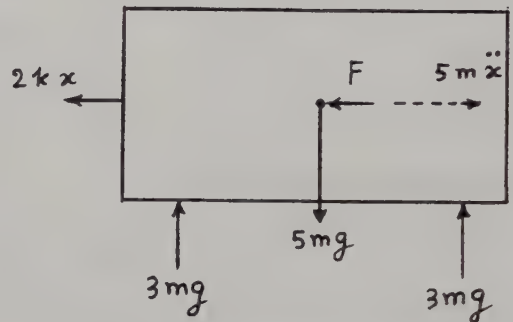
$$a_i > 0 \quad ; \quad i = 0, 1, 2, 3, 4$$

$$a_1 a_2 a_3 - a_1^2 a_4 - a_0 a_3^2 > 0$$

5.61



Free body diagram of bar



Free body diagram of trailer

Equations of motion of bar:

$$m \left(\ddot{x} - \frac{\ell}{2} \ddot{\theta} \right) = k \theta \ell + F \quad (1)$$

$$J_0 \ddot{\theta} - m \left(\ddot{x} - \frac{\ell}{2} \ddot{\theta} \right) \frac{\ell}{2} = -k \theta \ell (\ell) + m g \frac{\ell}{2} \sin \theta \quad (2)$$

Equation of motion of trailer:

$$5 m \ddot{x} = -F - 2 k x \quad \text{or} \quad F = -2 k x - 5 m \ddot{x} \quad (3)$$

Equations (1) and (2) can be rewritten as:

$$m \left(\ddot{x} - \frac{\ell}{2} \ddot{\theta} \right) - k \theta \ell + 2 k x + 5 m \ddot{x} = 0 \quad (5)$$

$$J_0 \ddot{\theta} - \frac{m \ell}{2} \left(\ddot{x} - \frac{\ell}{2} \ddot{\theta} \right) + k \theta \ell^2 - \frac{m g \ell \theta}{2} = 0 \quad (6)$$

$$\text{where } J_0 = \frac{1}{3} m \ell^2 \quad (7)$$

Equations (5) and (6) can be expressed in matrix form as:

$$\begin{bmatrix} 6m & -\frac{m\ell}{2} \\ -\frac{m\ell}{2} & \frac{7}{12}m\ell^2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 2k & -k\ell \\ 0 & (k\ell^2 - \frac{1}{2}mg\ell) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (8)$$

Assuming a solution of the form:

$$x(t) = X e^{st} \quad \text{and} \quad \theta(t) = \Theta e^{st} \quad (9)$$

Eq. (8) can be expressed as:

$$\left[s^2 \begin{bmatrix} 6m & -\frac{m\ell}{2} \\ -\frac{m\ell}{2} & \frac{m\ell^2}{3} \end{bmatrix} + \begin{bmatrix} 2k & -k\ell \\ 0 & -\left(\frac{mg\ell}{2} - k\ell^2\right) \end{bmatrix} \right] \begin{Bmatrix} X \\ \Theta \end{Bmatrix} e^{st} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (10)$$

By setting the determinant of the coefficient matrix in Eq. (10) equal to zero, we obtain:

$$\begin{vmatrix} (6ms^2 + 2k) & -\left(\frac{m\ell s^2}{2} + k\ell\right) \\ -\left(\frac{m\ell s^2}{2}\right) & \left(\frac{m\ell^2 s^2}{3} - \frac{mg\ell}{2} + k\ell^2\right) \end{vmatrix} = 0 \quad (11)$$

which, upon expansion, gives:

$$\left(\frac{7}{4}m^2\ell^2\right)s^4 + \left(\frac{37}{6}mk\ell^2 - 3m^2g\ell\right)s^2 + \left(-mk g\ell + 2k^2\ell^2\right) = 0 \quad (12)$$

A comparison of Eq. (12) with Eq. (5.43) gives:

$$\begin{aligned} a_0 &= \frac{7}{4}m^2\ell^2 \\ a_1 &= 0 \\ a_2 &= \frac{37}{6}mk\ell^2 - 3m^2g\ell \\ a_3 &= 0 \\ a_4 &= 2k^2\ell^2 - mk g\ell \end{aligned}$$

Conditions for the stability of the system:

1. All coefficients a_i must be positive:

$$a_2 \geq 0 \text{ or } k \geq \frac{18}{37} \frac{mg}{\ell}$$

$$a_4 \geq 0 \text{ or } k \geq \frac{1}{2} \frac{mg}{\ell}$$

2.

$$a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2$$

This is not applicable since both sides of the inequality are zero.

Thus the condition for stability is: $k \geq \frac{1}{2} \frac{mg}{\ell}$.

5.62

Equations of motion are (Eqs. (5.1) and (5.2))

$$2m \ddot{x}_1 + 3k x_1 - 2k x_2 = F_1(t)$$

$$m \ddot{x}_2 - 2k x_1 + 3k x_2 = 0$$

Hence $[m] = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix}$, $[k] = \begin{bmatrix} 3k & -2k \\ -2k & 3k \end{bmatrix} = \begin{bmatrix} 6000 & -4000 \\ -4000 & 6000 \end{bmatrix}$,

$$\vec{F} = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix}$$

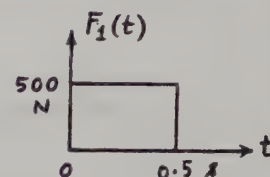
Frequency equation is $|\omega^2[m] + [k]| = \omega^4 - 900\omega^2 + 100000 = 0$

Hence $\omega_1 = 11.3949 \text{ rad/s}$, $\omega_2 = 27.7517 \text{ rad/s}$

and $\tau_1 = 0.5514 \text{ s}$, $\tau_2 = 0.2264 \text{ s}$.

We select $\Delta t = 0.02 \text{ s}$ and use central difference method for numerical solution (see chapter 11 for details).

The main program which calls CDIFF, the subroutine EXTFUN and the output are given below.



```

C =====
C
C PROGRAM
C MAIN PROGRAM WHICH CALLS CDIFF
C
C =====
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
  PEAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
  DIMENSION C(2,2),X1(2),XD1(2),XDD1(2),XM1(2),F(2),R(2),RR(2),
2  XMK(2),XMI(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
3  S(2),X(50,2),XD(50,2),XDD(50,2)
  DATA N,NSIEP,NSTEP1,DELT/2,49,50,0.02/
  DATA X1/0.0,0.0/
  DATA XD1/0.0,0.0/
  DATA M/20.0,0.0,0.0,10.0/
  DATA C/0.0,0.0,0.0,0.0/
  DATA K/6000.0,-4000.0,-4000.0,6000.0/
C END OF PROBLEM-DEPENDENT DATA
  CALL CDIFF (M,C,K,X1,XD1,XDD1,N,NSTEP,DELT,F,R,RR,XM1,XM2,XP1,
2  MC,MK,MCI,XMK,MIC,XMI,ZA,ZB,ZC,LA,LB,S,X,XD,XDD,NSTEP1)

```

```

WRITE (13,10)
10  FORMAT (//,38H SOLUTION BY CENTRAL DIFFERENCE METHOD,/)
    WRITE (13,20) N,NSTEP,DELT
20  FORMAT (12H GIVEN DATA:/,3H N=,I5,4X,7H NSTEP=,I5,4X,6H DELT=,
2    E15.8,/)
    WRITE (13,30)
30  FORMAT (10H SOLUTION:,,5H STEP,3X,5H TIME,3X,7H X(I,1),3X,
2    8H XD(I,1),2X,9H XDD(I,1),4X,7H X(I,2),3X,8H XD(I,2),2X,
3    9H XDD(I,2),/)
    DO 40, I=1,NSTEP1
    TIME=REAL(I-1)*DELT
40  WRITE (13,50) I,TIME,X(I,1),XD(I,1),XDD(I,1),X(I,2),XD(I,2),XDD(
2    I,2)
50  FORMAT (1X,I4,F8.4,6(1X,E10.4))
    STOP
    END

```

```

C =====
C
C SUBROUTINE EXTFUN
C THIS SUBROUTINE IS PROBLEM-DEPENDENT
C
C =====

```

```

SUBROUTINE EXTFUN (F,TIME,N)
DIMENSION F(N)
F(1)=0.0
F(2)=0.0
IF (TIME .LE. 0.5) F(1)=500.0
RETURN
END

```

SOLUTION BY CENTRAL DIFFERENCE METHOD

GIVEN DATA:

N= 2 NSTEP= 49 DELT= 0.20000000E-01

SOLUTION:

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.2500E+02	0.0000E+00	0.0000E+00	0.0000E+00
2	0.0200	0.5000E-02	0.0000E+00	0.2500E+02	0.0000E+00	0.0000E+00	0.0000E+00
3	0.0400	0.1940E-01	0.4850E+00	0.2350E+02	0.8000E-03	0.2000E-01	0.2000E+01
4	0.0600	0.4154E-01	0.9134E+00	0.1934E+02	0.4512E-02	0.1128E+00	0.7280E+01
5	0.0800	0.6905E-01	0.1241E+01	0.1344E+02	0.1379E-01	0.3247E+00	0.1391E+02
6	0.1000	0.9938E-01	0.1446E+01	0.7043E+01	0.3080E-01	0.6572E+00	0.1935E+02
7	0.1200	0.1302E+00	0.1530E+01	0.1347E+01	0.5632E-01	0.1063E+01	0.2127E+02
8	0.1400	0.1600E+00	0.1515E+01	-.2810E+01	0.8917E-01	0.1459E+01	0.1831E+02
9	0.1600	0.1877E+00	0.1436E+01	-.5164E+01	0.1262E+00	0.1747E+01	0.1050E+02
10	0.1800	0.2129E+00	0.1323E+01	-.6059E+01	0.1630E+00	0.1846E+01	-.6577E+00
:							
46	0.9000	0.2345E+00	-.1023E+01	-.8737E+01	0.1991E+00	-.5925E+00	-.2714E+02
47	0.9200	0.2101E+00	-.1165E+01	-.5533E+01	0.1715E+00	-.1120E+01	-.2565E+02
48	0.9400	0.1842E+00	-.1258E+01	-.3717E+01	0.1365E+00	-.1566E+01	-.1889E+02
49	0.9600	0.1571E+00	-.1325E+01	-.2966E+01	0.9808E-01	-.1837E+01	-.8195E+01
50	0.9800	0.1290E+00	-.1379E+01	-.2515E+01	0.6131E-01	-.1879E+01	0.3993E+01

5.63 (a) Frequency equation is

$$|-\omega^2 [m] + [k]| = \begin{vmatrix} -\omega^2 [0.2 & 0 \\ 0 & 0.2] + \begin{bmatrix} 36 & -18 \\ -18 & 18 \end{bmatrix} \end{vmatrix} = \omega^4 - 270\omega^2 + 8100 = 0$$

```

C =====
C
C PROGRAM 8
C MAIN PROGRAM FOR CALLING THE SUBROUTINE QUART
C
C =====
C SOLUTION OF: A(1)*(X**4)+A(2)*(X**3)+A(3)*(X**2)+A(4)*X+A(5)=0
C      DIMENSION A(5),RR(4),R1(4)
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
C      DATA A/1.0,0.0,-270.0,0.0,8100.0/
C END OF PROBLEM-DEPENDENT DATA
C      WRITE (26,10) (A(I),I=1,5)
10      FORMAT (//,31H SOLUTION OF A QUARTIC EQUATION,/,6H DATA:/,
2       7H A(1) =,E15.6,/,7H A(2) =,E15.6,/,7H A(3) =,E15.6,/,
3       7H A(4) =,E15.6,/,7H A(5) =,E15.6,/)
C      CALL QUART (A,RR,R1)
C      WRITE (26,20)
20      FORMAT (/,7H ROOTS:,,/,9H ROOT NO.,3X,10H REAL PART,5X,
2       15H IMAGINARY PART,/)
C      DO 30 I=1,4
30      WRITE (26,40) I,RR(I),R1(I)
40      FORMAT (I5,3X,E15.6,3X,E15.6)
C      STOP
C      END

```

SOLUTION OF A QUARTIC EQUATION

DATA:

A(1) = 0.100000E+01
 A(2) = 0.000000E+00
 A(3) = -0.270000E+03
 A(4) = 0.000000E+00
 A(5) = 0.810000E+04

ROOTS:

ROOT NO.	REAL PART	IMAGINARY PART
1	-0.153500E+02	0.000000E+00
2	-0.586319E+01	0.000000E+00
3	0.586319E+01	0.000000E+00
4	0.153500E+02	0.000000E+00

(b) $(-0.2 \omega_j^2 + 36) X_j^{(1)} - 18 X_j^{(2)} = 0 ; j=1,2$
 Writing $X_j^{(2)} = \left(\frac{-0.2 \omega_j^2 + 36}{18} \right) X_j^{(1)} \equiv r_j X_j^{(1)}$,

$$r_1 = \{-0.2 (5.86319)^2 + 36\} / 18 = 1.6180334$$

$$r_2 = \{-0.2 (15.35)^2 + 36\} / 18 = -0.6180278$$

Eqs. (5.18) give $X_1^{(1)} = 1.44722$, $X_1^{(2)} = 0.55279$, $\phi_1 = \phi_2 = 0$

Displacements of masses m_1 and m_2 are given by Eqs. (5.15):

$$x_1(t) = 1.44722 \cos(5.86319t) + 0.55279 \cos(15.35t)$$

$$x_2(t) = 2.34165 \cos(5.86319t) - 0.34164 \cos(15.35t)$$

5.64

```

C =====
C
C PROBLEM 5.64
C
C =====
      DIMENSION C(2,2),FZ(2)
      REAL K(2,2),M(2,2)
      COMPLEX Z(2,2),X(2),AA,BB,DEN
C INPUT DATA
      DATA 4/0.1,0.0,0.0,0.1/
      DATA C/1.0,0.0,0.0,0.0,0.0/
      DATA K/40.0,-20.0,-20.0,20.0/
      DATA FZ/1.0,2.0/
      OMF=5.0
C END OF INPUT DATA
      DO 10 I=1,2
      DO 10 J=1,2
      A=- (OMF**2)*M(I,J)+K(I,J)
      B=OMF*C(I,J)
10    Z(I,J)=CMPLX(A,B)
      DEN=Z(1,1)*Z(2,2)-Z(1,2)*Z(1,2)
      AA=Z(2,2)*CMPLX(FZ(1),0.0)
      BB=Z(1,2)*CMPLX(FZ(2),0.0)
      X(1)=(AA-BB)/DEN
      AA=Z(1,1)*CMPLX(FZ(2),0.0)
      BB=Z(1,2)*CMPLX(FZ(1),0.0)
      X(2)=(AA-BB)/DEN
      PRINT 20, X(1),X(2)
20    FORMAT (//,2X,25H SOLUTION OF PROBLEM 5.49,//,2X,
2 7H X(1) =,2E15.8,/,2X,7H X(2) =,2E15.8,/)
      PRINT 30, OMF
30    FORMAT (2X,6H OMF =,E15.8)
      STOP
      END

```

SOLUTION OF PROBLEM 5.64

$X(1) = 0.20095898E+00-0.68620138E-01$
 $X(2) = 0.34395310E+00-0.78423016E-01$
 $OMF = 0.50000000E+01$

5.65

The system shown in Fig. A can be drawn in equivalent form as shown in Fig. B where both pulleys have the same radius, r_1 .

The equivalent mass moment of inertia of pulley 2 can be computed in different speed ratios as :

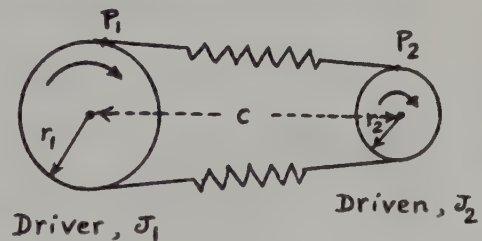


Fig. A

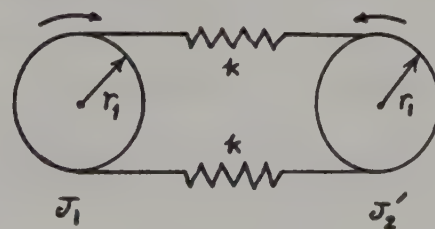


Fig. B

$$\begin{aligned}
 J_2' &= J_2 (\text{speed ratio})^2 \\
 &= J_2 \left(\frac{150}{350}\right)^2 ; J_2 \left(\frac{250}{350}\right)^2 ; \\
 &\quad J_2 \left(\frac{450}{350}\right)^2 ; J_2 \left(\frac{750}{350}\right)^2
 \end{aligned}$$

or $J_2' = 0.1837 J_2 ; 0.5102 J_2 ; 1.6531 J_2 ; 4.5918 J_2$

Stiffness of the belt (on each side) is given by

$$k = \frac{AE}{l}$$

where A = cross-sectional area of belt, E = Young's modulus and l = length of the belt. Length of the belt (distance $P_1 P_2$ in Fig. A) is given by

$$l = \frac{1}{2} [4c^2 - (D-d)^2]^{\frac{1}{2}}$$

In this example, $c = 5 \text{ m}$, $D = 1 \text{ m}$, $d = 0.25 \text{ m}$ and hence

$$l = \frac{1}{2} [4(5)^2 - (1 - 0.25)^2]^{\frac{1}{2}} = 4.9859 \text{ m}$$

$$\therefore k = \frac{A(10^{10})}{4.9859} = 2.0057 \times 10^9 A \text{ N/m}$$

Equation of motion:

$$J_1 \ddot{\theta}_1 + k_t (\theta_1 + \theta_2) = 0 \Rightarrow J_1 \ddot{\theta}_1 + k_t \theta_1 \left(1 + \frac{J_1}{J_2'}\right) = 0$$

$$J_2 \ddot{\theta}_2 + k_t (\theta_1 + \theta_2) = 0 \Rightarrow J_2 \ddot{\theta}_2 + k_t \theta_2 \left(\frac{J_2'}{J_1} + 1\right) = 0$$

$$\therefore \omega_n = \sqrt{k_t \left(\frac{J_1 + J_2'}{J_1 J_2'}\right)}$$

where $k_t = 2k r_1^2$ (see solution of problem 2.51).

Here $J_1 = 0.1 \text{ kg-m}^2$ and $J_2 = 0.2 \text{ kg-m}^2$. In order for the natural frequency ω_n to be away from the speeds 150, 250, 350, 450 and 750 rpm {or, 15.708, 26.180, 36.652, 47.124 and 78.540 rad/sec},

$$\omega_n \leq 15.708 \text{ rad/sec}$$

$$\omega_n \geq 78.540 \text{ rad/sec}$$

Since ω_n involves A (through k_t), it can be determined from the above inequalities.

5.66

Velocity of tup before impact is given by:

$$\frac{1}{2} m_{\text{tup}} v^2 = m_{\text{tup}} g h \quad \text{or} \quad v = \sqrt{2 g h} = \sqrt{2 (9.81) (2)} = 6.2642 \text{ m/sec}$$

(a) Impact is inelastic:

Conservation of momentum leads to:

$$m_{\text{tup}} v_{\text{tup}} + m_{\text{anvil}} (0) = (m_{\text{tup}} + m_{\text{anvil}}) v_0$$

$$\text{or} \quad v_0 = \frac{(1000) (6.2642)}{(1000 + 5000)} = 1.0440 \text{ m/sec}$$

(b) Natural frequencies:

$$\omega_{2,1}^2 = \frac{k_1 + k_2}{2 m_1} + \frac{k_2}{2 m_2} \pm \sqrt{\frac{1}{4} \left[\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right]^2 - \frac{k_1 k_2}{m_1 m_2}}$$

Thus the natural frequency requirement can be stated as:

$$f_1^2 = \frac{\omega_1^2}{(2 \pi)^2} = \frac{1}{(2 \pi)^2} \left\{ \frac{k_1 + k_2}{50000} + \frac{k_2}{10000} - \sqrt{\frac{1}{4} \left[\frac{k_1 + k_2}{25000} + \frac{k_2}{5000} \right]^2 - \frac{k_1 k_2}{125 (10^6)}} \right\} > (5^2) \quad (1)$$

(c) Free vibration response:

Initial conditions:

$$x_1(0) = x_2(0) = \dot{x}_1(0) = 0, \quad \dot{x}_2(0) = v_0 = 1.0440 \text{ m/sec}$$

Maximum forces in the springs:

$$F_1 = k_1 x_1 |_{\text{max}} \quad (2)$$

$$F_2 = k_2 (x_2 - x_1) |_{\text{max}} \quad (3)$$

For a helical spring, the shear stress (τ) under an axial force F is given by:

$$\tau = k_s \frac{8 F D}{\pi d^3} \quad (4)$$

where k_s = shear stress correction factor = $\frac{2D + d}{2D}$, D = mean coil diameter, and d = wire diameter.

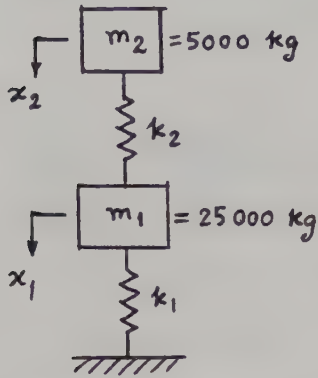
Ref: J. E. Shigley and C. R. Mischke, "Mechanical Engineering Design," 5th Ed., McGraw-Hill, New York, 1989.

Since stress is to be less than the yield stress with a factor of safety of 1.5, we have

$$\tau_1 \leq \frac{\tau_{\text{yield}}}{1.5} \quad (5)$$

$$\tau_2 \leq \frac{\tau_{\text{yield}}}{1.5} \quad (6)$$

where τ_1 and τ_2 denote the shear stresses induced in the springs k_1 and k_2 , respectively, and τ_{yield} is the shear stress corresponding to the yield stress of the material.



Chapter 6

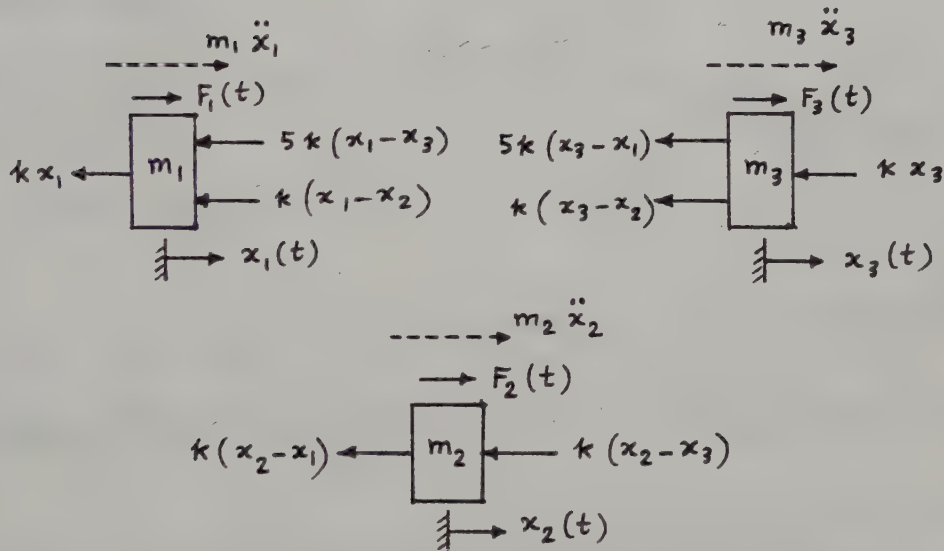
Multidegree of Freedom Systems

6.1 Equations of motion:

$$\begin{aligned} m_1 \ddot{x}_1 &= -k x_1 - 5k(x_1 - x_3) - k(x_1 - x_2) + F_1(t) \\ m_2 \ddot{x}_2 &= -k(x_2 - x_1) - k(x_2 - x_3) + F_2(t) \\ m_3 \ddot{x}_3 &= -5k(x_3 - x_1) - k(x_3 - x_2) - k x_3 + F_3(t) \end{aligned}$$

or

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k \begin{bmatrix} 7 & -1 & -5 \\ -1 & 2 & -1 \\ -5 & -1 & 7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{Bmatrix}$$



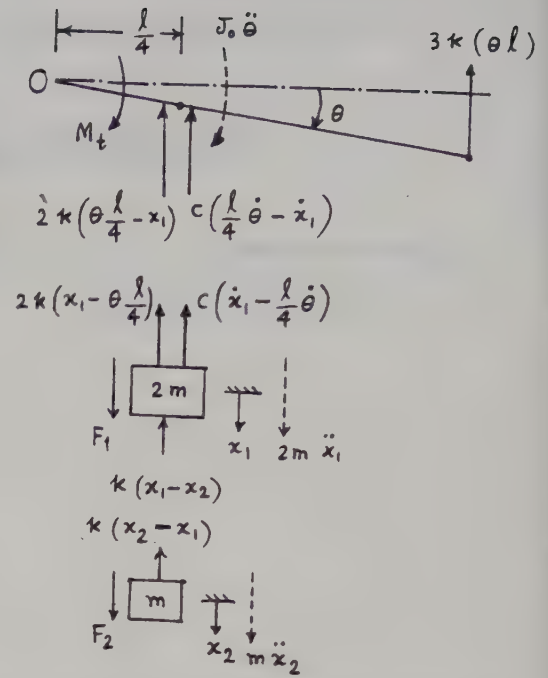
6.2 Equations of motion:

$$\begin{aligned} J_0 \ddot{\theta} &= -2k \left(\frac{\ell}{4} \theta - x_1 \right) \frac{\ell}{4} - c \left(\frac{\ell}{4} \dot{\theta} - \dot{x}_1 \right) \frac{\ell}{4} - 3k(\theta \ell) \ell + M_t \\ 2m \ddot{x}_1 &= -2k \left(x_1 - \frac{\ell}{4} \theta \right) - c \left(\dot{x}_1 - \frac{\ell}{4} \dot{\theta} \right) - k(x_1 - x_2) + F_1 \\ m \ddot{x}_2 &= -k(x_2 - x_1) + F_2 \end{aligned}$$

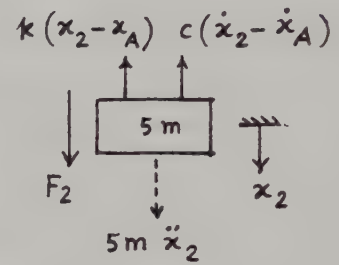
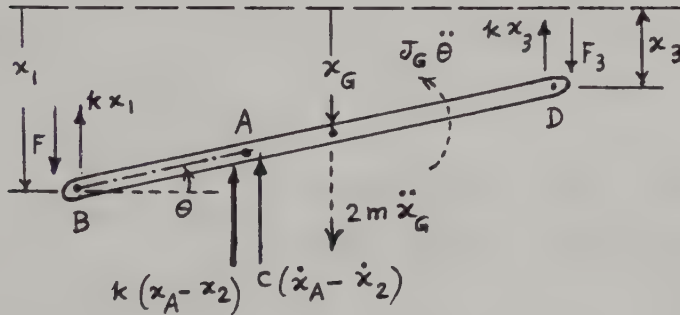
$$\text{where } J_0 = \frac{1}{3} (2m) \ell^2 = \frac{2}{3} m \ell^2$$

These equations can be stated in matrix form as:

$$\begin{bmatrix} \frac{2}{3} m \ell^2 & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} \frac{c \ell^2}{16} & -\frac{c \ell}{4} & 0 \\ -\frac{c \ell}{4} & c & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \frac{25 k \ell^2}{8} & -\frac{k \ell}{2} & 0 \\ -\frac{k \ell}{2} & 3k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} \theta \\ x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} M_t \\ F_1 \\ F_2 \end{Bmatrix}$$



6.3



Equation of motion for rotation about B:

$$J_G \ddot{\theta} - 2m \ddot{x}_G \left(\frac{5\ell}{2} \right) = k x_3 (5\ell) - F_3 (5\ell) + k (x_A - x_2) (2\ell) + c (\dot{x}_A - \dot{x}_2) (2\ell) \quad (1)$$

Equation of motion for rotation about D:

$$J_G \ddot{\theta} + 2m \ddot{x}_G \left(\frac{5\ell}{2} \right) = -k x_1 (5\ell) + F_1 (5\ell) - k (x_A - x_2) (3\ell) - c (\dot{x}_A - \dot{x}_2) (3\ell) \quad (2)$$

Equation of motion of mass 5m in vertical direction:

$$5m \ddot{x}_2 = -k (x_2 - x_A) - c (\dot{x}_2 - \dot{x}_A) + F_2 \quad (3)$$

Noting that

$$J_G = \frac{1}{12} (2m) (5\ell)^2 = \frac{25}{6} m \ell^2 ; \quad \theta = \frac{x_1 - x_3}{5\ell} ; \quad x_G = \frac{x_1 + x_3}{2}$$

$$\text{and } x_A = x_1 - 2\ell \theta = \frac{3}{5} x_1 + \frac{2}{5} x_3$$

Eqs. (1) to (3) can be rewritten as:

$$\begin{aligned} & \frac{m}{3} \ddot{x}_1 + \frac{2}{3} m \ddot{x}_3 + \frac{29}{25} k x_3 + \frac{6}{25} k x_1 \\ & - \frac{2}{5} k x_2 + \frac{6}{25} c \dot{x}_1 + \frac{4}{25} c \dot{x}_3 - \frac{2}{5} c \dot{x}_2 = F_3 \end{aligned} \quad (4)$$

$$\begin{aligned} & \frac{2}{3} m \ddot{x}_1 + \frac{m}{3} \ddot{x}_3 + \frac{34}{25} k x_1 - \frac{3}{5} k x_2 \\ & + \frac{6}{25} k x_3 + \frac{9}{25} c \dot{x}_1 - \frac{3}{5} c \dot{x}_2 + \frac{6}{25} c \dot{x}_3 = F_1 \end{aligned} \quad (5)$$

$$\begin{aligned} & 5 m \ddot{x}_2 - \frac{3}{5} k x_1 + k x_2 - \frac{2}{5} k x_3 \\ & - \frac{3}{5} c \dot{x}_1 + c \dot{x}_2 - \frac{2}{5} c \dot{x}_3 = F_2 \end{aligned} \quad (6)$$

6.4

Equations of motion:

Mass M: $M \ddot{x}_1 = -k x_1 + T - 2k(x_1 - x_3 - r\theta) + F_1$ (1)

Mass m: $m \ddot{x}_3 = -3k x_3 + 2k(x_1 - x_3 - r\theta) + F_3$ (2)

Mass 3m: $3m \ddot{x}_2 = F_2 - T$ (3)

Rotation of pulley:

$$J_0 \ddot{\theta} = T(3r) + r(2k)(x_1 - x_3 - r\theta) \quad (4)$$

Noting that

$$\theta = \frac{x_2 - x_1}{3r}$$

and

$$x_1 - x_3 - r\theta = x_1 - x_3 - r \left(\frac{x_2 - x_1}{3r} \right) = \frac{4}{3} x_1 - \frac{1}{3} x_2 - x_3$$

Eq. (4) can be used to find the tension T as:

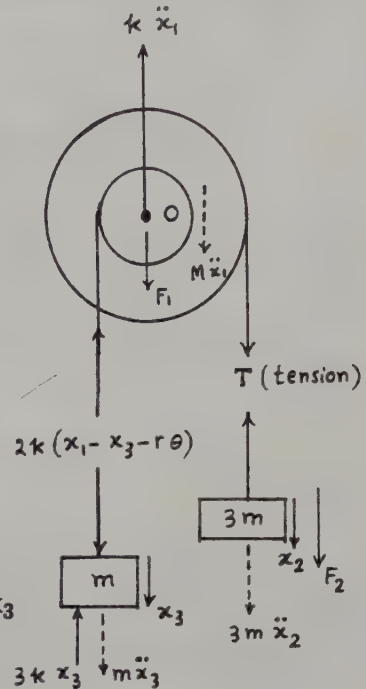
$$T = \left(\frac{J_0}{9r^2} \right) (\ddot{x}_2 - \ddot{x}_1) - \frac{8}{9} k x_1 + \frac{2}{9} k x_2 + \frac{2}{3} k x_3 \quad (5)$$

Using the expression of T, Eqs. (1) to (3) can be rewritten as

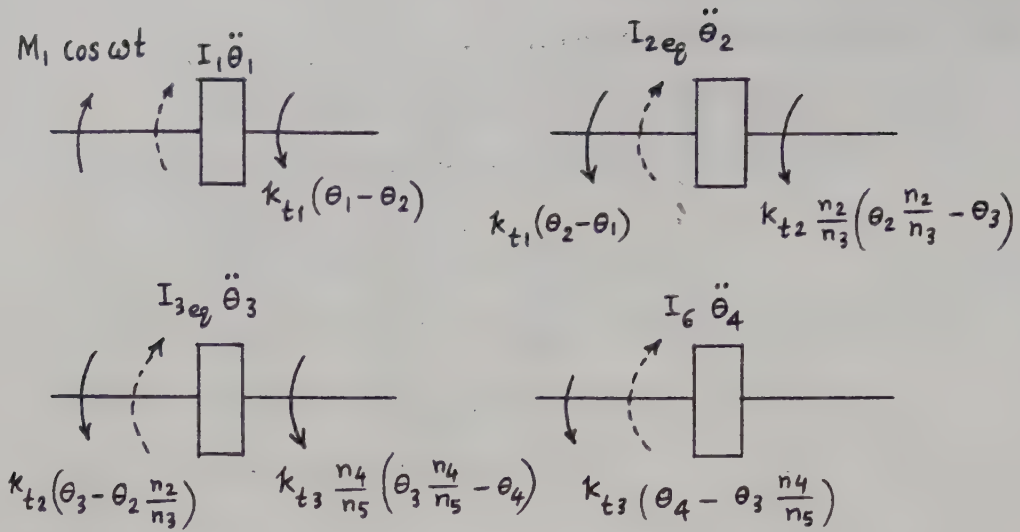
$$\left(M + \frac{J_0}{9r^2} \right) \ddot{x}_1 - \frac{J_0}{9r^2} \ddot{x}_2 + \frac{41}{9} k x_1 - \frac{8}{9} k x_2 - \frac{8}{3} k x_3 = F_1(t) \quad (6)$$

$$-\frac{J_0}{9r^2} \ddot{x}_1 + \left(3m + \frac{J_0}{9r^2} \right) \ddot{x}_2 - \frac{8}{9} k x_1 + \frac{2}{9} k x_2 + \frac{2}{3} k x_3 = F_2(t) \quad (7)$$

$$m \ddot{x}_3 - \frac{8}{3} k x_1 + \frac{2}{3} k x_2 + 5k x_3 = F_3(t) \quad (8)$$



6.5



$$I_{2eq} = I_2 + I_3 \left(\frac{n_2}{n_3} \right)^2 ; I_{3eq} = I_4 + I_5 \left(\frac{n_4}{n_5} \right)^2$$

Equations of motion:

$$\begin{aligned} I_1 \ddot{\theta}_1 + k_{t1} (\theta_1 - \theta_2) &= M_1 \cos \omega t \\ \left(I_2 + I_3 \frac{n_2^2}{n_3^2} \right) \ddot{\theta}_2 + k_{t1} (\theta_2 - \theta_1) + k_{t2} \frac{n_2}{n_3} (\theta_2 \frac{n_2}{n_3} - \theta_3) &= 0 \\ \left(I_4 + I_5 \frac{n_4^2}{n_5^2} \right) \ddot{\theta}_3 + k_{t2} (\theta_3 - \theta_2 \frac{n_2}{n_3}) + k_{t3} \frac{n_4}{n_5} (\theta_3 \frac{n_4}{n_5} - \theta_4) &= 0 \\ I_6 \ddot{\theta}_4 + k_{t3} (\theta_4 - \theta_3 \frac{n_4}{n_5}) &= 0 \end{aligned}$$

6.6

$$\begin{aligned} M \ddot{x}_3 &= -c_2 (\dot{x}_3 - \ell_1 \dot{\theta} - \dot{x}_1) - k_2 (x_3 - \ell_1 \theta - x_1) \\ &\quad - c_2 (\dot{x}_3 + \ell_1 \dot{\theta} - \dot{x}_2) - k_2 (x_3 + \ell_2 \theta - x_2) \end{aligned}$$

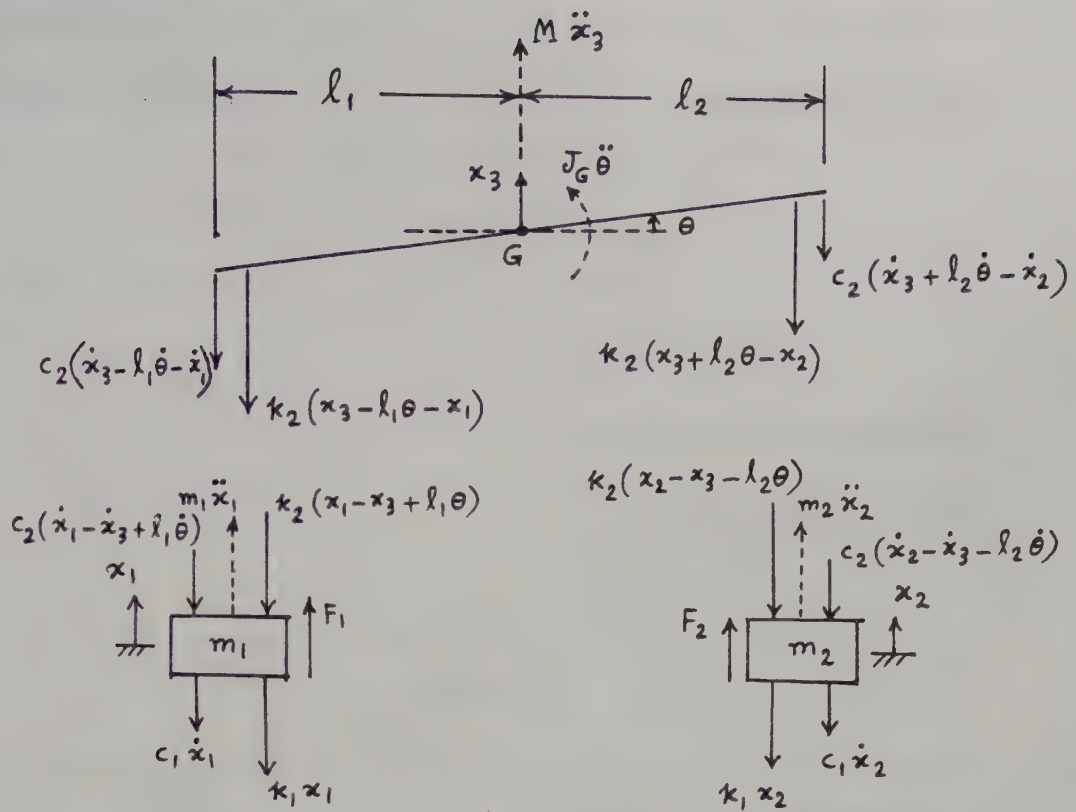
$$\begin{aligned} J_G \ddot{\theta} &= c_2 (\dot{x}_3 - \ell_1 \dot{\theta} - \dot{x}_1) \ell_1 + k_2 (x_3 - \ell_1 \theta - x_1) \ell_1 \\ &\quad - c_2 (\dot{x}_3 + \ell_2 \dot{\theta} - \dot{x}_2) \ell_2 - k_2 (x_3 + \ell_2 \theta - x_2) \ell_2 \end{aligned} \quad (2)$$

$$\begin{aligned} m_1 \ddot{x}_1 &= -c_2 (\dot{x}_1 - \dot{x}_3 + \ell_1 \dot{\theta}) - k_2 (x_1 - x_3 + \ell_1 \theta) \\ &\quad - c_1 \dot{x}_1 - k_1 x_1 + F_1 \end{aligned} \quad (3)$$

$$\begin{aligned} m_2 \ddot{x}_2 &= -c_2 (\dot{x}_2 - \dot{x}_3 - \ell_2 \dot{\theta}) - k_2 (x_2 - x_3 - \ell_2 \theta) \\ &\quad - k_1 x_2 - c_1 \dot{x}_2 + F_2 \end{aligned} \quad (4)$$

Eqs. (1) to (4) can be rewritten as

$$\begin{aligned} M_3 \dot{x}_3 + 2 c_2 \dot{x}_3 - c_2 \dot{x}_1 - c_2 \dot{x}_2 + 2 k_2 x_3 \\ + \theta (k_2 \ell_2 - k_2 \ell_1) - k_2 x_1 - k_2 x_2 &= 0 \end{aligned} \quad (5)$$

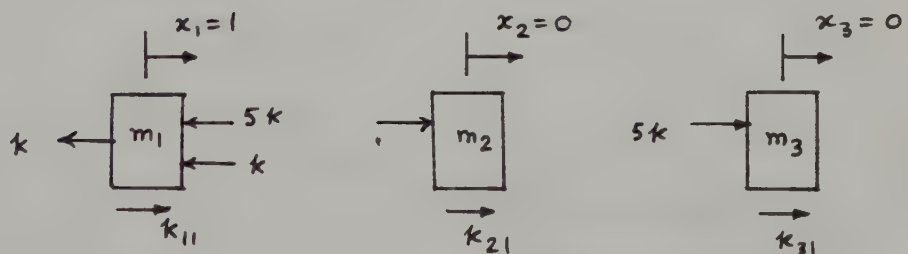


$$J_G \ddot{\theta} + (c_2 \ell_1^2 + c_2 \ell_2^2) \dot{\theta} + (-c_2 \ell_1 + c_2 \ell_2) \dot{x}_3 + c_2 \ell_1 \dot{x}_1 - c_2 \ell_2 \dot{x}_2 + (-k_2 \ell_1 + k_2 \ell_2) x_3 + k_2 \ell_1 x_1 - k_2 \ell_2 x_2 + (k_2 \ell_1^2 + k_2 \ell_2^2) \theta = 0 \quad (6)$$

$$m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 - c_2 \dot{x}_3 + c_2 \ell_1 \dot{\theta} + (k_1 + k_2) x_1 - k_2 x_3 + k_2 \ell_1 \theta = F_1 \quad (7)$$

$$m_2 \ddot{x}_2 + (c_1 + c_2) \dot{x}_2 - c_2 \dot{x}_3 - c_2 \ell_2 \dot{\theta} + (k_1 + k_2) x_2 - k_2 x_3 - k_2 \ell_2 \theta = F_2 \quad (8)$$

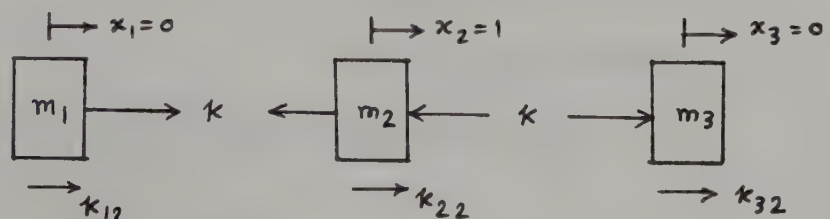
6.7



(i) Set $x_1 = 1, x_2 = x_3 = 0$:

Equilibrium of forces:

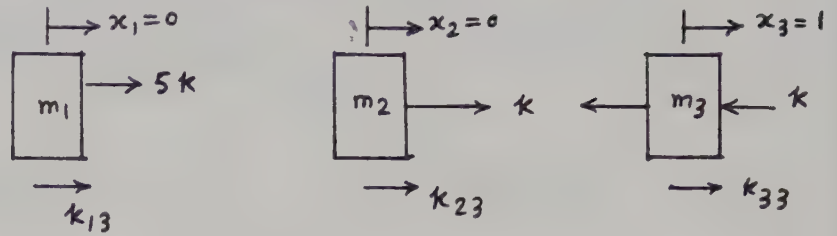
$$k_{11} - k - 5k - k = 0 \text{ or } k_{11} = 7k; k_{21} = -k; k_{31} = -5k$$



(ii) Set $x_2 = 1$, $x_1 = x_3 = 0$:

Equilibrium of forces:

$$k_{12} = -k ; k_{22} - k - k = 0 \text{ or } k_{22} = 2k ; k_{32} = -k$$

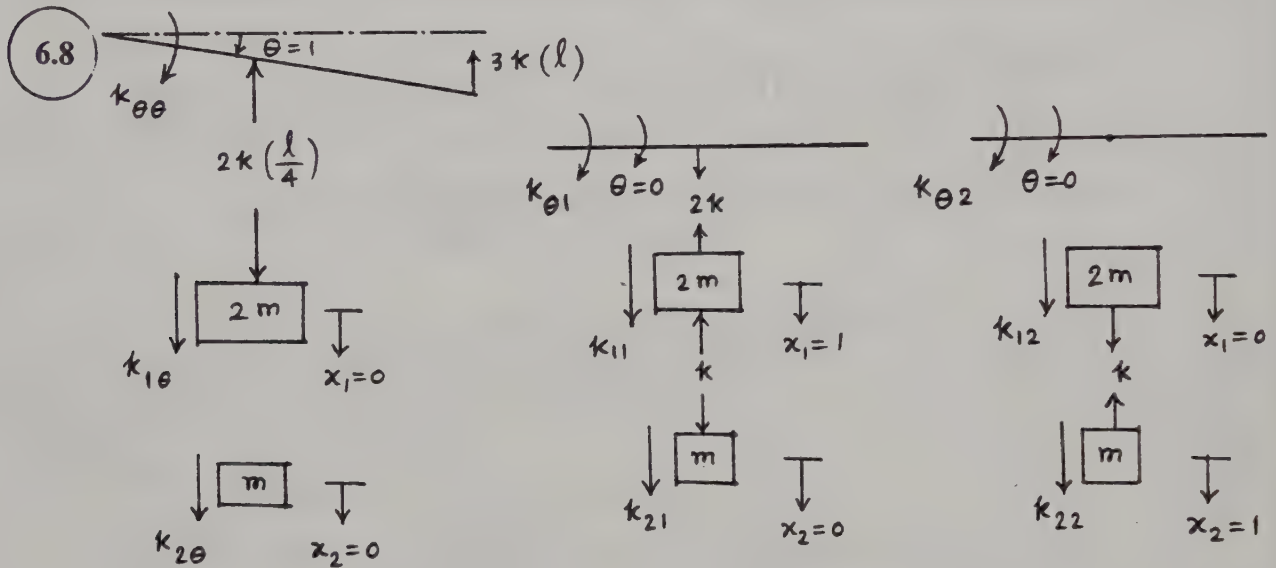


(iii) Set $x_3 = 1$, $x_1 = x_2 = 0$:

Equilibrium of forces:

$$k_{13} = -5k ; k_{23} = -k ; k_{33} = 7k$$

$$\therefore [k] = \begin{bmatrix} 7 & -1 & -5 \\ -1 & 2 & -1 \\ -5 & -1 & 7 \end{bmatrix}$$



(i) Give $\theta = 1$, $x_1 = x_2 = 0$:

Equilibrium equations:

$$k_{\theta\theta} = 3k\ell^2 + 2k(\ell/4)^2 = \frac{25}{8}k\ell^2 ; k_{1\theta} = -\frac{k\ell}{2} ; k_{2\theta} = 0$$

(ii) Give $x_1 = 1$, $\theta = x_2 = 0$:

Equilibrium equations:

$$k_{\theta 1} = -2k(\ell/4) = -\frac{k\ell}{2} ; k_{11} = 2k + k = 3k ; k_{21} = -k$$

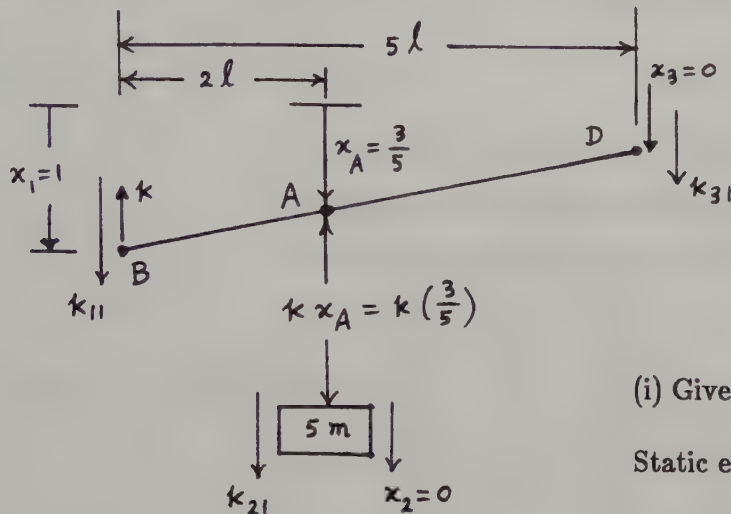
(iii) Give $x_2 = 1, \theta = x_1 = 0$:

Equilibrium equations:

$$k_{\theta 2} = 0 ; k_{12} = -k ; k_{22} = k$$

$$\therefore [k] = \begin{bmatrix} \frac{25 k \ell^2}{8} & -\frac{k \ell}{2} & 0 \\ -\frac{k \ell}{2} & 3 k & -k \\ 0 & -k & k \end{bmatrix}$$

6.9



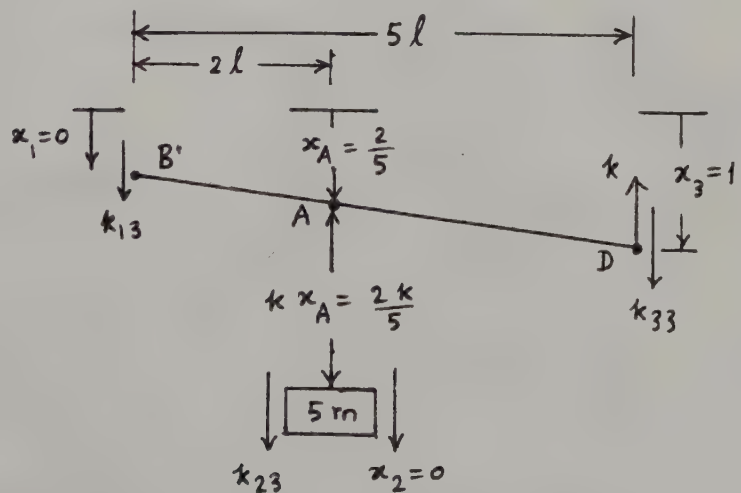
(i) Give $x_1 = 1, x_2 = x_3 = 0$:

Static equilibrium equations:

$$\sum M_B = 0 \text{ or } k_{31} (5 \ell) - \frac{3}{5} k (2 \ell) = 0 \text{ or } k_{31} = \frac{6}{25} k$$

$$\sum M_D = 0 \text{ or } k_{11} (5 \ell) - k (5 \ell) - \frac{3 k}{5} (3 \ell) = 0 \text{ or } k_{11} = \frac{34}{25} k$$

$$\sum F = 0 \text{ at mass } 5m \text{ or } k_{21} + \frac{3}{5} k = 0 \text{ or } k_{21} = -\frac{3}{5} k$$



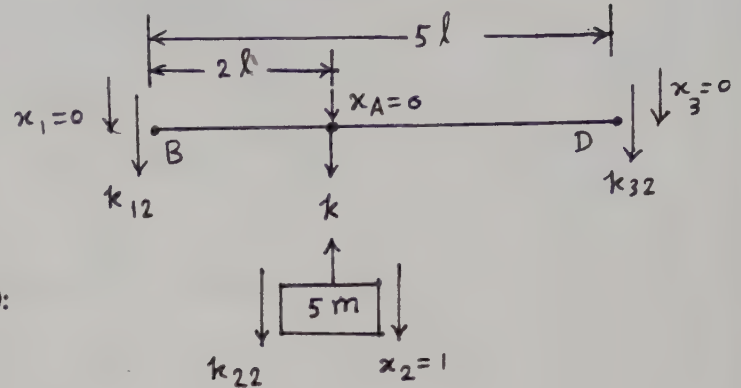
(ii) Give $x_3 = 1, x_1 = x_2 = 0$:

Static equilibrium equations:

$$\sum M_B = 0 \text{ or } k_{33} (5 \ell) - \frac{2 k}{5} (2 \ell) - k (5 \ell) = 0 \text{ or } k_{33} = \frac{29}{25} k$$

$$\sum M_D = 0 \text{ or } k_{13} (5\ell) - \frac{2k}{5} (3\ell) = 0 \text{ or } k_{13} = \frac{6}{25} k$$

$$\sum F = 0 \text{ at mass } 5m \text{ or } k_{23} + \frac{2}{5} k = 0 \text{ or } k_{23} = -\frac{2}{5} k$$



(iii) Give $x_3 = 1$, $x_1 = x_2 = 0$:

Static equilibrium equations:

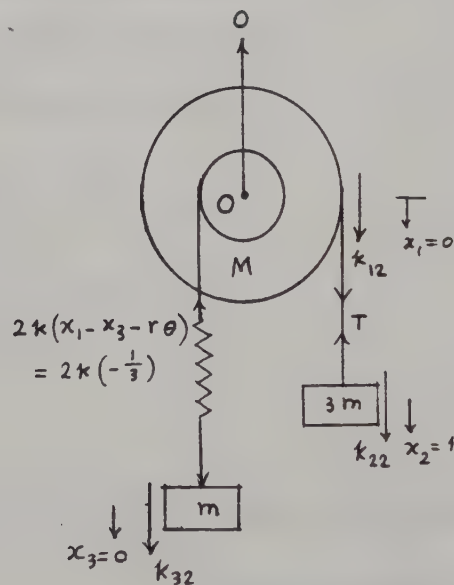
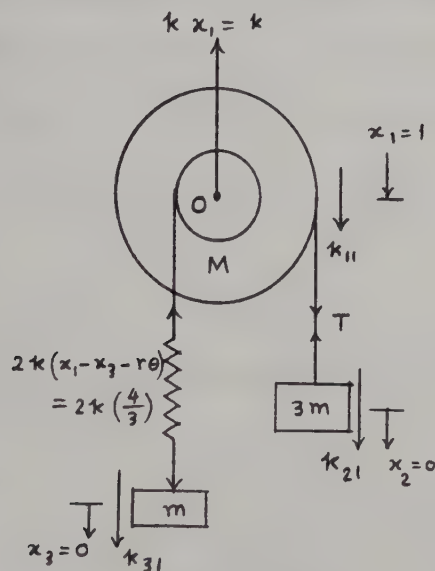
$$\sum M_B = 0 \text{ or } k_{32} (5\ell) + k (2\ell) = 0 \text{ or } k_{32} = -\frac{2}{5} k$$

$$\sum M_D = 0 \text{ or } k_{12} (5\ell) + k (3\ell) = 0 \text{ or } k_{12} = -\frac{3}{5} k$$

$$\sum F = 0 \text{ at mass } 5m \text{ or } k_{22} - k = 0 \text{ or } k_{22} = k$$

$$\therefore [k] = k \begin{bmatrix} \frac{34}{25} & -\frac{3}{5} & \frac{6}{25} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \\ \frac{6}{25} & -\frac{2}{5} & \frac{29}{25} \end{bmatrix}$$

6.10



$$\text{Here } \theta = \frac{x_2 - x_1}{3r}; \quad x_1 - x_3 - r\theta = \frac{4}{3} x_1 - \frac{1}{3} x_2 - x_3$$

(i) Give $x_1 = 1, x_2 = x_3 = 0$:

$$\sum F = 0 \text{ for mass } M: k_{11} - k + T - \frac{8k}{3} = 0 \quad (1)$$

$$\sum F = 0 \text{ for mass } m: k_{31} + \frac{8k}{3} = 0 \quad (2)$$

$$\sum F = 0 \text{ for mass } 3m: k_{21} = T \quad (3)$$

$$\sum M_0 = 0 \text{ for pulley: } T(3r) + 2k \left(\frac{4}{3} \right) r = 0 \text{ or } T = -\frac{8k}{9} \quad (4)$$

Eqs. (1) to (4) yield:

$$k_{21} = -\frac{8k}{9}, k_{31} = -\frac{8k}{3}, k_{11} = \frac{41}{9}k$$

(ii) Give $x_2 = 1, x_1 = x_3 = 0$:

$$\sum F = 0 \text{ for mass } M: k_{12} + T + \frac{2k}{3} = 0 \quad (5)$$

$$\sum F = 0 \text{ for mass } 3m: k_{22} - T = 0 \quad (6)$$

$$\sum F = 0 \text{ for mass } m: k_{32} - \frac{2k}{3} = 0 \quad (7)$$

$$\sum M_0 = 0 \text{ for pulley: } T(3r) + r \left(-\frac{2k}{3} \right) = 0 \quad (8)$$

Solution of Eqs. (5) to (8) yields:

$$k_{22} = T = \frac{2k}{9}, k_{32} = \frac{2k}{3}, k_{12} = -\frac{8k}{9}$$

(iii) Give $x_3 = 1, x_1 = x_2 = 0$:

$$\sum F = 0 \text{ for mass } M: k_{13} + T - (-2k) = 0 \quad (9)$$

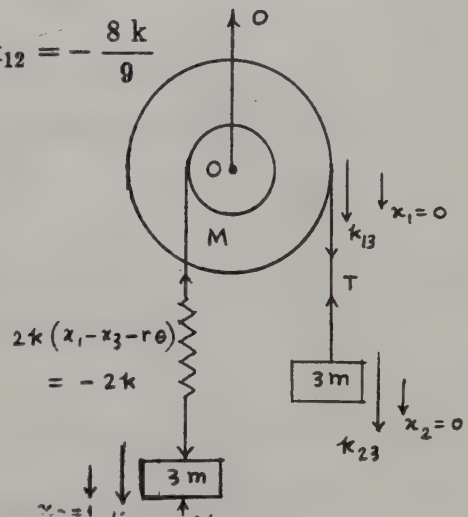
$$\sum F = 0 \text{ for mass } 3m: k_{23} - T = 0 \quad (10)$$

$$\sum F = 0 \text{ for mass } m: k_{33} - 2k - 3k = 0 \quad (11)$$

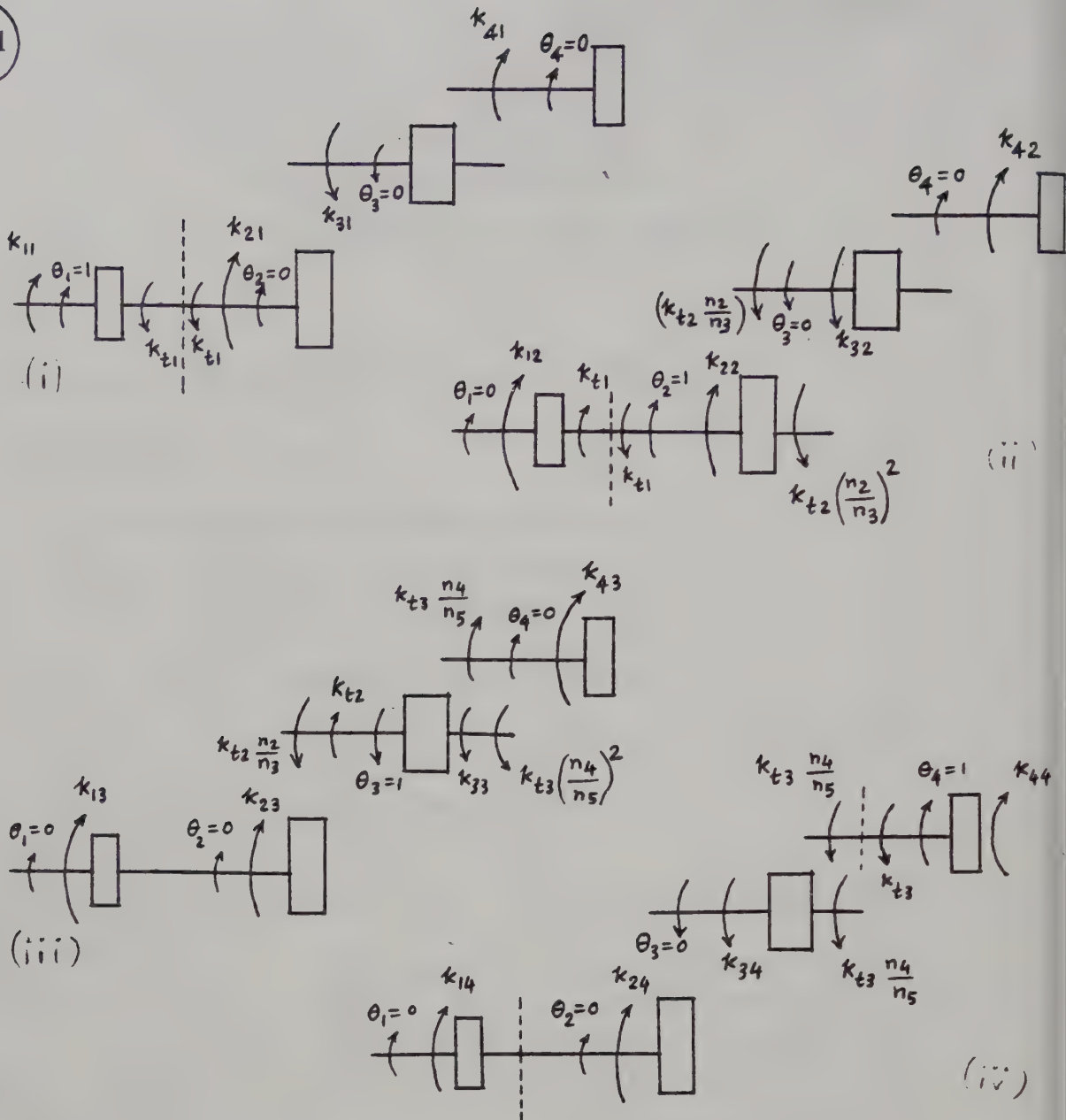
$$\sum M_0 = 0 \text{ for pulley: } T(3r) + r(-2k) = 0 \quad (12)$$

Solution of Eqs. (9) to (12) gives:

$$k_{33} = 5k, k_{23} = \frac{2k}{3}, k_{13} = -\frac{8k}{3}$$



$$\therefore [k] = k \begin{bmatrix} \frac{41}{9} & -\frac{8}{9} & -\frac{8}{3} \\ -\frac{8}{9} & \frac{2}{9} & \frac{2}{3} \\ -\frac{8}{3} & \frac{2}{3} & 5 \end{bmatrix}$$



(i) Set $\theta_1 = 1, \theta_2 = \theta_3 = \theta_4 = 0$:

Equilibrium equations give:

$$k_{11} = k_{t1}, \quad k_{21} = -k_{t1}, \quad k_{31} = k_{41} = 0$$

(ii) Set $\theta_2 = 1, \theta_1 = \theta_3 = \theta_4 = 0$:

Equilibrium equations yield:

$$k_{12} = -k_{t1}, \quad k_{22} = k_{t1} + k_{t2} \left(\frac{n_2}{n_3}\right)^2, \quad k_{32} = -k_{t2} \left(\frac{n_2}{n_3}\right), \quad k_{42} = 0$$

(iii) Set $\theta_3 = 1, \theta_1 = \theta_2 = \theta_4 = 0$:

Equilibrium equations provide:

$$k_{13} = 0, \quad k_{23} = -k_{t2} \left(\frac{n_2}{n_3} \right), \quad k_{33} = k_{t2} + k_{t3} \left(\frac{n_4}{n_5} \right)^2, \quad k_{43} = -k_{t3} \left(\frac{n_4}{n_5} \right)$$

(iv) Set $\theta_4 = 1, \theta_1 = \theta_2 = \theta_3 = 0$:

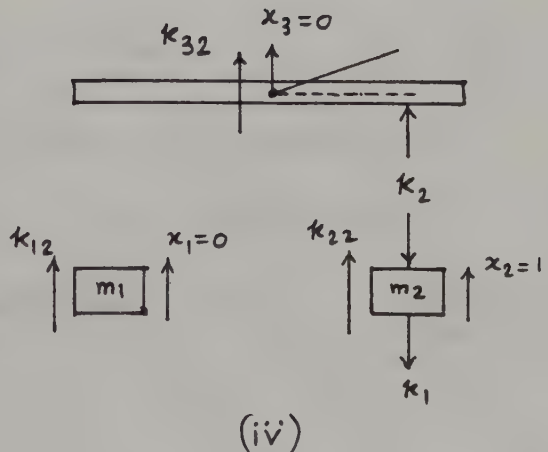
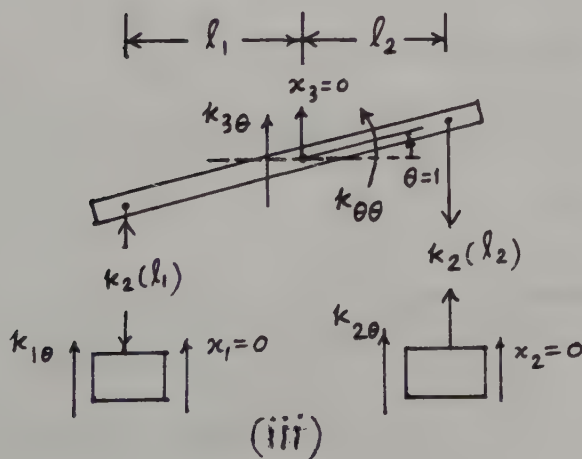
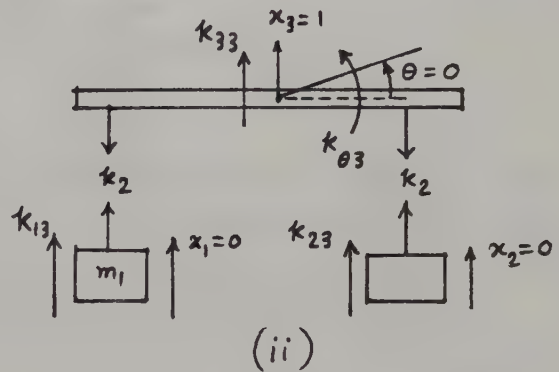
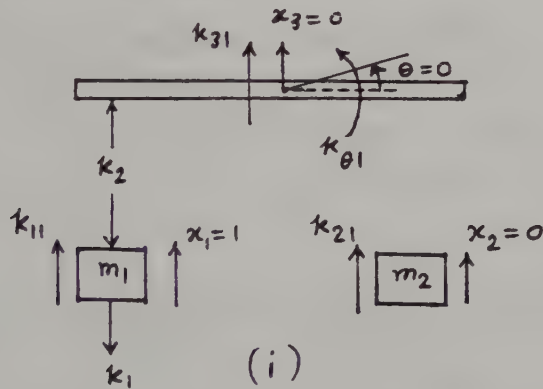
Equilibrium equations give:

$$k_{14} = k_{24} = 0, \quad k_{34} = -k_{t3} \left(\frac{n_4}{n_5} \right), \quad k_{44} = k_{t3}$$

Thus the stiffness matrix is:

$$[k] = \begin{bmatrix} k_{t1} & -k_{t1} & 0 & 0 \\ -k_{t1} & k_{t1} + k_{t2} \left(\frac{n_2}{n_3} \right)^2 & -k_{t2} \left(\frac{n_2}{n_3} \right) & 0 \\ 0 & -k_{t2} \left(\frac{n_2}{n_3} \right) & k_{t2} + k_{t3} \left(\frac{n_4}{n_5} \right)^2 & -k_{t3} \left(\frac{n_4}{n_5} \right) \\ 0 & 0 & -k_{t3} \left(\frac{n_4}{n_5} \right) & k_{t3} \end{bmatrix}$$

6.12



(i) Set $x_1 = 1, x_2 = x_3 = \theta = 0$:

Equilibrium equations:

$$k_{11} = k_1 + k_2, \quad k_{31} = -k_2, \quad k_{\theta 1} = k_2 \ell_1, \quad k_{21} = 0$$

(ii) Set $x_3 = 1, x_1 = x_2 = \theta = 0$:

Equilibrium equations:

$$k_{33} = 2k_2, \quad k_{13} = -k_2, \quad k_{23} = -k_2, \quad k_{\theta 3} = -k_2 \ell_1 + k_2 \ell_2$$

(iii) Set $\theta = 1, x_1 = x_2 = x_3 = 0$:

Equilibrium equations:

$$k_{\theta\theta} = k_2 (\ell_1^2 + \ell_2^2), \quad k_{\theta 3} = k_2 (\ell_2 - \ell_1), \quad k_{1\theta} = k_2 \ell_1, \quad k_{2\theta} = -k_2 \ell_2$$

(iv) Set $x_2 = 1, x_1 = x_3 = \theta = 0$:

Equilibrium equations:

$$k_{12} = 0, \quad k_{32} = -k_2, \quad k_{\theta 2} = -k_2 \ell_2, \quad k_{22} = k_1 + k_2$$

$$\therefore [k] = \begin{bmatrix} (k_1 + k_2) & 0 & -k_2 & k_2 \ell_1 \\ 0 & (k_1 + k_2) & -k_2 & -k_2 \ell_2 \\ -k_2 & -k_2 & 2k_2 & k_2 (\ell_2 - \ell_1) \\ k_2 \ell_1 & -k_2 \ell_2 & k_2 (\ell_2 - \ell_1) & k_2 (\ell_1^2 + \ell_2^2) \end{bmatrix}$$

6.13

(i) Give $F_x = 1, M_\theta = 0$:

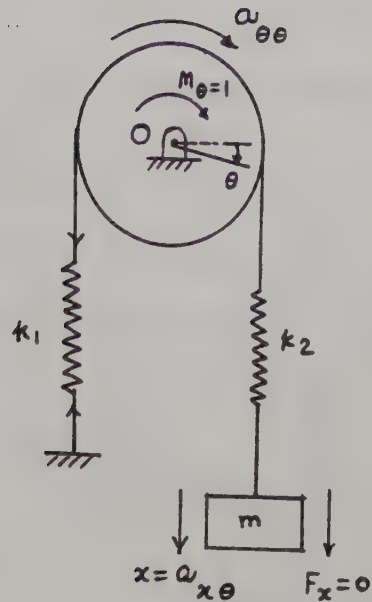
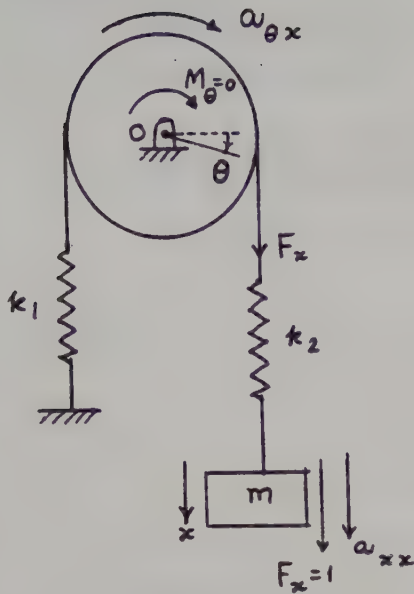
Same force of $F_x = 1$ is induced everywhere along the rope. Since k_1 and k_2 are in series,

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$a_{xx} = \frac{\text{force}}{k_{eq}} = \text{deflection at m due } F_x \text{ of } 1 = \frac{k_1 + k_2}{k_1 k_2}$$

Linear deflection of k_1 under $F_x = 1$ is $\frac{1}{k_1}$, angular deflection θ due to linear displacement of $\frac{1}{k_1} = a_{\theta x}$.

$$a_{\theta x} = \text{linear deflection of } k_1 = a_{\theta x} r = \frac{1}{k_1} ; \quad a_{\theta x} = \frac{1}{k_1 r}$$



(ii) Give $M_\theta = 1$, $F_x = 0$:

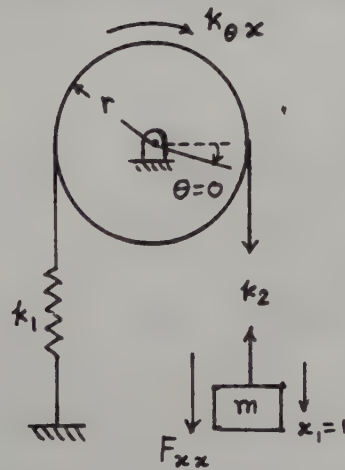
Extension of $k_1 = a_{\theta\theta} r$, spring force $= k_1 (a_{\theta\theta} r)$.

$$\sum M_\theta = 0 \text{ or } M_\theta = \left(k_1 a_{\theta\theta} r \right) r = 1 \text{ or } a_{\theta\theta} = \frac{1}{k_1 r^2}$$

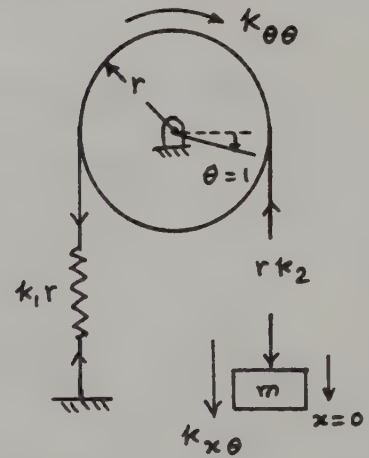
Displacement of $m = a_{x\theta} = a_{\theta\theta} r = \frac{1}{k_1 r}$

$$\therefore [a] = \begin{bmatrix} \frac{k_1 + k_2}{k_1 k_2} & \frac{1}{k_1 r} \\ \frac{1}{k_1 r} & \frac{1}{k_1 r^2} \end{bmatrix}$$

6.14



(i) Give $x = 1$, $\theta = 0$:



Equilibrium equations give:

$$k_{xx} - k_2 = 0 \text{ or } k_{xx} = k_2 ; k_{\theta x} + k_2 r = 0 \text{ or } k_{\theta x} = -k_2 r$$

(ii) Give $\theta = 1$, $x = 0$:

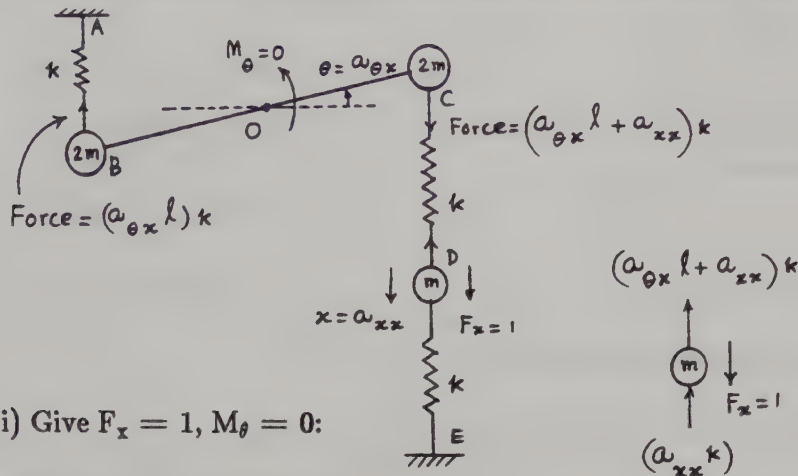
Equilibrium equations yield:

$$k_{x\theta} + r k_2 = 0 \text{ or } k_{x\theta} = -k_2 r$$

$$k_{\theta\theta} - r^2 k_2 - r^2 k_1 = 0 \text{ or } k_{\theta\theta} = r^2 (k_1 + k_2)$$

$$\therefore [k] = \begin{bmatrix} k_2 & -k_2 r \\ -k_2 r & (k_1 + k_2) r^2 \end{bmatrix}$$

6.15



(i) Give $F_x = 1$, $M_\theta = 0$:

Extension of spring AB = $a_{\theta x} \ell$.

Total extension of spring CD = $(a_{\theta x} \ell + a_{xx})$.

Compression of spring DE = a_{xx} .

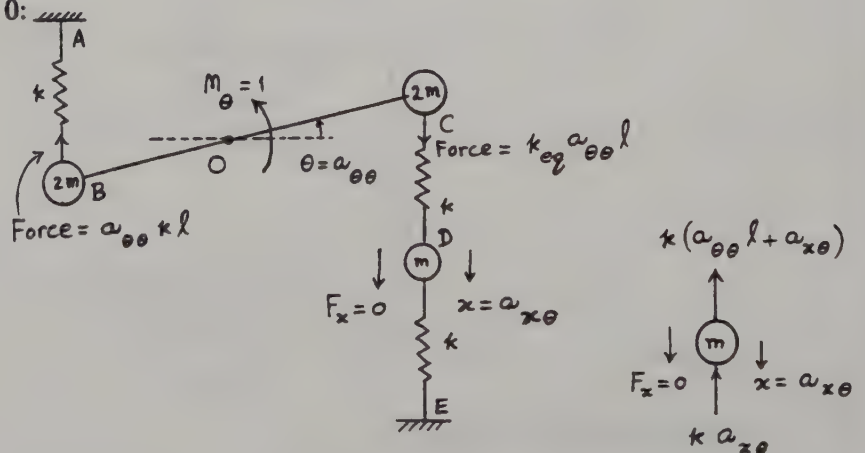
$$\sum F = 0 \text{ or } a_{xx} k + a_{\theta x} \ell k + a_{xx} k = 1 \text{ or } 2 a_{xx} k + a_{\theta x} k \ell = 1 \quad (1)$$

$$\sum M_0 = 0 \text{ or } a_{\theta x} k \ell + (a_{\theta x} k \ell + a_{xx} k) = 0 \text{ or } 2 a_{\theta x} k \ell + a_{xx} k = 0$$

Solution of Eqs. (1) and (2):

$$a_{xx} = \frac{2}{3 k} ; a_{\theta x} = -\frac{1}{3 k \ell}$$

(ii) Give $M_\theta = 1$, $F_x = 0$:



Extension of spring AB = $a_{\theta\theta} \ell$.

Total extension of springs CD and DE = $a_{\theta\theta} \ell$.

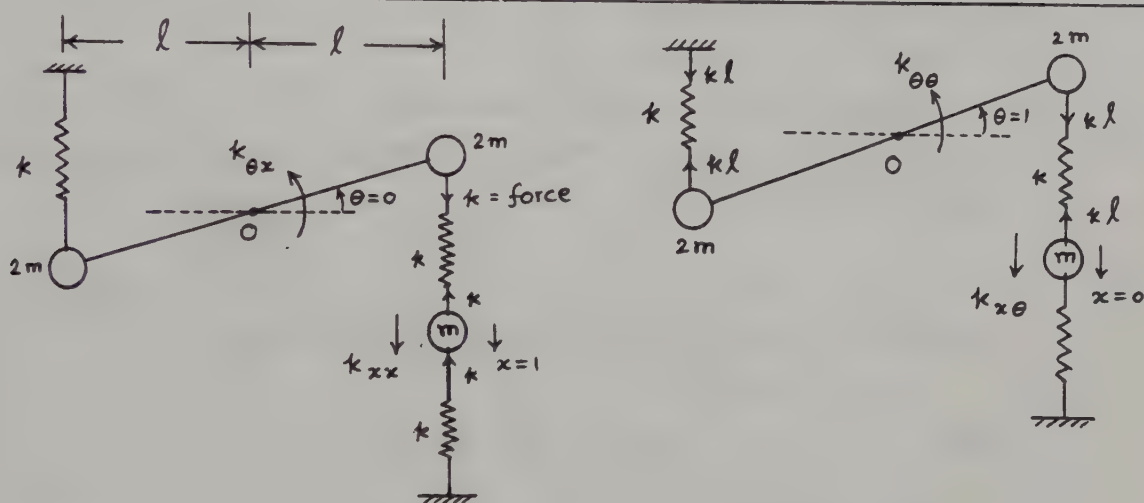
k_{eq} of series springs CD and DE = $\frac{k}{2}$.

$$\sum M_0 = 0 \text{ or } (a_{\theta\theta} k \ell) \ell + (k_{eq} a_{\theta\theta} \ell) \ell = 1 \text{ or } a_{\theta\theta} = \frac{2}{3 k \ell^2} \quad (3)$$

$$\sum F = 0 \text{ or } k a_{x\theta} + k a_{\theta\theta} \ell + k a_{x\theta} = 0 \text{ or } a_{x\theta} = -\frac{1}{3 k \ell} \quad (4)$$

$$\therefore [a] = \begin{bmatrix} \frac{2}{3 k} & -\frac{1}{3 k \ell} \\ -\frac{1}{3 k \ell} & \frac{2}{3 k \ell^2} \end{bmatrix}$$

6.16



(i) Give $x = 1, \theta = 0$:

$$\sum F = 0 \text{ or } k_{xx} - k - k = 0 \text{ or } k_{xx} = 2k$$

$$\sum M_0 = 0 \text{ or } k_{\theta x} - k \ell = 0 \text{ or } k_{\theta x} = k \ell$$

(ii) Give $\theta = 1, x = 0$:

$$\sum F = 0 \text{ or } k_{x\theta} - k \ell = 0 \text{ or } k_{x\theta} = k \ell$$

$$\sum M_0 = 0 \text{ or } k_{\theta\theta} - k \ell (\ell) - k \ell (\ell) = 0 \text{ or } k_{\theta\theta} = 2k \ell^2$$

$$\therefore [k] = \begin{bmatrix} 2k & k \ell \\ k \ell & 2k \ell^2 \end{bmatrix}$$

6.17

Kinetic energy of the system can be expressed as:

$$T = \frac{1}{2} (2m) (\ell \dot{\theta})^2 + \frac{1}{2} (2m) (\ell \dot{\theta})^2 + \frac{1}{2} m \dot{x}^2 = \frac{1}{2} (4m) (\ell \dot{\theta})^2 + \frac{1}{2} m \dot{x}^2$$

which can be expressed in matrix form as

$$T = \frac{1}{2} (\dot{x} \quad \dot{\theta}) [m] \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} \quad \text{where } [m] = \begin{bmatrix} m & 0 \\ 0 & 4 m \ell^2 \end{bmatrix}$$

6.18

Flexibility influence coefficients:

Spring constants of different sections of the shaft (k_i) are

$$k_i = \frac{(GJ)_i}{\ell_i}; i = 1, 2, 3, 4$$

where $(GJ)_i$ = torsional rigidity, J_i = polar moment of inertia, and ℓ_i = length of section i of shaft.

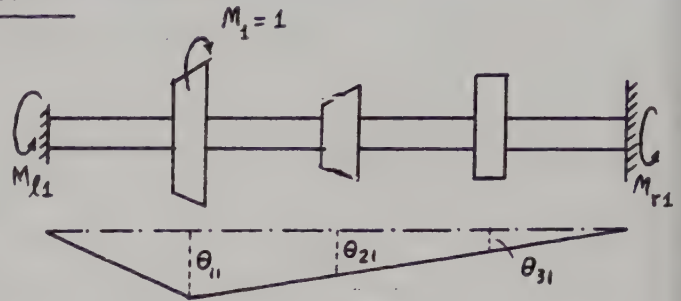


Fig. (A)

Consider disc 1. shaft to the left of disc 1 has a spring constant of k_1 while the shaft to the right side of disc 1 has an equivalent spring constant of

$$k_{e1} = \frac{1}{\sum_{i=2}^4 \left(\frac{1}{k_i} \right)}$$

If we apply unit torque to disc 1 ($M_1 = 1$) as shown in Fig. (A), reactive torques at left and right ends of the shaft are

$$M_{r1} = k_{e1} \theta_{11}, \quad M_{l1} = k_1 \theta_{11}$$

Since $M_{l1} + M_{r1} = M_1 = 1$, we get

$$\theta_{11} = \omega_{11} = \left\{ \sum_{i=2}^4 \left(\frac{1}{k_i} \right) \right\} / \left\{ k_1 \cdot \sum_{i=1}^4 \left(\frac{1}{k_i} \right) \right\}$$

$$M_{l1} = k_1 \theta_{11} = \frac{\sum_{i=2}^4 \left(\frac{1}{k_i} \right)}{\left\{ \sum_{i=1}^4 \left(\frac{1}{k_i} \right) \right\}}; \quad M_{r1} = \frac{\theta_{11}}{\sum_{i=2}^4 \left(\frac{1}{k_i} \right)} = \frac{1}{k_1 \cdot \sum_{i=1}^4 \left(\frac{1}{k_i} \right)}$$

Also,

$$\theta_{31} = \omega_{31} = \frac{M_{r1}}{k_4} = \frac{1}{k_1 k_4 \left\{ \sum_{i=1}^4 \left(\frac{1}{k_i} \right) \right\}}$$

$$\theta_{21} = \omega_{21} = M_{r1} / \left\{ \sum_{i=3}^4 \left(\frac{1}{k_i} \right) \right\} = \frac{1}{k_1 \cdot \left\{ \sum_{i=3}^4 \left(\frac{1}{k_i} \right) \right\} \cdot \left\{ \sum_{i=1}^4 \left(\frac{1}{k_i} \right) \right\}}$$

Consider disc 2. Shaft to the left side of disc 2 has an equivalent spring constant of k_{e2} and the shaft to its right side has an

equivalent spring constant of k_{e3} with

$$k_{e2} = \frac{1}{\sum_{i=1}^2 \left(\frac{1}{k_i} \right)}, \quad k_{e3} = \frac{1}{\sum_{i=3}^4 \left(\frac{1}{k_i} \right)}$$

If we apply a unit torque to disc 2 ($M_2 = 1$), reactive torques at the left and right ends of shaft are

$$M_{l2} = \theta_{22} k_{e2}, \quad M_{r2} = \theta_{22} k_{e3} \quad \text{with} \quad M_2 = M_{l2} + M_{r2} = 1$$

Hence

$$\theta_{22} = a_{22} = \frac{\left\{ \sum_{i=1}^2 \left(\frac{1}{k_i} \right) \right\} \cdot \left\{ \sum_{i=3}^4 \left(\frac{1}{k_i} \right) \right\}}{\left\{ \sum_{i=1}^4 \left(\frac{1}{k_i} \right) \right\}}$$

$$\theta_{32} = a_{32} = \frac{M_{r2}}{k_4} = \frac{\theta_{22}}{k_{e3} \cdot k_4} = \frac{\left\{ \sum_{i=1}^2 \left(\frac{1}{k_i} \right) \right\}}{\left[k_4 \left\{ \sum_{i=1}^4 \left(\frac{1}{k_i} \right) \right\} \right]}$$

$$\theta_{12} = a_{12} = \frac{M_{l2}}{k_1} = \frac{\theta_{22}}{k_{e2} \cdot k_1} = \frac{\left\{ \sum_{i=3}^4 \left(\frac{1}{k_i} \right) \right\}}{\left[k_1 \left\{ \sum_{i=1}^4 \left(\frac{1}{k_i} \right) \right\} \right]}$$

Consider disc 3. Apply unit torque to disc 3 ($M_3 = 1$) to obtain

$$M_{l3} = k_{e4} \theta_{33}, \quad M_{r3} = k_4 \theta_{33}, \quad M_{r3} + M_{l3} = M_3 = 1$$

$$\text{where } k_{e4} = \frac{1}{\sum_{i=1}^3 \left(\frac{1}{k_i} \right)}$$

Hence

$$\theta_{33} = a_{33} = \frac{\left\{ \sum_{i=1}^3 \left(\frac{1}{k_i} \right) \right\}}{\left[k_4 \left\{ \sum_{i=1}^4 \left(\frac{1}{k_i} \right) \right\} \right]}$$

$$\theta_{13} = a_{13} = \frac{M_{l3}}{k_1} = \frac{1}{k_1 k_4 \left\{ \sum_{i=1}^4 \left(\frac{1}{k_i} \right) \right\}}$$

$$\theta_{23} = a_{23} = \frac{M_{l3}}{\sum_{i=1}^2 \left(\frac{1}{k_i} \right)} = \frac{1}{k_4 \left\{ \sum_{i=1}^2 \left(\frac{1}{k_i} \right) \right\} \left\{ \sum_{i=1}^4 \left(\frac{1}{k_i} \right) \right\}}$$

$$\text{Flexibility matrix is } [a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Note: In this problem, it is much easier to derive the stiffness influence coefficients, k_{ij} , compared to a_{ij} . Hence it is advisable to find $[k]$ first and then find $[a]$ by inverting the matrix $[k]$.

Stiffness influence coefficients:

Let angular displacement of disc 1 be unity ($\theta_1 = 1$) and of discs 2 and 3 be zero as shown in Fig. (B). If the torque applied to disc 1 is M_{11} and the reactions are M_{l1} and M_{r1} , we have

$$M_{11} = k_1 \theta_{11} = k_1, \quad M_{21} = k_2 \theta_{11} = k_2$$

$$M_{11} = M_{11} + M_{21} = k_{11} = k_1 + k_2$$

$$k_{21} = -M_{21} = -k_2 \quad (\because \text{reactive torque is opposite to } M_{11} \text{ in direction})$$

$$k_{31} = M_{31} = 0 \quad (\because \text{disc 2 is fixed, no reactive torque is felt at disc 3})$$

Let displacement of disc 2 = 1 and displacements of discs 1 and 3 be zero.

$$k_{22} = k_2 + k_3, \quad k_{12} = -k_2, \quad k_{32} = -k_3$$

Similarly we can obtain

$$k_{33} = k_3 + k_4, \quad k_{13} = 0, \quad k_{23} = -k_3$$

stiffness matrix is $[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$

Equations of motion of the system:

$$[m] \ddot{\vec{\theta}} + [k] \vec{\theta} = \vec{M}_t$$

where $[m] = \begin{bmatrix} J_{d1} & 0 & 0 \\ 0 & J_{d2} & 0 \\ 0 & 0 & J_{d3} \end{bmatrix}$ = matrix of mass moments of inertia of the discs

$\vec{\theta} = \begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{Bmatrix}$ and $\vec{M}_t = \begin{Bmatrix} M_{t1}(t) \\ M_{t2}(t) \\ M_{t3}(t) \end{Bmatrix}$ = vector of external torques applied to discs = $\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

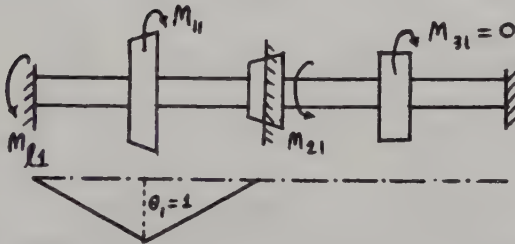


Fig. (B)

6.19

Stiffness influence coefficients:

Let $x_1 = 1, x_2 = x_3 = 0$.

Forces required at 1, 2, 3 are

$$F_1 = k_1 + k_2 = k_{11}; \quad F_2 = -k_2 = k_{21}; \quad F_3 = 0 = k_{31}$$

Let $x_2 = 1, x_1 = x_3 = 0$.

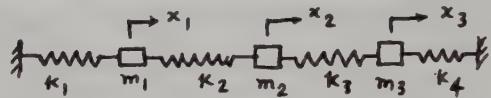
Forces required at 1, 2, 3 are

$$F_1 = -k_2 = k_{12}; \quad F_2 = k_2 + k_3 = k_{22}; \quad F_3 = -k_3 = k_{32}$$

Let $x_3 = 1, x_1 = x_2 = 0$.

Forces required at 1, 2, 3 are

$$F_1 = 0 = k_{13}; \quad F_2 = -k_3 = k_{23}; \quad F_3 = k_3 + k_4 = k_{33}$$



$$\therefore [k] = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & (k_3 + k_4) \end{bmatrix}$$

Flexibility influence coefficients:

Procedure and results are similar to those of problem 6.1.

Equations of motion: $[m] \ddot{\vec{x}} + [k] \vec{x} = \vec{F}$

6.20

From strength of materials, the deflection of the cantilever beam shown is given by

$$w(x) \Big|_{\text{in AB}} = \frac{Fx^2}{6EI} (-x + 3a) \quad \dots (E_1)$$

$$w(x) \Big|_{\text{in BC}} = \frac{Fa^2}{6EI} (-a + 3x) \quad \dots (E_2)$$

Apply $F_1 = 1, F_2 = F_3 = 0$: $a_{11} = (F = 1, x = l, a = l \text{ in } (E_1)) = l^3 / (3EI)$

$$a_{21} = (F = 1, x = 2l, a = l \text{ in } (E_2)) = 5l^3 / (6EI)$$

$$a_{31} = (F = 1, x = 3l, a = l \text{ in } (E_2)) = 4l^3 / (3EI)$$

Similarly apply $F_2 = 1, F_1 = F_3 = 0$ to get a_{22}, a_{32}, a_{12} and

$F_3 = 1, F_1 = F_2 = 0$ to get a_{33}, a_{13}, a_{23} . Result is

$$[a] = \frac{l^3}{EI} \begin{bmatrix} 1/3 & 5/6 & 4/3 \\ 5/6 & 8/3 & 14/3 \\ 4/3 & 14/3 & 9 \end{bmatrix}$$

Equations of motion:

$$[m] \ddot{\vec{w}} + [k] \vec{w} = \vec{0} \quad \text{or} \quad [a][m] \ddot{\vec{w}} + \vec{w} = \vec{0}$$

with $[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$ and $\vec{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$.

6.21

Deflection of a fixed-fixed beam is

$$y(x) \Big|_{\text{in AB}} = \frac{Fb^2x^2}{6EIL^3} \{-x(3a+b) + 3aL\} \quad \dots (E_1)$$

$$y(x) \Big|_{\text{in BC}} = \frac{Fa^2(L-x)^2}{6EIL^3} \{-(L-x)(3b+a) + 3bL\} \quad \dots (E_2)$$

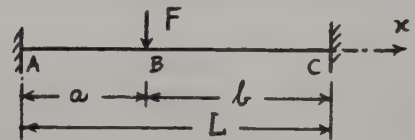
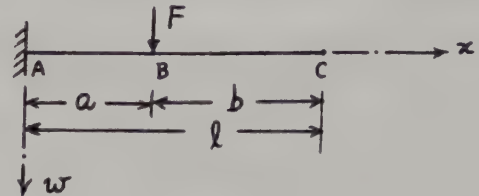
Apply $F_1 = 1, F_2 = F_3 = 0$:

$$a_{11} = (F = 1, a = l, b = 3l, x = l, L = 4l \text{ in } (E_1)) = 9l^3 / (64EI)$$

$$a_{21} = (F = 1, a = l, b = 3l, x = 2l, L = 4l \text{ in } (E_2)) = l^3 / (6EI)$$

$$a_{31} = (F = 1, a = l, b = 3l, x = 3l, L = 4l \text{ in } (E_2)) = 13l^3 / (192EI)$$

Similarly apply $F_2 = 1, F_1 = F_3 = 0$ to get a_{22}, a_{32}, a_{12} and



$F_3 = 1, F_1 = F_2 = 0$ to get a_{33}, a_{13}, a_{23} . Result is

$$[a] = \frac{l^3}{EI} \begin{bmatrix} 9/64 & 1/6 & 13/192 \\ 1/6 & 1/3 & 1/6 \\ 13/192 & 1/6 & 9/64 \end{bmatrix}$$

6.22

Flexibility matrix:

a_{11} = deflection of m_1 for a unit load on $m_1 = 1/k_1$

m_2 and m_3 get same displacement (as rigid body motion) as there are no other forces or constraints.

$a_{21} = a_{31} = \frac{1}{k_1}$. If we apply unit load to m_2 , equivalent stiffness is given by $\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$. $a_{22} = \frac{1}{k_{eq}} = \frac{k_1 + k_2}{k_1 k_2}$.

Mass m_3 follows deflection of m_2 . $a_{32} = a_{22}$.

If we apply unit load to m_3 , equivalent stiffness of springs is given by $\frac{1}{k_{eq}} = \frac{1}{k_3} + \frac{1}{k_1} + \frac{1}{k_2}$. $a_{33} = \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$.

Stiffness matrix:

Let $x_1 = 1, x_2 = x_3 = 0$. Forces required at 1, 2, 3 are

$$F_1 = k_1 + k_2 = k_{11}, F_2 = -k_2 = k_{21}, F_3 = 0 = k_{31}.$$

Let $x_2 = 1, x_1 = x_3 = 0$. Forces required at 1, 2, 3 are

$$F_1 = -k_2 = k_{12}, F_2 = k_2 + k_3 = k_{22}, F_3 = -k_3 = k_{23}.$$

Let $x_3 = 1, x_1 = x_2 = 0$. Forces required at 1, 2, 3 are

$$F_1 = 0 = k_{13}, F_2 = -k_3 = k_{23}, F_3 = k_3 = k_{33}.$$

$$[k] = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}; [a] = \begin{bmatrix} 1/k_1 & 1/k_1 & 1/k_1 \\ 1/k_1 & (\frac{1}{k_1} + \frac{1}{k_2}) & (\frac{1}{k_1} + \frac{1}{k_2}) \\ 1/k_1 & (\frac{1}{k_1} + \frac{1}{k_2}) & (\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}) \end{bmatrix}$$

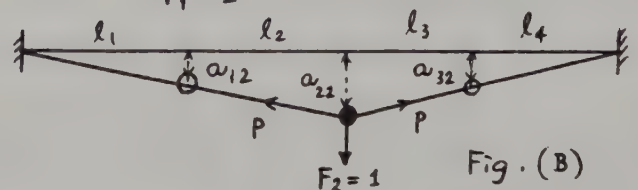
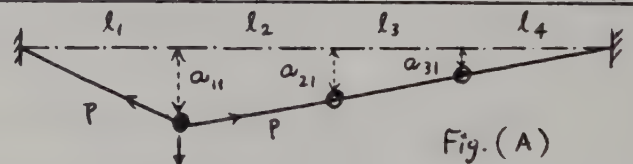
6.23

Assume small deflections; hence tension in spring (P) remains constant.

Let $F_1 = 1, F_2 = F_3 = 0$ as shown in Fig. (A).

Vertical force balance gives

$$F_1 = 1 = \left(\frac{a_{11}}{l_1}\right)P + \left(\frac{a_{11}}{l_2 + l_3 + l_4}\right)P$$



$$a_{11} = \frac{1}{P \left(\frac{1}{l_1} + \frac{1}{l_2 + l_3 + l_4} \right)}$$

From relations of triangles,

$$\frac{a_{11}}{l_2 + l_3 + l_4} = \frac{a_{21}}{l_3 + l_4} \quad \text{and} \quad \frac{a_{11}}{l_2 + l_3 + l_4} = \frac{a_{31}}{l_4}$$

$$a_{21} = \left(\frac{l_3 + l_4}{l_2 + l_3 + l_4} \right) a_{11}, \quad a_{31} = \left(\frac{l_4}{l_2 + l_3 + l_4} \right) a_{11}$$

When $F_2 = 1, F_1 = F_3 = 0$, vertical force balance gives (Fig. (B))

$$F_2 = 1 = \left(\frac{a_{22}}{l_1 + l_2} \right) P + \left(\frac{a_{22}}{l_3 + l_4} \right) P \Rightarrow a_{22} = \frac{1}{P \left\{ \frac{1}{l_1 + l_2} + \frac{1}{l_3 + l_4} \right\}}$$

From triangle relations

$$a_{12} = \left(\frac{l_1}{l_1 + l_2} \right) a_{22}, \quad a_{32} = \left(\frac{l_4}{l_3 + l_4} \right) a_{22}$$

When $F_3 = 1, F_1 = F_2 = 0$ (Fig. (C)), vertical force balance gives

$$F_3 = 1 = \left(\frac{a_{33}}{l_1 + l_2 + l_3} \right) P + \left(\frac{a_{33}}{l_4} \right) P; \quad a_{33} = \frac{1}{P \left(\frac{1}{l_1 + l_2 + l_3} + \frac{1}{l_4} \right)}$$

From triangle relations,

$$a_{23} = \left(\frac{l_1 + l_2}{l_1 + l_2 + l_3} \right) a_{33}, \quad a_{13} = \left(\frac{l_1}{l_1 + l_2 + l_3} \right) a_{33}$$

Equations of motion:

$$[a][m]\ddot{\vec{w}} + \vec{w} = \vec{0}$$

$$\text{with } [m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad \vec{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}.$$

6.24

Let x_1, x_2 and x_3 denote the displacements of top, middle and bottom masses. Equations of motion are

$$m \ddot{x}_1 = -k x_1 - k(x_1 - x_2) - 3k(x_1 - x_3)$$

$$2m \ddot{x}_2 = -2k x_2 - k(x_2 - x_1) - k(x_2 - x_3)$$

$$m \ddot{x}_3 = -k x_3 - 3k(x_3 - x_1) - k(x_3 - x_2)$$

$$\text{i.e. } \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 5k & -k & -3k \\ -k & 4k & -k \\ -3k & -k & 5k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

6.25

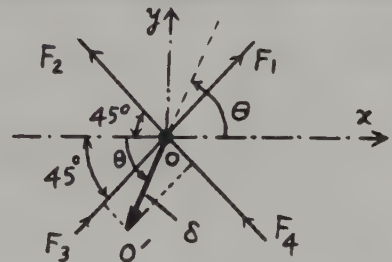
Let O move to O' with δ small.

$$\text{Then } F_1 = k \delta \cos(\theta - 45^\circ)$$

$$F_2 = k \delta \cos(135^\circ - \theta)$$

$$F_3 = k \delta \cos(\theta - 45^\circ)$$

$$F_4 = k \delta \cos(135^\circ - \theta)$$



Force along δ is:

$$\begin{aligned} F &= F_1 \cos(\theta - 45^\circ) + F_2 \cos(135^\circ - \theta) + F_3 \cos(\theta - 45^\circ) + F_4 \cos(135^\circ - \theta) \\ &= 2k\delta [\cos^2(\theta - 45^\circ) + \cos^2(135^\circ - \theta)] \\ &= 2k\delta \end{aligned}$$

$\therefore$ Stiffness influence coefficient of junction point in arbitrary direction = $F/\delta = 2k$

6.26

Stiffness matrix is given by Eq. (6.6):

$$[k] = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 & 0 & \dots & 0 \\ -k_2 & (k_2 + k_3) & -k_3 & 0 & \dots & 0 \\ 0 & -k_3 & (k_3 + k_4) & -k_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (k_n + k_{n+1}) \end{bmatrix}$$

All elements except those on the three diagonals are zero. Hence $[k]$ is a band matrix. In fact, it is a tri-diagonal matrix.

6.27

We use the expression of kinetic energy to derive the mass matrix. Let the generalized coordinates be x_1, x_2 and x_3 . The kinetic energy of the system is:

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

This can be expressed in matrix form as

$$T = \frac{1}{2} (\dot{x}_1 \quad \dot{x}_2 \quad \dot{x}_3) [m] \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix}$$

with the mass matrix given by

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

6.28

We use the expression of kinetic energy to derive the mass matrix. Using the generalized coordinates θ, x_1 and x_2 , the kinetic energy of the system can be expressed as:

$$T = \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} (2m) \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

where $J_0 = J_G + (2m) \left(\frac{\ell}{2}\right)^2 = \frac{2}{3} m \ell^2$. T can be expressed in matrix form as:

$$T = \frac{1}{2} (\dot{\theta} \quad \dot{x}_1 \quad \dot{x}_2) [m] \begin{Bmatrix} \dot{\theta} \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} \quad \text{with the mass matrix} \quad [m] = \begin{bmatrix} \frac{2}{3} m \ell^2 & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}$$

6.29

We derive the mass matrix from the kinetic energy expression. Using the coordinates x_1 , x_2 and x_3 , the kinetic energy of the system can be expressed as (see figure in the solution of Problem 6.3):

$$T = \frac{1}{2} J_G \dot{\theta}^2 + \frac{1}{2} (2m) \dot{x}_G^2 + \frac{1}{2} (5m) \dot{x}_2^2 \quad (1)$$

$$\text{Using } J_G = \frac{1}{12} (2m) (5\ell)^2 = \frac{25}{6} m \ell^2 \quad (2)$$

$$x_G = \frac{x_1 + x_3}{2}, \quad \theta = \frac{x_1 - x_3}{5\ell} \quad (3)$$

Eq. (1) can be rewritten as:

$$\begin{aligned} T &= \frac{1}{2} \left(\frac{25}{6} m \ell^2 \right) \left(\frac{\dot{x}_1 - \dot{x}_3}{5\ell} \right)^2 + \frac{1}{2} (2m) \left(\frac{\dot{x}_1 + \dot{x}_3}{2} \right)^2 + \frac{1}{2} (5m) \dot{x}_2^2 \\ &= \frac{1}{2} \frac{2m}{3} \dot{x}_1^2 + \frac{1}{2} \frac{2m}{3} \dot{x}_3^2 + \frac{1}{2} \frac{1}{3} m (2\dot{x}_1 \dot{x}_3) + \frac{1}{2} (5m) \dot{x}_2^2 \end{aligned} \quad (4)$$

Equation (4) can be expressed in matrix form as

$$T = \frac{1}{2} (\dot{x}_1 \quad \dot{x}_2 \quad \dot{x}_3) [m] \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} \quad \text{with the mass matrix} \quad [m] = m \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 5 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix}$$

6.30

The kinetic energy of the system can be expressed, in terms of the coordinates x_1 , x_2 and x_3 as:

$$T = \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} (3m) \dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2 \quad (1)$$

Using the relation $\theta = \frac{x_2 - x_1}{3r}$, Eq. (1) can be rewritten as

$$T = \frac{1}{2} \left(M + \frac{J_0}{9r^2} \right) \dot{x}_1^2 + \frac{1}{2} \left(\frac{J_0}{9r^2} + 3m \right) \dot{x}_2^2 - \frac{1}{2} \frac{J_0}{9r^2} (2\dot{x}_1 \dot{x}_2) + \frac{1}{2} m \dot{x}_3^2 \quad (2)$$

By expressing T in matrix form as

$$T = \frac{1}{2} (\dot{x}_1 \quad \dot{x}_2 \quad \dot{x}_3) [m] \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} \quad \text{the mass matrix} \quad [m] = \begin{bmatrix} \left(M + \frac{J_0}{9 r^2} \right) & -\frac{J_0}{9 r^2} & 0 \\ -\frac{J_0}{9 r^2} & \left(\frac{J_0}{9 r^2} + 3 m \right) & 0 \\ 0 & 0 & m \end{bmatrix}$$

6.31

The kinetic energy of the system can be expressed as

$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} I_3 \left(\dot{\theta}_2 \frac{n_2}{n_3} \right)^2 + \frac{1}{2} I_4 \dot{\theta}_3^2 + \frac{1}{2} I_5 \left(\dot{\theta}_3 \frac{n_4}{n_5} \right)^2 + \frac{1}{2} I_6 \dot{\theta}_4^2$$

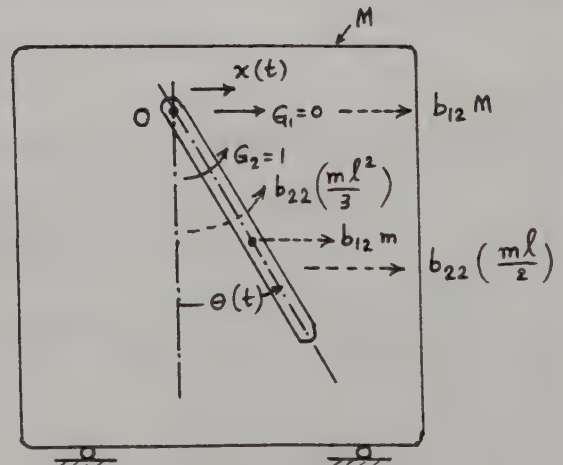
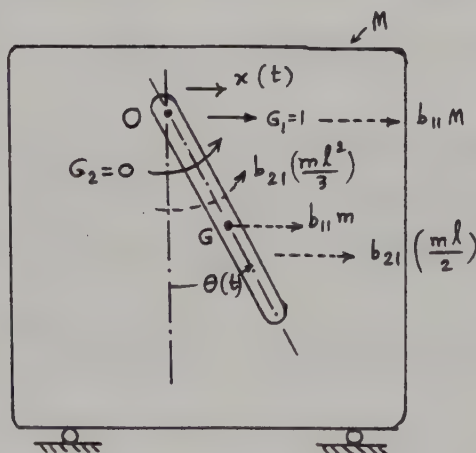
This can be expressed in matrix form as

$$T = \frac{1}{2} (\dot{\theta}_1 \quad \dot{\theta}_2 \quad \dot{\theta}_3 \quad \dot{\theta}_4) [m] \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{Bmatrix}$$

and the mass matrix can be identified as

$$[m] = \begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & \left(I_2 + I_3 \frac{n_2^2}{n_3^2} \right) & 0 & 0 \\ 0 & 0 & \left(I_4 + I_5 \frac{n_4^2}{n_5^2} \right) & 0 \\ 0 & 0 & 0 & I_6 \end{bmatrix}$$

6.32



Let $x(t)$ and $\theta(t)$ indicate the coordinates to define the linear and angular positions of the trailer (M) and the pendulum (m), respectively. To derive the inverse inertia influence coefficients, first we apply a unit linear impulse at point 1 (along $x(t)$) and write the impulse-momentum relations as (see Fig. 1):

Linear impulse-momentum relation along x :

$$\dot{x} = 1 = b_{11} (M + m) + b_{21} \left(\frac{m \ell}{2} \right) \quad (1)$$

Angular impulse-momentum relation along θ :

$$\dot{\theta} = 0 = b_{11} \left(\frac{m \ell}{2} \right) + b_{21} \left(\frac{m \ell^2}{3} \right) \quad (2)$$

Solution of Eqs. (1) and (2) gives:

$$b_{11} = \frac{4}{4M + m} ; b_{21} = -\frac{6}{4M\ell + m\ell} \quad (3)$$

Next we apply a unit angular impulse at point 2 (along $\theta(t)$) and write the impulse-momentum relations as (see Fig. 2):

Linear impulse-momentum relation along x :

$$\dot{x} = 0 = b_{12} (M + m) + b_{22} \left(\frac{m \ell}{2} \right) \quad (4)$$

Angular impulse-momentum about O along θ :

$$\dot{\theta} = 1 = b_{12} \left(\frac{m \ell}{2} \right) + b_{22} \left(\frac{m \ell^2}{3} \right) \quad (5)$$

Solution of Eqs. (4) and (5) gives:

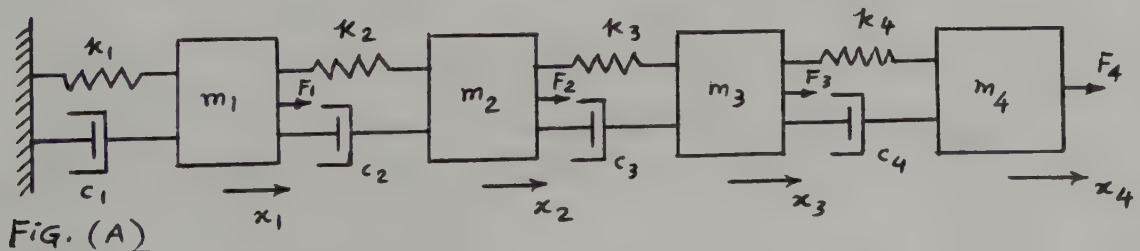
$$b_{12} = -\frac{6}{4M\ell + m\ell} ; b_{22} = \frac{12(M + m)}{4Mm\ell^2 + m^2\ell^2} \quad (6)$$

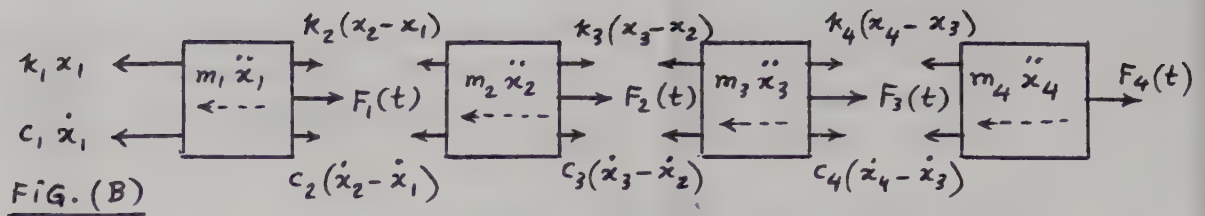
Thus the inverse mass matrix is given by

$$[b] = [m]^{-1} = \begin{bmatrix} \left(\frac{4}{4M + m} \right) & -\left(\frac{6}{4M\ell + m\ell} \right) \\ -\left(\frac{6}{4M\ell + m\ell} \right) & \left(\frac{12(M + m)}{4Mm\ell^2 + m^2\ell^2} \right) \end{bmatrix} \quad (7)$$

6.33

The shear building can be modeled as shown below:





(a)
Equations of motion (from free body diagrams in Fig. B):

$$\begin{aligned}
 m_1 \ddot{x}_1 + c_1 \dot{x}_1 + k_1 x_1 - c_2 (\dot{x}_2 - \dot{x}_1) - k_2 (x_2 - x_1) &= F_1(t) \\
 m_2 \ddot{x}_2 + c_2 (\dot{x}_2 - \dot{x}_1) + k_2 (x_2 - x_1) - c_3 (\dot{x}_3 - \dot{x}_2) - k_3 (x_3 - x_2) &= F_2(t) \\
 m_3 \ddot{x}_3 + c_3 (\dot{x}_3 - \dot{x}_2) + k_3 (x_3 - x_2) - c_4 (\dot{x}_4 - \dot{x}_3) - k_4 (x_4 - x_3) &= F_3(t) \\
 m_4 \ddot{x}_4 + c_4 (\dot{x}_4 - \dot{x}_3) + k_4 (x_4 - x_3) &= F_4(t) \quad \text{----- (E}_1\text{)}
 \end{aligned}$$

(b) Lagrange's equations are

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = Q_j \quad \text{----- (E}_2\text{)}$$

where T = kinetic energy, V = potential energy, R = Rayleigh's dissipation function, $Q_j = j^{\text{th}}$ generalized force and $q_j = j^{\text{th}}$ generalized coordinate:

$$\begin{aligned}
 T &= \frac{1}{2} \{ m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2 + m_3 \dot{x}_3^2 + m_4 \dot{x}_4^2 \} \\
 V &= \frac{1}{2} \{ k_1 x_1^2 + k_2 (x_2 - x_1)^2 + k_3 (x_3 - x_2)^2 + k_4 (x_4 - x_3)^2 \} \\
 R &= \frac{1}{2} \{ c_1 \dot{x}_1^2 + c_2 (\dot{x}_2 - \dot{x}_1)^2 + c_3 (\dot{x}_3 - \dot{x}_2)^2 + c_4 (\dot{x}_4 - \dot{x}_3)^2 \} \\
 Q_j &= F_j \quad ; \quad j = 1, 2, 3, 4
 \end{aligned}$$

Using $q_j = x_j$; $j = 1, 2, 3, 4$, the application of Eqs. (E₂) yields the equations of motion given in (E₁).

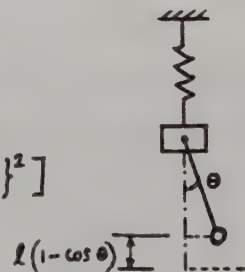
6.34

Coordinates of the bob are $(x + l \cos \theta, l \sin \theta)$

T = kinetic energy = kinetic energy of slider
+ kinetic energy of bob

$$\begin{aligned}
 &= \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 + \frac{1}{2} m \left[\left\{ \frac{d}{dt} (x + l \cos \theta) \right\}^2 + \left\{ \frac{d}{dt} (l \sin \theta) \right\}^2 \right] \\
 &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 (\sin^2 \theta + \cos^2 \theta) \\
 &\quad - \frac{1}{2} m (2 \dot{x} l \sin \theta \dot{\theta})
 \end{aligned}$$

$$= m \dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 - m \dot{x} \dot{\theta} l \sin \theta \simeq m \dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 \text{ for small } \theta.$$



$$V = \text{potential energy} = \text{potential energy of spring} + \text{potential energy of bob}$$

$$= \frac{1}{2} k x^2 + mgl (1 - \cos \theta)$$

(Note: Potential energy of slider need not be considered if $x=0$ corresponds to static equilibrium position)

$$\text{Since } \cos \theta \approx 1 - \frac{1}{2} \theta^2, \quad V = \frac{1}{2} k x^2 + \frac{1}{2} mgl \theta^2$$

As there are no external forces, Lagrange's equations become

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0; \quad j = 1, 2$$

Here $q_1 = x$ and $q_2 = \theta$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \dot{x}} = 2m\dot{x}, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) = 2m\ddot{x}, \quad \frac{\partial V}{\partial x} = kx$$

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = ml^2 \dot{\theta}, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) = ml^2 \ddot{\theta}, \quad \frac{\partial V}{\partial \theta} = mgl \theta$$

Lagrange's equations become

$$2m\ddot{x} + kx = 0; \quad ml^2 \ddot{\theta} + mgl \theta = 0 \quad \text{or} \quad l\ddot{\theta} + g\theta = 0$$

6.35

(1) With x_1 and x_2 as generalized coordinates:

Since $x_1 = x - l_1 \theta$ and $x_2 = x + l_2 \theta$,

$$x = \left(\frac{x_1 l_2 + x_2 l_1}{l_1 + l_2} \right) \quad \text{and} \quad \theta = \left(\frac{x_2 - x_1}{l_1 + l_2} \right)$$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} m \left(\frac{\dot{x}_1 l_2 + \dot{x}_2 l_1}{l_1 + l_2} \right)^2 + \frac{1}{2} J_0 \left(\frac{\dot{x}_2 - \dot{x}_1}{l_1 + l_2} \right)^2$$

$$\frac{\partial T}{\partial x_1} = 0, \quad \frac{\partial T}{\partial \dot{x}_1} = \frac{m}{2(l_1 + l_2)^2} (2l_2^2 \dot{x}_1 + 2l_1 l_2 \dot{x}_2) + \frac{J_0}{2(l_1 + l_2)^2} (2\dot{x}_1 - 2\dot{x}_2),$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = \frac{m}{(l_1 + l_2)^2} (l_2^2 \ddot{x}_1 + l_1 l_2 \ddot{x}_2) + \frac{J_0}{(l_1 + l_2)^2} (\ddot{x}_1 - \ddot{x}_2)$$

$$\frac{\partial T}{\partial x_2} = 0, \quad \frac{\partial T}{\partial \dot{x}_2} = \frac{m}{2(l_1 + l_2)^2} (2l_1^2 \dot{x}_2 + 2l_1 l_2 \dot{x}_1) + \frac{J_0}{2(l_1 + l_2)^2} (2\dot{x}_2 - 2\dot{x}_1),$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = \frac{m}{(l_1 + l_2)^2} (l_1^2 \ddot{x}_2 + l_1 l_2 \ddot{x}_1) + \frac{J_0}{(l_1 + l_2)^2} (\ddot{x}_2 - \ddot{x}_1)$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2$$

$$\frac{\partial V}{\partial x_1} = k_1 x_1, \quad \frac{\partial V}{\partial x_2} = k_2 x_2$$

Lagrange's equations, Eq. (6.41), give

$$\frac{m}{(l_1 + l_2)^2} (l_2^2 \ddot{x}_1 + l_1 l_2 \ddot{x}_2) + \frac{J_0}{(l_1 + l_2)^2} (\ddot{x}_1 - \ddot{x}_2) + k_1 x_1 = 0$$

$$\frac{m}{(l_1 + l_2)^2} (l_1^2 \ddot{x}_2 + l_1 l_2 \ddot{x}_1) + \frac{T_0}{(l_1 + l_2)^2} (\ddot{x}_2 - \ddot{x}_1) + k_2 x_2 = 0$$

i.e.

$$\ddot{x}_1 (m l_2^2 + T_0) + \ddot{x}_2 (m l_1 l_2 - T_0) + x_1 (l_1 + l_2)^2 k_1 = 0$$

$$\ddot{x}_1 (m l_1 l_2 - T_0) + \ddot{x}_2 (m l_1^2 + T_0) + x_2 (l_1 + l_2)^2 k_2 = 0$$

(2) With x and θ as generalized coordinates:

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} T_0 \dot{\theta}^2$$

$$V = \frac{1}{2} k_1 (x - l_1 \theta)^2 + \frac{1}{2} k_2 (x + l_2 \theta)^2$$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \dot{x}} = m \dot{x}, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) = m \ddot{x}, \quad \frac{\partial V}{\partial x} = k_1 (x - l_1 \theta) + k_2 (x + l_2 \theta)$$

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = T_0 \dot{\theta}, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) = T_0 \ddot{\theta}, \quad \frac{\partial V}{\partial \theta} = -k_1 l_1 (x - l_1 \theta) + k_2 l_2 (x + l_2 \theta)$$

Lagrange's equations give

$$m \ddot{x} + k_1 (x - l_1 \theta) + k_2 (x + l_2 \theta) = 0$$

$$T_0 \ddot{\theta} - k_1 l_1 (x - l_1 \theta) + k_2 l_2 (x + l_2 \theta) = 0$$

6.36

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 (x_3 - x_2)^2 + \frac{1}{2} k_4 x_3^2$$

$$\frac{\partial T}{\partial x_1} = 0, \quad \frac{\partial T}{\partial \dot{x}_1} = m_1 \dot{x}_1, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1, \quad \frac{\partial V}{\partial x_1} = k_1 x_1 - k_2 (x_2 - x_1)$$

$$\frac{\partial T}{\partial x_2} = 0, \quad \frac{\partial T}{\partial \dot{x}_2} = m_2 \dot{x}_2, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2, \quad \frac{\partial V}{\partial x_2} = k_2 (x_2 - x_1) - k_3 (x_3 - x_2)$$

$$\frac{\partial T}{\partial x_3} = 0, \quad \frac{\partial T}{\partial \dot{x}_3} = m_3 \dot{x}_3, \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_3} \right) = m_3 \ddot{x}_3, \quad \frac{\partial V}{\partial x_3} = k_3 (x_3 - x_2) + k_4 x_3$$

Lagrange's equations give

$$m_1 \ddot{x}_1 + x_1 (k_1 + k_2) - k_2 x_2 = 0$$

$$m_2 \ddot{x}_2 - k_2 x_1 + x_2 (k_2 + k_3) - k_3 x_3 = 0$$

$$m_3 \ddot{x}_3 - k_3 x_2 + x_3 (k_3 + k_4) = 0$$

6.37

Using θ_1 , θ_2 and θ_3 as generalized coordinates, we find

$$T \simeq \frac{1}{2} m_1 (l_1 \dot{\theta}_1)^2 + \frac{1}{2} m_2 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2)^2 + \frac{1}{2} m_3 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3)^2 \dots (E_1)$$

Let reference position correspond to $\theta_1 = \theta_2 = \theta_3 = 0$.

Vertical movement of m_1 is

$$l_1 (1 - \cos \theta_1) \simeq l_1 \left\{ 1 - \left(1 - \frac{\theta_1^2}{2} \right) \right\} = \frac{1}{2} l_1 \theta_1^2$$

vertical movement of $m_2 = \frac{1}{2} l_1 \dot{\theta}_1^2 + \frac{1}{2} l_2 \dot{\theta}_2^2$

vertical movement of $m_3 = \frac{1}{2} l_1 \dot{\theta}_1^2 + \frac{1}{2} l_2 \dot{\theta}_2^2 + \frac{1}{2} l_3 \dot{\theta}_3^2$

$$V = \frac{m_1 g l_1 \dot{\theta}_1^2}{2} + \frac{m_2 g}{2} (l_1 \dot{\theta}_1^2 + l_2 \dot{\theta}_2^2) + \frac{m_3 g}{2} (l_1 \dot{\theta}_1^2 + l_2 \dot{\theta}_2^2 + l_3 \dot{\theta}_3^2) \dots (E_2)$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = 0, \quad \frac{\partial T}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2) + m_3 l_1 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3),$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_1} \right) = (m_1 + m_2 + m_3) l_1^2 \ddot{\theta}_1 + (m_2 + m_3) l_1 l_2 \ddot{\theta}_2 + m_3 l_1 l_3 \ddot{\theta}_3,$$

$$\frac{\partial V}{\partial \theta_1} = (m_1 + m_2 + m_3) g l_1 \theta_1$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = 0, \quad \frac{\partial T}{\partial \dot{\theta}_2} = m_2 l_2 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2) + m_3 l_2 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3),$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_2} \right) = (m_2 + m_3) l_1 l_2 \ddot{\theta}_1 + (m_2 + m_3) l_2^2 \ddot{\theta}_2 + m_3 l_2 l_3 \ddot{\theta}_3,$$

$$\frac{\partial V}{\partial \theta_2} = m_2 g l_2 \theta_2 + m_3 g l_2 \theta_2 = (m_2 + m_3) g l_2 \theta_2$$

$$\frac{\partial T}{\partial \dot{\theta}_3} = 0, \quad \frac{\partial T}{\partial \dot{\theta}_3} = m_3 l_3 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3),$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_3} \right) = m_3 l_3 (l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2 + l_3 \ddot{\theta}_3), \quad \frac{\partial V}{\partial \theta_3} = m_3 g l_3 \theta_3$$

Lagrange's equations give the equations of motion

$$\begin{bmatrix} (m_1 + m_2 + m_3) l_1^2 & (m_2 + m_3) l_1 l_2 & m_3 l_1 l_3 \\ (m_2 + m_3) l_1 l_2 & (m_2 + m_3) l_2^2 & m_3 l_2 l_3 \\ m_3 l_1 l_3 & m_3 l_2 l_3 & m_3 l_3^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} (m_1 + m_2 + m_3) g l_1 & 0 & 0 \\ 0 & (m_2 + m_3) g l_2 & 0 \\ 0 & 0 & m_3 g l_3 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \dots (E_3)$$

6.38

(a) Let generalized coordinates be $q_1 = x$ and $q_2 = \theta$

Displacement of mass M is $x + l\theta$.

Kinetic energy of airplane is $T = \frac{1}{2} M_0 \dot{x}^2 + 2 \left\{ \frac{1}{2} M (\dot{x} + l\dot{\theta})^2 \right\}$

Potential energy is $V = 2 \left(\frac{1}{2} k_t \theta^2 \right)$

Lagrange's equations are $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i; i = 1, 2$

Here

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \dot{x}} = M_0 \dot{x} + 2 M (\dot{x} + l\dot{\theta}), \quad \frac{\partial V}{\partial x} = 0$$

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = 2 M l (\dot{x} + l \dot{\theta}), \quad \frac{\partial V}{\partial \theta} = 2 k_t \theta, \quad Q_1 = Q_2 = 0$$

Hence Lagrange's equations become

$$\left. \begin{aligned} M_0 \ddot{x} + 2 M (\ddot{x} + l \ddot{\theta}) &= 0 \\ 2 M l (\ddot{x} + l \ddot{\theta}) + 2 k_t \theta &= 0 \end{aligned} \right\} \quad (E_1)$$

(b) If $x(t) = X \cos(\omega t + \phi)$ and $\theta(t) = \Theta \cos(\omega t + \phi)$, the equations of motion become

$$\left. \begin{aligned} (-M_0 \omega^2 - 2 M \omega^2) X - 2 M l \omega^2 \Theta &= 0 \\ -2 M l \omega^2 X - 2 M l^2 \omega^2 \Theta + 2 k_t \Theta &= 0 \end{aligned} \right\} \quad (E_2)$$

which yields the frequency equation:

$$\begin{vmatrix} M_0 \omega^2 + 2 M \omega^2 & 2 M l \omega^2 \\ 2 M l \omega^2 & 2 M l^2 \omega^2 - 2 k_t \end{vmatrix} = 0$$

$$\text{i.e., } \omega^2 [2 M_0 M l^2 \omega^2 - 2 M_0 k_t - 4 M k_t] = 0$$

$$\text{i.e., } \omega^2 = 0; \quad \omega^2 = \left(\frac{2 M_0 k_t + 4 M k_t}{2 M_0 M l^2} \right)$$

Mode shapes: Eg. (E₂) gives

$$\frac{\Theta}{X} = \frac{2 M l \omega^2}{-2 M l^2 \omega^2 + 2 k_t}$$

$$\text{For } \omega_1 = 0; \quad \left. \frac{\Theta}{X} \right|_{\omega_1} = \frac{0}{2 k_t} = 0 \Rightarrow \Theta = 0$$

This corresponds to rigid body translation in x (vertical) direction.

$$\begin{aligned} \text{For } \omega_2; \quad \left. \frac{\Theta}{X} \right|_{\omega_2} &= \frac{2 M l \left(\frac{2 M_0 k_t + 4 M k_t}{2 M_0 M l^2} \right)}{-2 M l^2 \left(\frac{2 M_0 k_t + 4 M k_t}{2 M_0 M l^2} \right) + 2 k_t} \\ &= - \left(\frac{M_0}{2 M l} + \frac{1}{l} \right) \end{aligned}$$

(c) For $\omega_2 > 4 \pi \text{ rad/sec}$,

$$\left(\frac{2 M_0 k_t + 4 M k_t}{2 M M_0 l^2} \right) > 16 \pi^2 \quad (E_3)$$

When $M_0 = 1000 \text{ kg}$, $M = 500 \text{ kg}$ and $l = 6 \text{ m}$, inequality (E_3) becomes

$$\frac{2000 k_t + 2000 k_t}{(1 \times 10^6) (36)} > 16 \pi^2$$

i.e., $k_t > 1.4212 \times 10^6 \text{ N-m/rad.}$

6.39

Generalized coordinates: x_1, x_2, x_3 :

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} k (x_3 - x_2)^2 + \frac{1}{2} k x_3^2 + \frac{1}{2} (5k) (x_3 - x_1)^2$$

$$Q_i = F_i ; i = 1, 2, 3$$

$$\frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i ; i = 1, 2, 3$$

$$\frac{\partial V}{\partial x_1} = 7 k x_1 - k x_2 - 5 k x_3$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_i} \right) = m_i \ddot{x}_i ; i = 1, 2, 3$$

$$\frac{\partial V}{\partial x_2} = -k x_1 + 2 k x_2 - k x_3$$

$$\frac{\partial T}{\partial x_i} = 0 ; i = 1, 2, 3$$

$$\frac{\partial V}{\partial x_3} = -5 k x_1 - k x_2 + 7 k x_3$$

Lagrange's equations yield the equations of motion:

$$m_1 \ddot{x}_1 + 7 k x_1 - k x_2 - 5 k x_3 = F_1(t)$$

$$m_2 \ddot{x}_2 - k x_1 + 2 k x_2 - k x_3 = F_2(t)$$

$$m_3 \ddot{x}_3 - 5 k x_1 - k x_2 + 7 k x_3 = F_3(t)$$

6.40

Generalized coordinates: θ, x_1, x_2 :

$$T = \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} (2m) \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$\text{where } J_0 = \frac{1}{3} (2m) \ell^2 = \frac{2}{3} m \ell^2.$$

$$V = \frac{1}{2} (2k) \left(x_1 - \frac{\ell}{4} \theta \right)^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} (3k) (\theta \ell)^2$$

$$R = \frac{1}{2} c \left(\dot{x}_1 - \frac{\ell}{4} \dot{\theta} \right)^2$$

$$Q_\theta = M_t(t) ; Q_1 = F_1(t) ; Q_2 = F_2(t)$$

$$\frac{\partial T}{\partial \dot{\theta}} = J_0 \dot{\theta} ; \frac{\partial T}{\partial \dot{x}_1} = 2 m \dot{x}_1 ; \frac{\partial T}{\partial \dot{x}_2} = m \dot{x}_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) = J_0 \ddot{\theta} ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = 2 m \ddot{x}_1 ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m \ddot{x}_2$$

$$\frac{\partial T}{\partial \theta} = 0 ; \quad \frac{\partial T}{\partial x_i} = 0 ; \quad i = 1, 2$$

$$\frac{\partial R}{\partial \dot{\theta}} = -c \frac{\ell}{4} (\dot{x}_1 - \dot{\theta} \frac{\ell}{4}) ; \quad \frac{\partial R}{\partial \dot{x}_1} = c (\dot{x}_1 - \dot{\theta} \frac{\ell}{4}) ; \quad \frac{\partial R}{\partial \dot{x}_2} = 0$$

$$\frac{\partial V}{\partial \theta} = \left(\frac{25}{8} k \ell^2 \right) \theta - \frac{1}{2} k \ell x_1$$

$$\frac{\partial V}{\partial x_1} = -\frac{1}{2} k \ell \theta + 3 k x_1 - k x_2$$

$$\frac{\partial V}{\partial x_2} = -k x_1 + k x_2$$

Lagrange's equations, Eqs. (6.118), yield the equations of motion as:

$$\begin{aligned} \frac{2}{3} m \ell^2 \ddot{\theta} + \frac{1}{16} c \ell^2 \dot{\theta} - \frac{1}{4} c \ell \dot{x}_1 + \frac{25}{8} k \ell^2 \theta - \frac{1}{2} k \ell x_1 &= M_t(t) \\ 2 m \ddot{x}_1 - \frac{1}{4} c \ell \dot{\theta} + c \dot{x}_1 - \frac{1}{2} k \ell \theta + 3 k x_1 - k x_2 &= F_2(t) \\ m \ddot{x}_2 - k x_1 + k x_2 &= F_2(t) \end{aligned}$$

6.41

Generalized coordinates: x_i ; $i = 1, 2, 3$:

$$\begin{aligned} T &= \frac{1}{2} J_G \dot{\theta}^2 + \frac{1}{2} (2m) \dot{x}_G^2 + \frac{1}{2} (5m) \dot{x}_2^2 \\ &= \frac{1}{2} \left(\frac{25}{6} m \ell^2 \right) \left(\frac{\dot{x}_1 - \dot{x}_3}{5 \ell} \right)^2 + \frac{1}{2} (2m) \left(\frac{\dot{x}_1 + \dot{x}_3}{2} \right)^2 + \frac{1}{2} (5m) \dot{x}_2^2 \end{aligned}$$

where the subscript G denotes the mass center of the bar with

$$J_G = \frac{1}{12} (2m) (5 \ell)^2 = \frac{25}{6} m \ell^2 ; \quad \theta = \frac{x_1 - x_3}{5 \ell} ; \quad x_G = \frac{x_1 + x_3}{2}$$

$$\begin{aligned} V &= \frac{1}{2} k x_1^2 + \frac{1}{2} k x_3^2 + \frac{1}{2} k (x_A - x_2)^2 \\ &= \frac{1}{2} k x_1^2 + \frac{1}{2} k x_3^2 + \frac{1}{2} k \left(\frac{3 x_1 + 2 x_3}{5} - x_2 \right)^2 \end{aligned}$$

$$\text{since } x_A = x_1 - \frac{2}{5} (x_1 - x_3) = \frac{3 x_1 + 2 x_3}{5}$$

$$R = \frac{1}{2} c (\dot{x}_A - \dot{x}_2)^2 = \frac{1}{2} c \left(\frac{3 \dot{x}_1 + 2 \dot{x}_3}{5} - \dot{x}_2 \right)^2$$

$$Q_i(t) = F_i(t) ; i = 1, 2, 3$$

$$\frac{\partial T}{\partial \dot{x}_1} = \left(\frac{25 m \ell^2}{6} \right) \left(\frac{1}{5 \ell} \right) \left(\frac{\dot{x}_1 - \dot{x}_3}{5 \ell} \right) + (2m) \frac{1}{2} \left(\frac{\dot{x}_1 + \dot{x}_3}{2} \right)$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = \left(\frac{25 m \ell^2}{6} \right) \left(\frac{1}{25 \ell^2} \right) (\ddot{x}_1 - \ddot{x}_3) + \frac{m}{2} (\ddot{x}_1 + \ddot{x}_3)$$

$$\frac{\partial T}{\partial \dot{x}_2} = (5m) \dot{x}_2 ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = (5m) \ddot{x}_2$$

$$\frac{\partial T}{\partial \dot{x}_3} = - \left(\frac{25 m \ell^2}{6} \right) \left(\frac{1}{5 \ell} \right) \left(\frac{\dot{x}_1 - \dot{x}_3}{5 \ell} \right) + (2m) \frac{1}{2} \left(\frac{\dot{x}_1 + \dot{x}_3}{2} \right)$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_3} \right) = - \left(\frac{25 m \ell^2}{6} \right) \left(\frac{1}{25 \ell^2} \right) (\ddot{x}_1 - \ddot{x}_3) + \frac{m}{2} (\ddot{x}_1 + \ddot{x}_3)$$

$$\frac{\partial T}{\partial x_i} = 0 ; i = 1, 2, 3$$

$$\frac{\partial R}{\partial \dot{x}_1} = \frac{3}{5} c \left(\frac{3 \dot{x}_1 + 2 \dot{x}_3}{5} - \dot{x}_2 \right) \quad \frac{\partial V}{\partial x_1} = k x_1 + \frac{3}{5} k \left(\frac{3 x_1 + 2 x_3}{5} - x_2 \right)$$

$$\frac{\partial R}{\partial \dot{x}_3} = \frac{2}{5} c \left(\frac{3 \dot{x}_1 + 2 \dot{x}_3}{5} - \dot{x}_2 \right) \quad \frac{\partial V}{\partial x_2} = -k \left(\frac{3 x_1 + 2 x_3}{5} - x_2 \right)$$

$$\frac{\partial R}{\partial \dot{x}_2} = -c \left(\frac{3 \dot{x}_1 + 2 \dot{x}_3}{5} - \dot{x}_2 \right) \quad \frac{\partial V}{\partial x_3} = k x_3 + \frac{2}{5} k \left(\frac{3 x_1 + 2 x_3}{5} - x_2 \right)$$

Application of Lagrange's equations gives the equations of motion as:

$$\begin{aligned} \frac{2}{3} m \ddot{x}_1 + \frac{1}{3} m \ddot{x}_3 + \frac{9}{25} c \dot{x}_1 - \frac{3}{5} c \dot{x}_2 + \frac{6}{25} c \dot{x}_3 + \frac{34}{25} k x_1 - \frac{3}{5} k x_2 + \frac{6}{25} k x_3 &= F_1(t) \\ 5 m \ddot{x}_2 - \frac{3}{5} c \dot{x}_1 + c \dot{x}_2 - \frac{2}{5} c \dot{x}_3 - \frac{3}{5} k x_1 + k x_2 - \frac{2}{5} k x_3 &= F_2(t) \\ \frac{1}{3} m \ddot{x}_1 + \frac{2}{3} m \ddot{x}_3 + \frac{6}{25} c \dot{x}_1 - \frac{2}{5} c \dot{x}_2 + \frac{4}{25} c \dot{x}_3 + \frac{6}{25} k x_1 - \frac{2}{5} k x_2 + \frac{29}{25} k x_3 &= F_3(t) \end{aligned}$$

6.42

$$\begin{aligned} T &= \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} (3m) \dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2 \\ V &= \frac{1}{2} k x_1^2 + \frac{1}{2} (2k) (x_1 - x_3 - r \theta)^2 + \frac{1}{2} (3k) x_2^2 \\ Q_i &= F_i ; i = 1, 2, 3 \end{aligned}$$

Noting that $\theta = \left(\frac{x_2 - x_1}{3 r} \right)$, we can express

$$\frac{\partial T}{\partial \dot{x}_1} = M \dot{x}_1 + J_0 \left(-\frac{1}{3r} \right) \left(\frac{\dot{x}_2 - \dot{x}_1}{3r} \right) ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = M \ddot{x}_1 - \frac{J_0}{3r} \left(\frac{\ddot{x}_2 - \ddot{x}_1}{3r} \right)$$

$$\frac{\partial T}{\partial \dot{x}_2} = J_0 \left(\frac{1}{3r} \right) \left(\frac{\dot{x}_2 - \dot{x}_1}{3r} \right) + 3m \dot{x}_2 ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = \frac{J_0}{9r^2} (\ddot{x}_2 - \ddot{x}_1) + 3m \ddot{x}_2$$

$$\frac{\partial T}{\partial \dot{x}_3} = m \dot{x}_3 ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_3} \right) = m \ddot{x}_3$$

$$\frac{\partial V}{\partial x_1} = k x_1 + (2k) \frac{4}{3} \left(\frac{4}{3} x_1 - \frac{1}{3} x_2 - x_3 \right)$$

$$\frac{\partial V}{\partial x_2} = -\frac{1}{3} (2k) \left(\frac{4}{3} x_1 - \frac{1}{3} x_2 - x_3 \right)$$

$$\frac{\partial V}{\partial x_3} = -(2k) \left(\frac{4}{3} x_1 - \frac{1}{3} x_2 - x_3 \right) + (3k) x_3$$

Application of Lagrange's equations give the equations of motion:

$$\begin{aligned} \left(M + \frac{J_0}{9r^2} \right) \ddot{x}_1 - \frac{J_0}{9r^2} \ddot{x}_2 + \frac{41}{9} k x_1 - \frac{8}{9} k x_2 - \frac{8}{3} k x_3 &= F_1(t) \\ -\frac{J_0}{9r^2} \ddot{x}_1 + \left(3m + \frac{J_0}{9r^2} \right) \ddot{x}_2 - \frac{8}{9} k x_1 + \frac{2}{9} k x_2 + \frac{2}{3} k x_3 &= F_2(t) \\ m \ddot{x}_3 - \frac{8}{3} k x_1 + \frac{2}{3} k x_2 + 5k x_3 &= F_3(t) \end{aligned}$$

6.43

Generalized coordinates: θ_i ; $i = 1, 2, 3, 4$:

$$\begin{aligned} T &= \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} \left[I_2 + I_3 \frac{n_2^2}{n_3^2} \right] \dot{\theta}_2^2 + \frac{1}{2} \left[I_4 + I_5 \frac{n_4^2}{n_5^2} \right] \dot{\theta}_3^2 + \frac{1}{2} I_6 \dot{\theta}_4^2 \\ V &= \frac{1}{2} k_{t1} (\theta_2 - \theta_1)^2 + \frac{1}{2} k_{t2} \left(\theta_3 - \theta_2 \frac{n_2}{n_3} \right)^2 + \frac{1}{2} k_{t3} \left(\theta_4 - \theta_3 \frac{n_4}{n_5} \right)^2 \end{aligned}$$

$$Q_1 = M_1 \cos \omega t$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = I_1 \dot{\theta}_1 ; \quad \frac{\partial T}{\partial \dot{\theta}_2} = \left[I_2 + I_3 \frac{n_2^2}{n_3^2} \right] \dot{\theta}_2$$

$$\frac{\partial T}{\partial \dot{\theta}_3} = \left[I_4 + I_5 \frac{n_4^2}{n_5^2} \right] \dot{\theta}_3 ; \quad \frac{\partial T}{\partial \dot{\theta}_4} = I_6 \dot{\theta}_4$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_1} \right) = I_1 \ddot{\theta}_1 ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_2} \right) = \left[I_2 + I_3 \frac{n_2^2}{n_3^2} \right] \ddot{\theta}_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_3} \right) = \left[I_4 + I_5 \frac{n_4^2}{n_5^2} \right] \ddot{\theta}_3 ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_4} \right) = I_6 \ddot{\theta}_4$$

$$\begin{aligned}\frac{\partial V}{\partial \theta_1} &= -k_{t1} (\theta_2 - \theta_1) \\ \frac{\partial V}{\partial \theta_2} &= k_{t1} (\theta_2 - \theta_1) - k_{t2} \left(\theta_3 - \theta_2 \frac{n_2}{n_3} \right) \frac{n_2}{n_3} \\ \frac{\partial V}{\partial \theta_3} &= k_{t2} \left(\theta_3 - \theta_2 \frac{n_2}{n_3} \right) - k_{t3} \left(\theta_4 - \theta_3 \frac{n_4}{n_5} \right) \frac{n_4}{n_5} \\ \frac{\partial V}{\partial \theta_4} &= k_{t3} \left(\theta_4 - \theta_3 \frac{n_4}{n_5} \right)\end{aligned}$$

Equations of motion (from Lagrange's equations):

$$\begin{aligned}I_1 \ddot{\theta}_1 - k_{t1} (\theta_2 - \theta_1) &= M_1 \cos \omega t \\ \left(I_2 + I_3 \frac{n_2^2}{n_3^2} \right) \ddot{\theta}_2 + k_{t1} (\theta_2 - \theta_1) - k_{t2} \left(\theta_3 - \theta_2 \frac{n_2}{n_3} \right) \frac{n_2}{n_3} &= 0 \\ \left(I_4 + I_5 \frac{n_4^2}{n_5^2} \right) \ddot{\theta}_3 + k_{t2} \left(\theta_3 - \theta_2 \frac{n_2}{n_3} \right) - k_{t3} \left(\theta_4 - \theta_3 \frac{n_4}{n_5} \right) \frac{n_4}{n_5} &= 0 \\ I_6 \ddot{\theta}_4 + k_{t3} \left(\theta_4 - \theta_3 \frac{n_4}{n_5} \right) &= 0\end{aligned}$$

6.44 Equations of motion for the system of Fig. 6.8(a) are

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = 0 \quad \text{--- (E}_1\text{)}$$

$$m_2 \ddot{x}_2 - k_2 (x_1 - x_2) + k_3 (x_2 - x_3) = 0 \quad \text{--- (E}_2\text{)}$$

$$m_3 \ddot{x}_3 - k_3 (x_2 - x_3) = 0 \quad \text{--- (E}_3\text{)}$$

Let $\varrho_1 = x_1$, $\varrho_2 = x_2 - x_1$ and $\varrho_3 = x_3 - x_2$. Eqs. (E₁) to (E₃) become

$$m_1 \ddot{\varrho}_1 + k_1 \varrho_1 - k_2 \varrho_2 = 0 \Rightarrow \ddot{\varrho}_1 + \frac{k_1}{m_1} \varrho_1 - \frac{k_2}{m_1} \varrho_2 = 0 \quad \text{--- (E}_4\text{)}$$

$$m_2 \ddot{\varrho}_2 + k_2 \varrho_2 - k_3 \varrho_3 = 0 \Rightarrow \ddot{\varrho}_2 + \frac{k_2}{m_2} \varrho_2 - \frac{k_3}{m_2} \varrho_3 = 0 \quad \text{--- (E}_5\text{)}$$

$$m_3 \ddot{\varrho}_3 + k_3 \varrho_3 = 0 \Rightarrow \ddot{\varrho}_3 + \frac{k_3}{m_3} \varrho_3 = 0 \quad \text{--- (E}_6\text{)}$$

$$(E_4) \text{ minus } (E_5) \text{ gives } (\ddot{\varrho}_2 - \ddot{\varrho}_1) - \frac{k_1}{m_1} \varrho_1 + \left(\frac{k_2}{m_2} + \frac{k_2}{m_1} \right) \varrho_2 - \frac{k_3}{m_2} \varrho_3 = 0 \quad \text{--- (E}_7\text{)}$$

$$(E_5) \text{ minus } (E_6) \text{ gives } (\ddot{\varrho}_3 - \ddot{\varrho}_2) - \frac{k_2}{m_2} \varrho_2 + \varrho_3 \left(\frac{k_3}{m_3} + \frac{k_3}{m_2} \right) = 0 \quad \text{--- (E}_8\text{)}$$

(E₄), (E₇) and (E₈) can be expressed as

$$\left. \begin{aligned}\ddot{\varrho}_1 + \frac{k_1}{m_1} \varrho_1 - \frac{k_2}{m_1} \varrho_2 &= 0 \\ \ddot{\varrho}_2 - \frac{k_1}{m_1} \varrho_1 + \left(\frac{k_2}{m_2} + \frac{k_2}{m_1} \right) \varrho_2 - \frac{k_3}{m_2} \varrho_3 &= 0 \\ \ddot{\varrho}_3 - \frac{k_2}{m_2} \varrho_2 + \left(\frac{k_3}{m_3} + \frac{k_3}{m_2} \right) \varrho_3 &= 0\end{aligned} \right\} \quad \text{--- (E}_9\text{)}$$

For $k_i = k$ and $m_i = m$ ($i = 1, 2, 3$), Eqs. (E₉) reduce to

$$\left. \begin{aligned} \ddot{v}_1 + \frac{k}{m} v_1 - \frac{k}{m} v_2 &= 0 \\ \ddot{v}_2 - \frac{k}{m} v_1 + 2 \frac{k}{m} v_2 - \frac{k}{m} v_3 &= 0 \\ \ddot{v}_3 - \frac{k}{m} v_1 + 2 \frac{k}{m} v_3 &= 0 \end{aligned} \right\} \quad \text{--- (E}_{10}\text{)}$$

For $v_i(t) = Q_i \cos(\omega t + \phi)$; $i = 1, 2, 3$, (E₁₀) give

$$\begin{bmatrix} (-\omega^2 + \frac{k}{m}) & -\frac{k}{m} & 0 \\ -\frac{k}{m} & (-\omega^2 + 2\frac{k}{m}) & -\frac{k}{m} \\ 0 & -\frac{k}{m} & (-\omega^2 + 2\frac{k}{m}) \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- (E}_{11}\text{)}$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 + \frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & -\omega^2 + 2\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & -\omega^2 + 2\frac{k}{m} \end{vmatrix} = \omega^6 - 5\omega^4 \frac{k}{m} + 6\omega^2 \frac{k^2}{m^2} - \frac{k^3}{m^3} = 0$$

i.e. $\alpha^3 - 5\alpha^2 + 6\alpha - 1 = 0$ where $\alpha = \frac{\omega^2 m}{k}$.

Roots of this equation give

$$\alpha_1 = 0.19806, \quad \omega_1 = 0.44504 \sqrt{\frac{k}{m}}$$

$$\alpha_2 = 1.5553, \quad \omega_2 = 1.2471 \sqrt{\frac{k}{m}}$$

$$\alpha_3 = 3.2490, \quad \omega_3 = 1.8025 \sqrt{\frac{k}{m}}$$

It can be seen that the eigenvalues are same in both problems.

6.45

Equations of motion (from problem 6.14):

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For harmonic motion $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2, 3$, we get

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 & 0 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & -\omega^2 m_3 + k_3 + k_4 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 & 0 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & -\omega^2 m_3 + k_3 + k_4 \end{vmatrix} = 0$$

i.e. $(-\omega^2 m_1 + k_1 + k_2) \{ (-\omega^2 m_2 + k_2 + k_3)(-\omega^2 m_3 + k_3 + k_4) - k_3^2 \} + k_2 \{ -k_3(-\omega^2 m_3 + k_3 + k_4) \} = 0$

i.e.
$$\omega^6 (m_1 m_2 m_3) - \omega^4 [m_1 m_2 (k_3 + k_4) + m_2 m_3 (k_1 + k_2) + m_1 m_3 (k_2 + k_3)] + \omega^2 [m_1 (k_2 + k_3)(k_3 + k_4) + m_2 (k_1 + k_2)(k_3 + k_4) + m_3 (k_1 + k_2)(k_2 + k_3) - m_1 k_3^2 - m_3 k_2^2] - [(k_1 + k_2)(k_2 + k_3)(k_3 + k_4) + (k_1 + k_2) k_3^2 + (k_3 + k_4) k_2^2] = 0$$

6.46

Equations of motion:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- (E}_1\text{)}$$

For harmonic motion, we get

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 & 0 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & -\omega^2 m_3 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- (E}_2\text{)}$$

This gives the frequency equation (for $k_1 = k$, $k_2 = 2k$, $k_3 = 3k$, $m_1 = m$, $m_2 = 2m$ and $m_3 = 3m$)

$$\begin{vmatrix} -\omega^2 m + 3k & -2k & 0 \\ -2k & -2\omega^2 m + 5k & -3k \\ 0 & -3k & -3\omega^2 m + 3k \end{vmatrix} = 0 \quad \text{--- (E}_3\text{)}$$

i.e.
$$(-\omega^2 m + 3k) [(-2\omega^2 m + 5k)(-3\omega^2 m + 3k) - 9k^2] + 2k [-2k(-3\omega^2 m + 3k)] = 0$$

i.e.
$$2\alpha^3 - 13\alpha^2 + 19\alpha - 2 = 0 \quad \text{where } \alpha = \frac{\omega^2 m}{k} \quad \text{--- (E}_4\text{)}$$

This gives the roots

$$\alpha_1 = 0.113992, \quad \alpha_2 = 2.00002, \quad \alpha_3 = 4.38600$$

$$\omega_1 = 0.337627 \sqrt{\frac{k}{m}}, \quad \omega_2 = 1.414221 \sqrt{\frac{k}{m}}, \quad \omega_3 = 2.094278 \sqrt{\frac{k}{m}}$$

Mode shape in j^{th} mode:

Eg. (E₂) gives
$$\frac{x_2^{(j)}}{x_1^{(j)}} = \frac{-\omega_j^2 m_1 + k_1 + k_2}{k_2} = \frac{-\omega_j^2 m + 3k}{2k}$$

$$\frac{x_3^{(j)}}{x_2^{(j)}} = \frac{k_3}{-\omega_j^2 m_3 + k_3} = \frac{3k}{-3\omega_j^2 m + 3k}$$

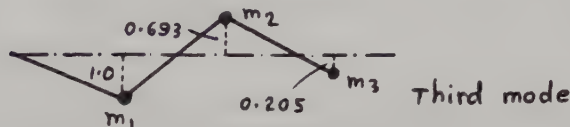
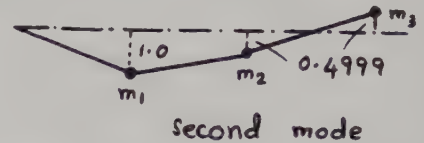
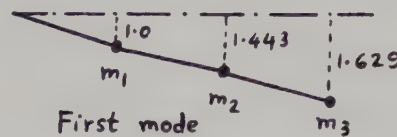
$$\frac{x_3^{(j)}}{x_1^{(j)}} = \frac{x_3^{(j)}}{x_2^{(j)}} \cdot \frac{x_2^{(j)}}{x_1^{(j)}} = \frac{3(-\omega_j^2 m + 3k)}{2(-3\omega_j^2 m + 3k)}$$

$$\vec{X}^{(j)} = \begin{Bmatrix} X_1^{(j)} \\ X_2^{(j)} \\ X_3^{(j)} \end{Bmatrix} = X_1^{(j)} \begin{Bmatrix} 1 \\ (-\omega_j^2 m + 3k)/(2k) \\ 3(-\omega_j^2 m + 3k)/[2(-3\omega_j^2 m + 3k)] \end{Bmatrix} \quad \text{--- (E5)}$$

Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.443004 \\ 1.628659 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.49999 \\ -0.49998 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -0.693 \\ 0.204666 \end{Bmatrix}$$

Mode shapes:



6.47

When $k_1 = 3k$, $k_2 = k_3 = k$, $m_1 = 3m$ and $m_2 = m_3 = m$, Eq. (E2) of problem 6.46 gives the frequency equation

$$\begin{vmatrix} -3m\omega^2 + 4k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + k \end{vmatrix} = 0$$

i.e. $(-3m\omega^2 + 4k)[(-m\omega^2 + 2k)(-m\omega^2 + k) - k^2] + k[-k(-m\omega^2 + k)] = 0$

i.e. $3\alpha^3 - 13\alpha^2 + 14\alpha - 3 = 0 \quad \text{--- (E1)}$

where $\alpha = m\omega^2/k$. Roots of (E1) are

$$\alpha_1 = 0.284515, \quad \alpha_2 = 1.26053, \quad \alpha_3 = 2.78829$$

$$\omega_1 = 0.533399 \sqrt{\frac{k}{m}}, \quad \omega_2 = 1.122733 \sqrt{\frac{k}{m}}, \quad \omega_3 = 1.669817 \sqrt{\frac{k}{m}}$$

Eq. (E5) of problem 6.46 gives the j th mode shape as

$$\vec{X}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-3m\omega_j^2 + 4k)/k \\ (-3m\omega_j^2 + 4k)/(-m\omega_j^2 + k) \end{Bmatrix}$$

Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 3.146455 \\ 4.397653 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.21841 \\ -0.83833 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -4.36487 \\ 2.44081 \end{Bmatrix}$$

Orthogonality of normal modes:

$$[m] = m \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{X}^{(1)T} [m] \vec{X}^{(1)} = 1 \Rightarrow \begin{pmatrix} 1.0 & 3.146455 & 4.397653 \end{pmatrix} \begin{Bmatrix} 3.0 \\ 3.146455 \\ 4.397653 \end{Bmatrix} m X_1^{(1)2} = 1$$

$$\Rightarrow 32.239531 m X_1^{(1)2} = 1 \quad \text{or} \quad X_1^{(1)} = \frac{1}{\sqrt{m}} (0.176119)$$

$$\vec{X}^{(1)} = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.176119 \\ 0.554151 \\ 0.774510 \end{Bmatrix}$$

$$\vec{X}^{(2)T} [m] \vec{X}^{(2)} = 1 \Rightarrow \begin{pmatrix} 1.0 & 0.21841 & -0.83833 \end{pmatrix} \begin{Bmatrix} 3.0 \\ 0.21841 \\ -0.83833 \end{Bmatrix} m X_1^{(2)2} = 1$$

$$\Rightarrow 3.7505 m X_1^{(2)2} = 1 \quad \text{or} \quad X_1^{(2)} = \frac{1}{\sqrt{m}} (0.516363)$$

$$\vec{X}^{(2)} = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.516363 \\ 0.112779 \\ -0.432883 \end{Bmatrix}$$

$$\vec{X}^{(3)T} [m] \vec{X}^{(3)} = 1 \Rightarrow \begin{pmatrix} 1.0 & -4.36487 & 2.44081 \end{pmatrix} \begin{Bmatrix} 3.0 \\ -4.36487 \\ 2.44081 \end{Bmatrix} m X_1^{(3)2} = 1$$

$$\Rightarrow 28.00964 m X_1^{(3)2} = 1 \quad \text{or} \quad X_1^{(3)} = \frac{1}{\sqrt{m}} (0.18895)$$

$$\vec{X}^{(3)} = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.18895 \\ -0.824742 \\ 0.461191 \end{Bmatrix}$$

It can be verified that

$$\vec{X}^{(1)T} [m] \vec{X}^{(2)} = (0.176119 \quad 0.554151 \quad 0.774510) \begin{Bmatrix} 1.549089 \\ 0.112779 \\ -0.432883 \end{Bmatrix} = 0$$

$$\vec{X}^{(1)T} [m] \vec{X}^{(3)} = (0.176119 \quad 0.554151 \quad 0.774510) \begin{Bmatrix} 0.56685 \\ -0.824742 \\ 0.461191 \end{Bmatrix} = 0$$

$$\vec{X}^{(2)T} [m] \vec{X}^{(3)} = (0.516363 \quad 0.112779 \quad -0.432883) \begin{Bmatrix} 0.56685 \\ -0.824742 \\ 0.461191 \end{Bmatrix} = 0$$

6.48

For $l_1 = 0.2 \text{ m}$, $l_2 = 0.3 \text{ m}$, $l_3 = 0.4 \text{ m}$, $m_1 = 1 \text{ kg}$, $m_2 = 2 \text{ kg}$ and $m_3 = 3 \text{ kg}$,

Eg. (E₃) of problem 6.37 gives the equations of motion

$$\begin{bmatrix} 6(0.04) & 5(0.06) & 3(0.08) \\ 5(0.06) & 5(0.09) & 3(0.12) \\ 3(0.08) & 3(0.12) & 3(0.16) \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} 6(9.81)(0.2) & 0 & 0 \\ 0 & 5(9.81)(0.3) & 0 \\ 0 & 0 & 3(9.81)(0.4) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- (E}_1\text{)}$$

For harmonic motion, (E₁) becomes

$$-\omega^2 \begin{bmatrix} 0.24 & 0.30 & 0.24 \\ 0.30 & 0.45 & 0.36 \\ 0.24 & 0.36 & 0.48 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} + \begin{bmatrix} 11.772 & 0 & 0 \\ 0 & 14.715 & 0 \\ 0 & 0 & 11.772 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- (E}_2\text{)}$$

This gives the frequency equation

$$\begin{vmatrix} \omega^2(0.24) - 11.772 & \omega^2(0.30) & \omega^2(0.24) \\ \omega^2(0.30) & \omega^2(0.45) - 14.715 & \omega^2(0.36) \\ \omega^2(0.24) & \omega^2(0.36) & \omega^2(0.48) - 11.772 \end{vmatrix} = 0$$

$$\begin{aligned} \text{i.e. } & (0.24\omega^2 - 11.772)[(0.45\omega^2 - 14.715)(0.48\omega^2 - 11.772) - (0.36)^2\omega^4] \\ & - 0.3\omega^2[0.3\omega^2(0.48\omega^2 - 11.772) - (0.24)(0.36)\omega^4] \\ & + 0.24\omega^2[(0.30)(0.36)\omega^4 - 0.24\omega^2(0.45\omega^2 - 14.715)] = 0 \end{aligned}$$

$$\text{i.e. } \omega^6 - 600.8625\omega^4 + 54132.806\omega^2 - 590047.6 = 0$$

Roots of this equation are

$$\omega_1^2 = 12.6335, \quad \omega_1 = 3.554364 \text{ rad/s}$$

$$\omega_2^2 = 94.6116, \quad \omega_2 = 9.726849 \text{ rad/s}$$

$$\omega_3^2 = 493.619, \quad \omega_3 = 22.217538 \text{ rad/s}.$$

6.49 (a) By replacing l by $\frac{l}{4}$ in problem 6.21, we obtain

$$[a] = \frac{l^3}{64EI} \begin{bmatrix} \frac{9}{64} & \frac{1}{6} & \frac{13}{192} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{13}{192} & \frac{1}{6} & \frac{9}{64} \end{bmatrix}$$

$$[m] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; [D] = [a][m] = \frac{ml^3}{EI} \begin{bmatrix} 0.0021975 & 0.0026042 & 0.0010579 \\ 0.0026042 & 0.0052083 & 0.0026042 \\ 0.0010579 & 0.0026042 & 0.0021973 \end{bmatrix}$$

Frequency equation is

$$|[D] - \lambda[I]| = 0 \quad \text{where } \lambda = \frac{1}{\omega^2}$$

$$\text{i.e. } \begin{vmatrix} 0.0021973 - \alpha & 0.0026042 & 0.0010579 \\ 0.0026042 & 0.0052083 - \alpha & 0.0026042 \\ 0.0010579 & 0.0026042 & 0.0021973 - \alpha \end{vmatrix} = 0 \quad \text{--- (E}_1\text{)}$$

$$\text{where } \alpha = \frac{EI}{ml^3\lambda} = \frac{EI}{ml^3\omega^2}. \quad E_8 \cdot (E_1) \text{ gives}$$

$$\begin{aligned} & (0.0021973 - \alpha)[(0.0052083 - \alpha)(0.0021973 - \alpha) - (0.0026042)^2] \\ & - (0.0026042)[0.0026042(0.0021973 - \alpha) - (0.0010579)(0.0026042)] \\ & + (0.0010579)[(0.0026042)^2 - (0.0010579)(0.0052083 - \alpha)] = 0 \end{aligned}$$

$$\text{i.e. } \alpha^3 - 0.96029 \times 10^{-2} \alpha^2 + 0.1303355 \times 10^{-4} \alpha - 0.0038623 \times 10^{-6} = 0$$

Roots are:

$$\alpha_1 = 0.000421453, \quad \omega_1 = 48.71082 \sqrt{EI/(ml^3)}$$

$$\alpha_2 = 0.00113955, \quad \omega_2 = 29.62329 \sqrt{EI/(ml^3)}$$

$$\alpha_3 = 0.00804192, \quad \omega_3 = 11.15116 \sqrt{EI/(m l^3)}$$

(b) $m = 10 \text{ kg}, \quad l = 0.5 \text{ m}, \quad E = 2.07 \times 10^{11} \text{ N/m}^2,$

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64} \left(\frac{2.5}{100} \right)^4 = 1.9175 \times 10^{-8} \text{ m}^4$$

$$\sqrt{\frac{EI}{m l^3}} = \sqrt{\frac{(2.07 \times 10^{11})(1.9175 \times 10^{-8})}{10 (0.5)^3}} = 56.3505$$

$$\omega_3 = 48.71082 (56.3505) = 2744.8791 \text{ rad/sec}$$

$$\omega_2 = 29.62329 (56.3505) = 1669.2872 \text{ rad/sec}$$

$$\omega_1 = 11.15116 (56.3505) = 628.3734 \text{ rad/sec}$$

(c) In order to have the same natural frequencies, we need to have the same value of I .

(i) For solid circular cross-section of diameter 2.5 cm,

$$I = 1.9175 \times 10^{-8} \text{ m}^4$$

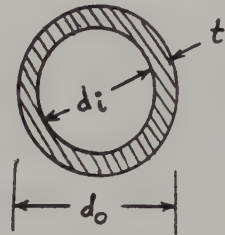
(ii) For hollow circular section:

$$\text{let } d_o = 5t$$

$$I = \frac{\pi}{64} (d_o^4 - d_i^4) = \frac{\pi}{64} (625 - 81) t^4$$

$$= 26.7036 t^4 = 1.9175 \times 10^{-8}$$

$$t = 0.5177 \text{ cm}, \quad d_o = 2.5885 \text{ cm}, \quad d_i = 1.5531 \text{ cm}.$$

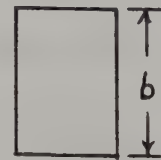


(iii) For solid rectangular section:

$$\text{Let } b = 2a.$$

$$I = \frac{1}{12} (a) b^3 = \frac{2}{3} a^4 = 1.9175 \times 10^{-8}$$

$$a = 1.3023 \text{ cm}, \quad b = 2.6046 \text{ cm}.$$



(iv) For hollow rectangular section:

$$\text{Let } b = 5t, \quad b_o = 3t$$

$$a = 2.5t, \quad a_o = 0.5t$$

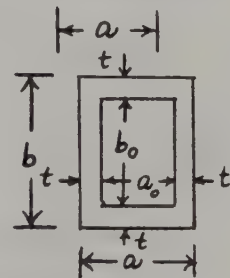
$$I = \frac{1}{12} [a b^3 - a_o b_o^3]$$

$$= \frac{1}{12} [2.5(125) t^4 - (0.5)(27) t^4]$$

$$= 24.9167 t^4 = 1.9175 \times 10^{-8} \text{ m}^4$$

$$t = 0.5267 \text{ cm}, \quad a = 1.3167 \text{ cm}, \quad b = 2.6335 \text{ cm},$$

$$a_o = 0.2634 \text{ cm}, \quad b_o = 1.5801 \text{ cm}.$$



Weights:

Weights are proportional to cross-sectional areas.

(i) For solid circular section:

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} (2.5)^2 = 4.90875 \text{ cm}^2$$

(ii) For hollow circular section:

$$A = \frac{\pi}{4} (d_o^2 - d_i^2) = \frac{\pi}{4} [2.5885^2 - 1.5531^2] = 3.3680 \text{ cm}^2$$

(iii) For solid rectangular section:

$$A = ab = (1.3023)(2.6046) = 3.3920 \text{ cm}^2$$

(iv) For hollow rectangular section:

$$A = ab - a_o b_o = (1.3167)(2.6335) - (0.2634)(1.5801) = 3.0513 \text{ cm}^2$$

$\therefore$ Least weight beam will have a hollow rectangular section.

6.50

$$\begin{vmatrix} \lambda - 5 & -3 & -2 \\ -3 & \lambda - 6 & -4 \\ -1 & -2 & \lambda - 6 \end{vmatrix} = 0$$

$$\begin{aligned} \text{i.e. } & (\lambda - 5)[(\lambda - 6)^2 - (-2)(-4)] - (-3)[-3(\lambda - 6) - (-1)(-4)] \\ & + (-2)[-3(-2) - (-1)(\lambda - 6)] = 0 \end{aligned}$$

$$\text{i.e. } \lambda^3 - 17\lambda^2 + 77\lambda - 98 = 0$$

$$\text{Roots give: } \lambda_1 = 2.21398, \lambda_2 = 4.16929, \lambda_3 = 10.6168$$

6.51

From problem 6.19, for $k_i = k$; $i = 1, 2, 3, 4$,

$$[k] = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad [m] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equations of motion:

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

These become, for harmonic motion,

$$\begin{bmatrix} -m\omega^2 + 2k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + 2k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \dots (E_1)$$

Frequency equation:

$$(-m\omega^2 + 2k)[(-m\omega^2 + 2k)^2 - k^2] + k[-k(-m\omega^2 + 2k)] = 0$$

$$\text{i.e. } (-\alpha + 2)(\alpha^2 - 4\alpha + 2) = 0 \quad \text{where } \alpha = \frac{m\omega^2}{k}.$$

This gives

$$\alpha_1 = 2 - \sqrt{2} = 0.585786, \quad \alpha_2 = 2, \quad \alpha_3 = 3.414214$$

$$\omega_1 = 0.765367 \sqrt{\frac{k}{m}}, \quad \omega_2 = 1.414214 \sqrt{\frac{k}{m}}, \quad \omega_3 = 1.847759 \sqrt{\frac{k}{m}}$$

(E₁) gives

$$x_2^{(j)} = \left(\frac{-m \omega_j^2 + 2k}{k} \right) x_1^{(j)}$$

$$-k x_1^{(j)} + (-m \omega_j^2 + 2k) x_2^{(j)} - k x_3^{(j)} = 0$$

$$\text{or } \left[-k + (-m \omega_j^2 + 2k)^2 \cdot \frac{1}{k} \right] x_1^{(j)} - k x_3^{(j)} = 0$$

$$\text{or } x_3^{(j)} = \left\{ \frac{(-m \omega_j^2 + 2k)^2 - k^2}{k^2} \right\} x_1^{(j)}$$

$$j^{\text{th}} \text{ mode} = \vec{x}^{(j)} = \begin{Bmatrix} x_1^{(j)} \\ x_2^{(j)} \\ x_3^{(j)} \end{Bmatrix} = x_1^{(j)} \begin{Bmatrix} 1 \\ (-m \omega_j^2 + 2k)/k \\ [(-m \omega_j^2 + 2k)^2 - k^2]/k^2 \end{Bmatrix}$$

Mode shapes are:

$$\vec{x}^{(1)} = \begin{Bmatrix} 1 \\ 1.414214 \\ 1 \end{Bmatrix} x_1^{(1)}, \quad \vec{x}^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} x_1^{(2)}, \quad \vec{x}^{(3)} = \begin{Bmatrix} 1 \\ -1.414214 \\ 1 \end{Bmatrix} x_1^{(3)}$$

6.52

From problem 6.19,

$$[k] = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}, \quad [m] = \begin{bmatrix} 2m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & 2m \end{bmatrix}$$

Equations of motion for harmonic motion:

$$\begin{bmatrix} -2m \omega^2 + 2k & -k & 0 \\ -k & -3m \omega^2 + 2k & -k \\ 0 & -k & -2m \omega^2 + 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- (E}_1\text{)}$$

Frequency equation:

$$(-2m \omega^2 + 2k) [(-3m \omega^2 + 2k)(-2m \omega^2 + 2k) - k^2] + k [-k (-2m \omega^2 + 2k)] = 0$$

$$\text{i.e. } (-m \omega^2 + k) [3m^2 \omega^4 - 5km \omega^2 + k^2] = 0$$

$$\text{i.e. } (-\alpha + 1) (3\alpha^2 - 5\alpha + 1) = 0 \quad \text{where } \alpha = \frac{\omega^2 m}{k}$$

$$\therefore \alpha_1 = 0.232408, \quad \omega_1 = 0.482087 \sqrt{\frac{k}{m}}$$

$$\alpha_2 = 1.0, \quad \omega_2 = \sqrt{\frac{k}{m}}$$

$$\alpha_3 = 1.434258, \quad \omega_3 = 1.197605 \sqrt{\frac{k}{m}}$$

(E₁) gives

$$\frac{x_2^{(j)}}{x_1^{(j)}} = \frac{-2m \omega_j^2 + 2k}{k}$$

$$(-3m\omega_j^2 + 2\kappa) X_2^{(j)} - \kappa X_3^{(j)} = \kappa X_1^{(j)}$$

$$\text{or } (-3m\omega_j^2 + 2\kappa) \left(\frac{-2m\omega_j^2 + 2\kappa}{\kappa} \right) X_1^{(j)} - \kappa X_1^{(j)} = \kappa X_3^{(j)}$$

$$\text{or } \frac{X_3^{(j)}}{X_1^{(j)}} = \frac{(-3m\omega_j^2 + 2\kappa)(-2m\omega_j^2 + 2\kappa) - \kappa^2}{\kappa^2}$$

$$j^{\text{th}} \text{ mode} = \vec{x}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-2m\omega_j^2 + 2\kappa)/\kappa \\ \{(-3m\omega_j^2 + 2\kappa)(-2m\omega_j^2 + 2\kappa) - \kappa^2\}/\kappa^2 \end{Bmatrix}$$

This leads to:

$$\vec{x}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1 \\ 1.535184 \\ 1 \end{Bmatrix}, \quad \vec{x}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \vec{x}^{(3)} = X_1^{(3)} \begin{Bmatrix} -1 \\ -0.868516 \\ 1 \end{Bmatrix}$$

6.53

For $l_i = l$ and $m_i = m$ ($i = 1, 2, 3$), problem 6.37 gives

$$\begin{bmatrix} 3ml^2 & 2ml^2 & ml^2 \\ 2ml^2 & 2ml^2 & ml^2 \\ ml^2 & ml^2 & ml^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} 3mgl & 0 & 0 \\ 0 & 2mgl & 0 \\ 0 & 0 & mgl \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For harmonic motion,

$$\begin{bmatrix} -3l\omega^2 + 3g & -2l\omega^2 & -l\omega^2 \\ -2l\omega^2 & -2l\omega^2 + 2g & -l\omega^2 \\ -l\omega^2 & -l\omega^2 & -l\omega^2 + g \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Dividing throughout by $-g$ and defining $\alpha = \frac{\omega^2 l}{g}$, this gives

$$\begin{bmatrix} 3\alpha - 3 & 2\alpha & \alpha \\ 2\alpha & 2\alpha - 2 & \alpha \\ \alpha & \alpha & \alpha - 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{----- (E}_1\text{)}$$

Frequency equation:

$$(3\alpha - 3) [(2\alpha - 2)(\alpha - 1) - \alpha^2] - 2\alpha [2\alpha(\alpha - 1) - \alpha^2] + \alpha [2\alpha^2 - \alpha(2\alpha - 2)] = 0$$

$$\text{i.e. } \alpha^3 - 9\alpha^2 + 18\alpha - 6 = 0$$

Roots are:

$$\alpha_1 = 0.415764, \quad \omega_1 = 0.644798 \sqrt{\frac{g}{l}}$$

$$\alpha_2 = 2.29431, \quad \omega_2 = 1.514698 \sqrt{g/l}$$

$$\alpha_3 = 6.28995, \quad \omega_3 = 2.507977 \sqrt{g/l}$$

Mode shapes: (E₁) gives

$$\theta_2^{(j)} = \left\{ \frac{-2\alpha_j^2 + 6\alpha_j - 3}{\alpha_j(\alpha_j - 2)} \right\} \theta_1^{(j)}, \quad \theta_3^{(j)} = \left(\frac{\alpha_j - 3}{\alpha_j - 2} \right) \theta_1^{(j)}$$

$$j^{\text{th}} \text{ mode} = \vec{u}^{(j)} = u_1^{(j)} \begin{Bmatrix} 1 \\ (-2\alpha_j^2 + 6\alpha_j - 3)/(\alpha_j^2 - 2\alpha_j) \\ (\alpha_j - 3)/(\alpha_j - 2) \end{Bmatrix}$$

Hence

$$\vec{u}^{(1)} = u_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.2922 \\ 1.6312 \end{Bmatrix}, \quad \vec{u}^{(2)} = u_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.3527 \\ -2.3978 \end{Bmatrix}, \quad \vec{u}^{(3)} = u_1^{(3)} \begin{Bmatrix} 1.0 \\ -1.6450 \\ 0.7669 \end{Bmatrix}$$

6.54

From problem 6.22 we find, for $m_1 = m_3 = m$, $m_2 = 2m$, $k_1 = k_2 = k$ and

$$k_3 = 2k,$$

$$[k] = \begin{bmatrix} 2k & -k & 0 \\ -k & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix}, \quad [m] = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}$$

Equations of motion for harmonic motion

$$\begin{bmatrix} -\omega^2 m + 2k & -k & 0 \\ -k & -2m\omega^2 + 3k & -2k \\ 0 & -2k & -\omega^2 m + 2k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- (E}_1\text{)}$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 m + 2k & -k & 0 \\ -k & -2m\omega^2 + 3k & -2k \\ 0 & -2k & -\omega^2 m + 2k \end{vmatrix} = 0$$

$$\text{or } (-\alpha + 2)(2\alpha^2 - 7\alpha + 1) = 0 \quad \text{with } \alpha = \frac{m\omega^2}{k}$$

$$\therefore \alpha_1 = 0.149219, \quad \omega_1 = 0.386289 \sqrt{k/m}$$

$$\alpha_2 = 2.0, \quad \omega_2 = 1.414214 \sqrt{k/m}$$

$$\alpha_3 = 3.350781, \quad \omega_3 = 1.830514 \sqrt{k/m}$$

Eg. (E₁) gives, for ω_j ,

$$\frac{X_2^{(j)}}{X_1^{(j)}} = \frac{-\omega_j^2 m + 2k}{k}$$

$$\frac{X_3^{(j)}}{X_1^{(j)}} = \frac{(-2m\omega_j^2 + 3k)(-\omega_j^2 m + 2k) - k^2}{2k^2}$$

$$j^{\text{th}} \text{ mode} = \vec{X}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-\omega_j^2 m + 2k)/k \\ \{(-2m\omega_j^2 + 3k)(-\omega_j^2 m + 2k) - k^2\}/(2k^2) \end{Bmatrix}$$

Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.850781 \\ 2.0 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.0 \\ -0.5 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -1.350781 \\ 2.0 \end{Bmatrix}$$

6.55

For $k_1 = 3k$, $k_2 = k_3 = k$, $m_1 = 4m$, $m_2 = 2m$ and $m_3 = m$, Eq. (E₂) of problem 6.46 gives

$$\begin{bmatrix} -4m\omega^2 + 4k & -k & 0 \\ -k & -2m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- (E}_1\text{)}$$

Frequency equation:

$$(-4m\omega^2 + 4k) \{ (-2m\omega^2 + 2k)(-m\omega^2 + k) - k^2 \} + k \{ -k(-m\omega^2 + k) \} = 0$$

i.e. $(-\alpha + 1)(8\alpha^2 - 16\alpha + 3) = 0$ with $\alpha = \frac{m\omega^2}{k}$

$$\therefore \alpha_1 = 0.209431, \quad \omega_1 = 0.457636 \sqrt{k/m}$$

$$\alpha_2 = 1.0, \quad \omega_2 = \sqrt{k/m}$$

$$\alpha_3 = 1.790569, \quad \omega_3 = 1.338121 \sqrt{k/m}$$

Eq. (E₁) gives

$$X_2^{(j)} = \left\{ \frac{-4m\omega_j^2 + 4k}{k} \right\} X_1^{(j)}$$

$$X_3^{(j)} = \left[-1 + (-2m\omega_j^2 + 2k) \left(\frac{-4m\omega_j^2 + 4k}{k^2} \right) \right] X_1^{(j)}$$

$$j^{\text{th}} \text{ mode} = \vec{x}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-4m\omega_j^2 + 4k)/k \\ \{ (-2m\omega_j^2 + 2k)(-4m\omega_j^2 + 4k) - k^2 \} / k^2 \end{Bmatrix}$$

Thus

$$\vec{x}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 3.162276 \\ 4.0 \end{Bmatrix}, \quad \vec{x}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.0 \\ -1.0 \end{Bmatrix}, \quad \vec{x}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -3.162276 \\ 4.0 \end{Bmatrix}$$

6.56

For $m_1 = 2m$, $m_2 = m$, $m_3 = 3m$ and $l_i = l$ for all i , problem 6.23 gives

$$a_{11} = \frac{1}{P(\frac{1}{l} + \frac{1}{3l})} = \frac{3l}{4P}, \quad a_{21} = \frac{2}{3} a_{11} = \frac{1}{2} \frac{l}{P}, \quad a_{31} = \frac{1}{3} a_{11} = \frac{1}{4} \frac{l}{P}$$

$$a_{22} = \frac{1}{P(\frac{1}{2l} + \frac{1}{2l})} = \frac{l}{P}, \quad a_{32} = \frac{1}{2} a_{22} = \frac{1}{2} \frac{l}{P}, \quad a_{33} = \frac{1}{P(\frac{1}{3l} + \frac{1}{l})} = \frac{3l}{4P}$$

$$[a] = \frac{l}{4P} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad [m] = m \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad [a][m] = \frac{lm}{4P} \begin{bmatrix} 6 & 2 & 3 \\ 4 & 4 & 6 \\ 2 & 2 & 9 \end{bmatrix}$$

Equations of motion:

$$[a][m] \ddot{\vec{x}} + [I] \dot{\vec{x}} = \vec{0}$$

$$\text{Frequency equation: } | -[a][m] \omega^2 + [I] | = 0$$

$$\text{i.e.} \quad \left| -\frac{\omega^2 l m}{4 P} \begin{bmatrix} 6 & 2 & 3 \\ 4 & 4 & 6 \\ 2 & 2 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

with $\alpha = \omega^2 l m / (4 P)$, this equation becomes

$$\begin{vmatrix} 6\alpha - 1 & 2\alpha & 3\alpha \\ 4\alpha & 4\alpha - 1 & 6\alpha \\ 2\alpha & 2\alpha & 9\alpha - 1 \end{vmatrix} = 96\alpha^3 - 88\alpha^2 + 19\alpha - 1 = 0$$

Roots are:

$$\begin{aligned} \alpha_1 &= 0.079126, & \omega_1 &= 0.562587 \sqrt{\frac{P}{l m}} \\ \alpha_2 &= 0.209671, & \omega_2 &= 0.915797 \sqrt{\frac{P}{l m}} \\ \alpha_3 &= 0.627872, & \omega_3 &= 1.584767 \sqrt{\frac{P}{l m}} \end{aligned}$$

6.57 For $(GJ)_i = GJ$, $J_{di} = J_0$ and $l_i = l$ for all i , problem 6.18 gives

$$[*] = \frac{GJ}{l} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad [J_d] = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equations of motion for harmonic oscillation:

$$\begin{bmatrix} -\omega^2 J_0 + 2 \frac{GJ}{l} & -\frac{GJ}{l} & 0 \\ -\frac{GJ}{l} & -\omega^2 J_0 + 2 \frac{GJ}{l} & -\frac{GJ}{l} \\ 0 & -\frac{GJ}{l} & -\omega^2 J_0 + 2 \frac{GJ}{l} \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---}(E_1)$$

Dividing throughout by GJ/l and defining $\alpha = \frac{\omega^2 J_0 l}{GJ}$, (E_1) gives

$$\begin{bmatrix} -\alpha + 2 & -1 & 0 \\ -1 & -\alpha + 2 & -1 \\ 0 & -1 & -\alpha + 2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---}(E_2)$$

Frequency equation:

$$\begin{vmatrix} -\alpha + 2 & -1 & 0 \\ -1 & -\alpha + 2 & -1 \\ 0 & -1 & -\alpha + 2 \end{vmatrix} = (-\alpha + 2)(\alpha^2 - 4\alpha + 2) = 0$$

Roots are:

$$\begin{aligned} \alpha_1 &= 2 - \sqrt{2} = 0.585786, & \alpha_2 &= 2, & \alpha_3 &= 2 + \sqrt{2} = 3.414214 \\ \omega_1 &= 0.765367 \sqrt{\frac{GJ}{l J_0}}, & \omega_2 &= 1.414214 \sqrt{\frac{GJ}{l J_0}}, & \omega_3 &= 1.847759 \sqrt{\frac{GJ}{l J_0}} \end{aligned}$$

Noting that $E_g.(E_1)$ is similar to $E_g.(E_1)$ of problem 6.51, we can use the same modeshapes:

$$\vec{\Phi}^{(1)} = \Phi_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.414214 \\ 1.0 \end{Bmatrix}, \quad \vec{\Phi}^{(2)} = \Phi_1^{(2)} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \vec{\Phi}^{(3)} = \Phi_1^{(3)} \begin{Bmatrix} 1.0 \\ -1.414214 \\ 1.0 \end{Bmatrix}$$

6.58 Equations of motion

$$\frac{\rho A l}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \frac{2AE}{l} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- (E}_1\text{)}$$

This gives, for harmonic motion,

$$\begin{bmatrix} \left(-\frac{\omega^2 \rho A l}{4} + \frac{2AE}{l}\right) & -\frac{2AE}{l} & 0 \\ -\frac{2AE}{l} & \left(-\frac{\omega^2 \rho A l}{4} + \frac{4AE}{l}\right) & -\frac{2AE}{l} \\ 0 & -\frac{2AE}{l} & \left(-\frac{\omega^2 \rho A l}{4} + \frac{2AE}{l}\right) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Dividing throughout by $\frac{2AE}{l}$ and defining $\alpha = \frac{\omega^2 \rho A l}{8AE}$, --- (E₂)

$$\begin{bmatrix} -\alpha+1 & -1 & 0 \\ -1 & -2\alpha+2 & -1 \\ 0 & -1 & -\alpha+1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- (E}_3\text{)}$$

Frequency equation:

$$\begin{vmatrix} -\alpha+1 & -1 & 0 \\ -1 & -2\alpha+2 & -1 \\ 0 & -1 & -\alpha+1 \end{vmatrix} = 2\alpha(-\alpha+1)(\alpha-2) = 0$$

$$\therefore \alpha_1 = 0, \quad \omega_1 = 0$$

$$\alpha_2 = 1, \quad \omega_2 = \sqrt{\frac{8AE}{\rho A l^2}}$$

$$\alpha_3 = 2, \quad \omega_3 = \sqrt{\frac{16AE}{\rho A l^2}}$$

Principal modes:

$$E_g.(E_3) \text{ gives } x_2^{(j)} = (-\alpha_j + 1) x_1^{(j)}$$

$$x_3^{(j)} = [-1 + (-2\alpha_j + 2)(-\alpha_j + 1)] x_1^{(j)}$$

$$j^{\text{th}} \text{ mode: } \vec{x}^{(j)} = x_1^{(j)} \begin{Bmatrix} 1.0 \\ (-\alpha_j + 1) \\ 2(-\alpha_j + 1)^2 - 1 \end{Bmatrix}$$

Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

6.59

For orthonormalization, $\vec{X}^{(i)T} [m] \vec{X}^{(i)} = 1$; $i = 1, 2, 3$

Let new $\vec{X}^{(1)} = a_1 \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$, $\vec{X}^{(2)} = a_2 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ and $\vec{X}^{(3)} = a_3 \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix}$

$$\vec{X}^{(1)T} [m] \vec{X}^{(1)} = a_1^2 (1 \quad -1 \quad 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} = 4 a_1^2 = 1 \Rightarrow a_1 = \frac{1}{2}$$

$$\vec{X}^{(2)T} [m] \vec{X}^{(2)} = a_2^2 (1 \quad 1 \quad 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 4 a_2^2 = 1 \Rightarrow a_2 = \frac{1}{2}$$

$$\vec{X}^{(3)T} [m] \vec{X}^{(3)} = a_3^2 (0 \quad 1 \quad 2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix} = 6 a_3^2 = 1 \Rightarrow a_3 = \frac{1}{\sqrt{6}}$$

$$[m]\text{-orthonormal modal matrix} = [X] = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & \sqrt{2/3} \\ 1 & 1 & \sqrt{8/3} \end{bmatrix}$$

6.60

Stiffness matrix:

Let $x_1=1, x_2=x_3=0$. $F_1=2+1+1=4=k_{11}$, $F_2=-1=k_{21}$, $F_3=-1=k_{31}$

Let $x_2=1, x_1=x_3=0$. $F_2=1+1=2=k_{22}$, $F_1=-1=k_{12}$, $F_3=0=k_{32}$

Let $x_3=1, x_1=x_2=0$. $F_3=1+1=2=k_{33}$, $F_1=-1=k_{13}$, $F_2=0=k_{23}$

$$\therefore [k] = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Mass matrix:

$$[m] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

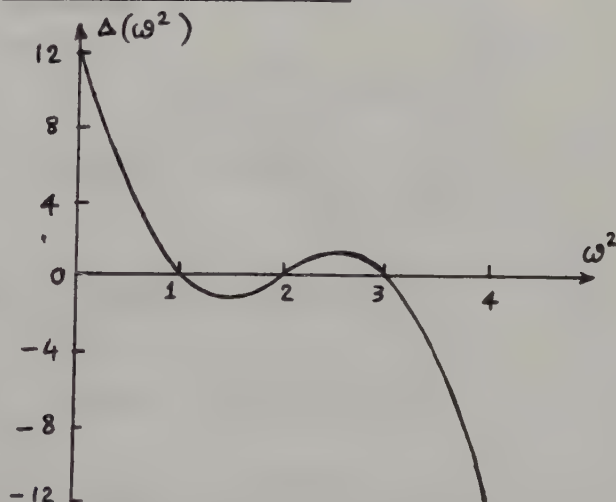
(a) Characteristic polynomial:

$$\begin{vmatrix} (-2\omega^2+4) & -1 & -1 \\ -1 & (-\omega^2+2) & 0 \\ -1 & 0 & (-\omega^2+2) \end{vmatrix} = 0$$

$$\text{i.e. } 2(-\omega^2+1)(-\omega^2+2)(-\omega^2+3) = 0$$

$$\therefore \Delta(\omega^2) = 2(-\omega^2+1)(-\omega^2+2)(-\omega^2+3)$$

(b) Plot of $\Delta(\omega^2)$:



(c) Roots of equation:

$$\left. \begin{matrix} \omega_1^2 = 1 \\ \omega_2^2 = 2 \\ \omega_3^2 = 3 \end{matrix} \right\} \text{from the graph.}$$

$$(6.61) \quad (a) \quad \vec{X}^{(1)} = \begin{Bmatrix} 0.2754946 \\ 0.3994672 \\ 0.4490562 \end{Bmatrix}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 0.6916979 \\ 0.2974301 \\ -0.3389320 \end{Bmatrix}, \quad [m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\vec{X}^{(1)T} [m] \vec{X}^{(2)} = (0.2754946 \quad 0.3994672 \quad 0.4490562) \begin{Bmatrix} 0.6916979 \\ 0.5948602 \\ -1.0167960 \end{Bmatrix} \approx 0$$

$$\text{Let } \vec{X}^{(3)} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\text{Then } \vec{X}^{(3)T} [m] \vec{X}^{(3)} = 1, \quad \vec{X}^{(1)T} [m] \vec{X}^{(3)} = 0, \quad \vec{X}^{(2)T} [m] \vec{X}^{(3)} = 0$$

These relations give

$$a_1^2 + 2a_2^2 + 3a_3^2 = 1 \quad \text{--- (E}_1\text{)}$$

$$0.2754946 a_1 + 0.7989344 a_2 + 1.3471686 a_3 = 0 \quad \text{--- (E}_2\text{)}$$

$$0.6916979 a_1 + 0.5948602 a_2 - 1.0167960 a_3 = 0 \quad \text{--- (E}_3\text{)}$$

From (E₂) and (E₃),

$$a_1 = -2.9000002 a_2 - 4.89 a_3 = -0.86 a_2 + 1.4700001 a_3$$

$$\text{or } a_2 = -3.1176468 a_3 \quad \text{--- (E}_4\text{)}$$

$$\text{and } a_1 = -2.9000002(-3.1176468 a_3) - 4.89 a_3 = 4.1511763 a_3 \quad \text{--- (E}_5\text{)}$$

(E₁), (E₄) and (E₅) give

$$a_3^2 (17.232265 + 9.7197216 + 1) = 1 \Rightarrow a_3 = \pm 0.1891445$$

$$\text{Hence } a_2 = \mp 0.5896857, \quad a_1 = \pm 0.7851722$$

$$\therefore \vec{X}^{(3)} = \begin{Bmatrix} 0.7851722 \\ -0.5896857 \\ 0.1891445 \end{Bmatrix}$$

$$(b) \quad \omega_i^2 = \vec{X}^{(i)T} [k] \vec{X}^{(i)} \quad ; \quad [k] = \begin{bmatrix} 6 & -4 & 0 \\ -4 & 10 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\omega_1^2 = (0.2754946 \quad 0.3994672 \quad 0.4490562) \begin{Bmatrix} 0.05509889 \\ 2.8926938 \\ 2.6943374 \end{Bmatrix} = 2.3806248$$

$$\omega_2^2 = (0.6916979 \quad 0.2974301 \quad -0.3389320) \begin{Bmatrix} 2.9604671 \\ 0.2075095 \\ -2.0335920 \end{Bmatrix} = 2.7987180$$

$$\omega_3^2 = (0.7851722 \quad -0.5896857 \quad 0.1891445) \begin{Bmatrix} 7.0697761 \\ -9.0375462 \\ 1.1348671 \end{Bmatrix} = 11.094957$$

$$\therefore \omega_1 = 1.5429274, \quad \omega_2 = 1.6729369, \quad \omega_3 = 3.3309095.$$

(6.62)

From solution of Problem 6.1, we find

$$[m] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad [k] = k \begin{bmatrix} 7 & -1 & -5 \\ -1 & 2 & -1 \\ -5 & -1 & 7 \end{bmatrix}$$

Frequency equation:

$$\left| -\omega^2 [m] + [k] \right| = 0$$

$$\text{or } \begin{vmatrix} (-\alpha + 7) & -1 & -5 \\ -1 & (-\alpha + 2) & -1 \\ -5 & -1 & (-\alpha + 7) \end{vmatrix} = 0$$

where $\alpha = \frac{\omega^2 m}{k}$. Expansion of the frequency equation gives:

$$(-\alpha + 7) \left\{ (-\alpha + 2)(-\alpha + 7) - 1 \right\} + 1 \left\{ -(-\alpha + 7) - 5 \right\} - 5 \left\{ 1 + 5(-\alpha + 2) \right\} = 0$$

$$\text{or } \alpha^3 - 16\alpha^2 + 50\alpha - 24 = 0$$

Roots of this equation give (using Program):

$$\alpha_1 = 0.58576 ; \omega_1 = 0.7653 \sqrt{\frac{k}{m}}$$

$$\alpha_2 = 4.41428 ; \omega_2 = 1.8478 \sqrt{\frac{k}{m}}$$

$$\alpha_3 = 12.0 ; \omega_3 = 3.4641 \sqrt{\frac{k}{m}}$$

6.63

From the solution of Problem 6.2, we obtain:

$$[m] = \begin{bmatrix} \frac{2m\ell^2}{3} & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} = \begin{bmatrix} 0.6667 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[k] = \begin{bmatrix} \frac{25k\ell^2}{8} & -\frac{k\ell}{2} & 0 \\ -\frac{k\ell}{2} & 3k & -k \\ 0 & -k & k \end{bmatrix} = \begin{bmatrix} 3125 & -500 & 0 \\ -500 & 3000 & -1000 \\ 0 & -1000 & 1000 \end{bmatrix}$$

Frequency equation:

$$\left| -\omega^2 [m] + [k] \right| = 0$$

$$\text{or } \begin{vmatrix} (3125 - 0.6667\omega^2) & -500 & 0 \\ -500 & (-2\omega^2 + 3000) & -1000 \\ 0 & -1000 & (1000 - \omega^2) \end{vmatrix} = 0$$

$$\text{or } (3125 - 0.6667 \omega^2) \left\{ (3000 - 2 \omega^2) (1000 - \omega^2) - 1000^2 \right\} + 500 \left\{ -500 (1000 - \omega^2) - 0 \right\} = 0$$

$$\text{or } -1.3334 \omega^6 + 9583.4 \omega^4 - 16.7083 (10^6) \omega^2 + 6.0 (10^9) = 0$$

Defining $\alpha = \left(\frac{\omega^2}{1000} \right)$, the above equation can be rewritten as

$$\alpha^3 - 7.1872 \alpha^2 + 12.5306 \alpha - 4.4998 = 0$$

Roots of this equation are (using Program):

$$\alpha_1 = 0.484831 ; \omega_1 = 22.0189 \text{ rad/sec}$$

$$\alpha_2 = 1.95501 ; \omega_2 = 44.2155 \text{ rad/sec}$$

$$\alpha_3 = 4.74738 ; \omega_3 = 68.9012 \text{ rad/sec}$$

6.64

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 3m \end{bmatrix}, \quad [k] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & (k_1 + k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

Equations of motion for harmonic motion:

$$\begin{bmatrix} (-m\omega^2 + k) & -k & 0 \\ -k & (-2m\omega^2 + 2k) & -k \\ 0 & -k & (-3m\omega^2 + k) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (\text{E.1})$$

Defining $\alpha = \frac{m\omega^2}{k}$, (E.1) can be rewritten as

$$\begin{bmatrix} (-\alpha + 1) & -1 & 0 \\ -1 & (-2\alpha + 2) & -1 \\ 0 & -1 & (-3\alpha + 1) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (\text{E.2})$$

Frequency equation is

$$2\alpha (3\alpha^2 - 7\alpha + 3) = 0$$

Roots are

$$\alpha_1 = 0 ; \omega_1 = 0$$

$$\alpha_2 = 0.565741 ; \omega_2 = 0.752158 \sqrt{k/m}$$

$$\alpha_3 = 1.767592 ; \omega_3 = 1.329508 \sqrt{k/m}$$

Eqs. (E.2) give

$$x_2^{(j)} = (-\alpha_j + 1) x_1^{(j)}, \quad x_3^{(j)} = \left(\frac{1}{-3\alpha_j + 1} \right) x_2^{(j)}$$

$$\therefore \vec{x}^{(j)} = \begin{Bmatrix} 1.0 \\ (-\alpha_j + 1) \\ \left(\frac{-\alpha_j + 1}{-3\alpha_j + 1} \right) \end{Bmatrix} x_1^{(j)}$$

Hence

$$\vec{x}^{(1)} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} x_1^{(1)}, \quad \vec{x}^{(2)} = \begin{Bmatrix} 1 \\ 0.434259 \\ -0.622841 \end{Bmatrix} x_1^{(2)}, \quad \vec{x}^{(3)} = \begin{Bmatrix} 1 \\ -0.767592 \\ 0.178395 \end{Bmatrix} x_1^{(3)}$$

6.65

$$[k_t] = k_t \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}, \quad [J] = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With $\alpha = \frac{\omega^2 J_0}{k_t}$, the equations of motion for harmonic motion become

$$\begin{bmatrix} (-\alpha+1) & -1 & 0 \\ -1 & (-\alpha+3) & -2 \\ 0 & -2 & (-\alpha+2) \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (E.1)$$

Frequency equation is $\alpha(\alpha^2 - 6\alpha + 6) = 0$

Roots are: $\alpha_1 = 0, \quad \alpha_2 = 1.267949, \quad \alpha_3 = 4.732051$

$$\omega_1 = 0, \quad \omega_2 = 1.126032 \sqrt{k_t/J_0}, \quad \omega_3 = 2.175328 \sqrt{k_t/J_0}$$

Eq. (E.1) gives

$$\Theta_2^{(j)} = (-\alpha_j + 1) \Theta_1^{(j)}, \quad \Theta_3^{(j)} = \frac{2}{(-\alpha_j + 2)} \Theta_1^{(j)}$$

$$\vec{\Theta}^{(j)} = \begin{Bmatrix} 1 \\ (-\alpha_j + 1) \\ \left(\frac{-2\alpha_j + 2}{-\alpha_j + 2} \right) \end{Bmatrix} \Theta_1^{(j)}$$

$$\therefore \vec{\Theta}^{(1)} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \Theta_1^{(1)}, \quad \vec{\Theta}^{(2)} = \begin{Bmatrix} 1 \\ -0.267949 \\ -0.732050 \end{Bmatrix} \Theta_1^{(2)}, \quad \vec{\Theta}^{(3)} = \begin{Bmatrix} 1 \\ -3.732051 \\ 2.732051 \end{Bmatrix} \Theta_1^{(3)}$$

Normalize the eigenvectors with respect to the inertia matrix as

$$\vec{\Theta}^{(1)T} [J] \vec{\Theta}^{(1)} = \Theta_1^{(1)2} J_0 (1 \ 1 \ 1) \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 3 J_0 \Theta_1^{(1)2} = 1$$

$$\Theta_1^{(1)} = 0.57735 / \sqrt{J_0}$$

$$\begin{aligned} \vec{\Theta}^{(2)T} [J] \vec{\Theta}^{(2)} &= \Theta_1^{(2)2} J_0 (1 \ -0.267949 \ -0.732050) \begin{Bmatrix} 1 \\ -0.267949 \\ -0.732050 \end{Bmatrix} \\ &= \Theta_1^{(2)2} J_0 (1.607694) = 1 \end{aligned}$$

$$\Theta_1^{(2)} = 0.788675 / \sqrt{J_0}$$

$$\begin{aligned} \vec{\Theta}^{(3)T} [J] \vec{\Theta}^{(3)} &= \Theta_1^{(3)2} J_0 (1 \ -3.732051 \ 2.732051) \begin{Bmatrix} 1 \\ -3.732051 \\ 2.732051 \end{Bmatrix} \\ &= \Theta_1^{(3)2} J_0 (22.392307) \end{aligned}$$

$$\Theta_1^{(3)} = 0.211325 / \sqrt{J_0}$$

Modal matrix is

$$[X] = \frac{1}{\sqrt{J_0}} \begin{bmatrix} 0.57735 & 0.788675 & 0.211325 \\ 0.57735 & -0.211325 & -0.788676 \\ 0.57735 & -0.577349 & 0.577351 \end{bmatrix}$$

6.66

From solution of Problem 6.52, the natural frequencies and mode shapes are given by:

$$\omega_1 = 0.482087 \sqrt{\frac{k}{m}} ; \omega_2 = \sqrt{\frac{k}{m}} ; \omega_3 = 1.197605 \sqrt{\frac{k}{m}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.535184 \\ 1.0 \end{Bmatrix} ; \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} ; \vec{X}^{(3)} = \begin{Bmatrix} 1.0 \\ 0.868516 \\ -1 \end{Bmatrix}$$

Initial conditions:

$$x_1(0) = x_{10}, x_2(0) = 0, x_3(0) = 0, \dot{x}_i(0) = 0 ; i = 1, 2, 3$$

Equations (6.98) and (6.99) yield:

$$A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3 = x_{10} \quad (1)$$

$$1.5352 A_1 \cos \phi_1 + 0.8685 A_3 \cos \phi_3 = 0 \quad (2)$$

$$A_1 \cos \phi_1 - A_2 \cos \phi_2 - A_3 \cos \phi_3 = 0 \quad (3)$$

$$-0.4821 \sqrt{\frac{k}{m}} A_1 \sin \phi_1 - \sqrt{\frac{k}{m}} A_2 \sin \phi_2 - 1.1976 \sqrt{\frac{k}{m}} A_3 \sin \phi_3 = 0 \quad (4)$$

$$-0.4821 \sqrt{\frac{k}{m}} (1.5352) A_1 \sin \phi_1 - 1.1976 \sqrt{\frac{k}{m}} (0.8685) A_3 \sin \phi_3 = 0 \quad (5)$$

$$-0.4821 \sqrt{\frac{k}{m}} (1.0) A_1 \sin \phi_1 - \sqrt{\frac{k}{m}} (-1) A_2 \sin \phi_2 - 1.1976 \sqrt{\frac{k}{m}} (-1) A_3 \sin \phi_3 = 0 \quad (6)$$

Solution of Eqs. (4) to (6):

$$\phi_i = 0 ; i = 1, 2, 3$$

Solution of Eqs. (1) to (3) gives:

$$A_1 = 0.5 x_{10} ; A_2 = 0.3838 x_{10} ; A_3 = -0.8838 x_{10}$$

Thus the free vibration solution of the system is given by:

$$\begin{aligned} x_1(t) &= x_{10} (0.5 \cos 0.4821 \sqrt{\frac{k}{m}} t + 0.3838 \cos \sqrt{\frac{k}{m}} t \\ &\quad - 0.8838 \cos 1.1976 \sqrt{\frac{k}{m}} t) \\ x_2(t) &= x_{10} \left\{ 0.7676 \cos 0.4821 \sqrt{\frac{k}{m}} t - 0.7676 \cos 1.1976 \sqrt{\frac{k}{m}} t \right\} \\ x_3(t) &= x_{10} (0.5 \cos 0.4821 \sqrt{\frac{k}{m}} t - 0.3838 \cos \sqrt{\frac{k}{m}} t \\ &\quad + 0.8838 \cos 1.1976 \sqrt{\frac{k}{m}} t) \end{aligned}$$

From solution of Problem 6.53, we find that:

$$\omega_1 = 0.6448 \sqrt{\frac{g}{\ell}} ; \omega_2 = 1.5147 \sqrt{\frac{g}{\ell}} ; \omega_3 = 2.5080 \sqrt{\frac{g}{\ell}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.2922 \\ 1.6312 \end{Bmatrix} ; \vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ 0.3527 \\ -2.3978 \end{Bmatrix} ; \vec{X}^{(3)} = \begin{Bmatrix} 1.0 \\ -1.6450 \\ 0.7669 \end{Bmatrix}$$

Equations (6.98) and (6.99) can be written, for the stated initial conditions, as:

$$A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3 = 0 \quad (1)$$

$$1.2922 A_1 \cos \phi_1 + 0.3527 A_2 \cos \phi_2 - 1.6450 A_3 \cos \phi_3 = 0 \quad (2)$$

$$1.6312 A_1 \cos \phi_1 - 2.3978 A_2 \cos \phi_2 + 0.7669 A_3 \cos \phi_3 = \theta_{30} \quad (3)$$

$$\begin{aligned} & -0.6448 \sqrt{\frac{g}{\ell}} A_1 \sin \phi_1 - 1.5147 \sqrt{\frac{g}{\ell}} A_2 \sin \phi_2 \\ & - 2.5080 \sqrt{\frac{g}{\ell}} A_3 \sin \phi_3 = 0 \end{aligned} \quad (4)$$

$$\begin{aligned} & -0.6448 \sqrt{\frac{g}{\ell}} (1.2922) A_1 \sin \phi_1 - 1.5147 \sqrt{\frac{g}{\ell}} (0.3527) A_2 \sin \phi_2 \\ & - 2.5080 \sqrt{\frac{g}{\ell}} (-1.6450) A_3 \sin \phi_3 = 0 \end{aligned} \quad (5)$$

$$\begin{aligned} & -0.6448 \sqrt{\frac{g}{\ell}} (1.6312) A_1 \sin \phi_1 - 1.5147 \sqrt{\frac{g}{\ell}} (-2.3978) A_2 \sin \phi_2 \\ & - 2.5080 \sqrt{\frac{g}{\ell}} (0.7669) A_3 \sin \phi_3 = 0 \end{aligned} \quad (6)$$

Equations (4) to (6) yield:

$$\phi_i = 0 ; i = 1, 2, 3$$

Equations (1) to (3) give (solution found using Program 12):

$$A_1 = 0.1812 x_{30} ; A_2 = -0.2665 x_{30} ; A_3 = 0.08524 x_{30}$$

Thus the free vibration solution can be expressed as (see Eq. (6.96)):

$$\begin{aligned} x_1(t) = x_{30} & (0.1812 \cos 0.6448 \sqrt{\frac{g}{\ell}} t - 0.2665 \cos 1.5147 \sqrt{\frac{g}{\ell}} t \\ & + 0.08524 \cos 2.5080 \sqrt{\frac{g}{\ell}} t) \end{aligned} \quad (7)$$

$$\begin{aligned} x_2(t) = x_{30} & (0.2341 \cos 0.6448 \sqrt{\frac{g}{\ell}} t - 0.09399 \cos 1.5147 \sqrt{\frac{g}{\ell}} t \\ & - 0.1402 \cos 2.5080 \sqrt{\frac{g}{\ell}} t) \end{aligned} \quad (8)$$

$$\begin{aligned} x_3(t) = x_{30} & (0.2956 \cos 0.6448 \sqrt{\frac{g}{\ell}} t + 0.6390 \cos 1.5147 \sqrt{\frac{g}{\ell}} t \\ & + 0.06537 \cos 2.5080 \sqrt{\frac{g}{\ell}} t) \end{aligned} \quad (9)$$

From solution of Problem 6.56, we obtain

$$\begin{aligned}\omega_1 &= 0.5626 \sqrt{\frac{P}{\ell m}} ; \omega_2 = 0.9158 \sqrt{\frac{P}{\ell m}} ; \omega_3 = 1.5848 \sqrt{\frac{P}{\ell m}} \\ \alpha_1 &= \frac{\omega_1^2 \ell m}{4 P} = 0.079126 ; \alpha_2 = \frac{\omega_2^2 \ell m}{4 P} = 0.209671 ; \alpha_3 = \frac{\omega_3^2 \ell m}{4 P} = 0.627872\end{aligned}$$

The mode shapes can be determined from the equations:

$$(6 \alpha - 1) X_1 + 2 \alpha X_2 + 3 \alpha X_3 = 0 \quad (1)$$

$$4 \alpha X_1 + (4 \alpha - 1) X_2 + 6 \alpha X_3 = 0 \quad (2)$$

$$2 \alpha X_1 + 2 \alpha X_2 + (9 \alpha - 1) X_3 = 0 \quad (3)$$

$$\text{Let } X_1 = 1 \quad (4)$$

Then Eqs. (1) and (2) yield

$$X_2 = 2 - 8 \alpha \quad (5)$$

$$X_3 = \frac{1 - 10 \alpha + 16 \alpha^2}{3 \alpha} \quad (6)$$

Using the values of α_1 , α_2 and α_3 , we obtain, from Eqs. (4) to (6):

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.3670 \\ 1.3014 \end{Bmatrix} ; \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0.3226 \\ -0.6253 \end{Bmatrix} ; \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -3.0230 \\ 0.5462 \end{Bmatrix}$$

The stated initial conditions give, using Eqs. (6.98) and (6.99),

$$A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3 = 0 \quad (7)$$

$$1.3670 A_1 \cos \phi_1 + 0.3226 A_2 \cos \phi_2 - 3.0230 A_3 \cos \phi_3 = x_{20} \quad (8)$$

$$1.3014 A_1 \cos \phi_1 - 0.6253 A_2 \cos \phi_2 + 0.5462 A_3 \cos \phi_3 = 0 \quad (9)$$

$$\begin{aligned}0.5626 \sqrt{\frac{P}{\ell m}} A_1 \sin \phi_1 + 0.9158 \sqrt{\frac{P}{\ell m}} A_2 \sin \phi_2 \\ + 1.5848 \sqrt{\frac{P}{\ell m}} A_3 \sin \phi_3 = 0\end{aligned} \quad (10)$$

$$\begin{aligned}0.5626 \sqrt{\frac{P}{\ell m}} (1.3670) A_1 \sin \phi_1 + 0.9158 \sqrt{\frac{P}{\ell m}} (0.3226) A_2 \sin \phi_2 \\ - 1.5848 \sqrt{\frac{P}{\ell m}} (3.0230) A_3 \sin \phi_3 = 0\end{aligned} \quad (11)$$

$$\begin{aligned}0.5626 \sqrt{\frac{P}{\ell m}} (1.3014) A_1 \sin \phi_1 - 0.9158 \sqrt{\frac{P}{\ell m}} (0.6253) A_2 \sin \phi_2 \\ + 1.5848 \sqrt{\frac{P}{\ell m}} (0.5462) A_3 \sin \phi_3 = 0\end{aligned} \quad (12)$$

Equations (10) to (12) yield:

$$\phi_i = 0 ; i = 1, 2, 3$$

Equations (7) to (9) give (using Program 12):

$$A_1 = 0.1527 x_{20} ; A_2 = 0.09847 x_{20} ; A_3 = -0.2512 x_{20}$$

$$x_1(t) = x_{20} (0.1527 \cos 0.5626 \sqrt{\frac{P}{\ell m}} t + 0.09847 \cos 0.9158 \sqrt{\frac{P}{\ell m}} t - 0.2512 \cos 1.5848 \sqrt{\frac{P}{\ell m}} t) \quad (13)$$

$$x_2(t) = x_{20} (0.2087 \cos 0.5626 \sqrt{\frac{P}{\ell m}} t + 0.03177 \cos 0.9158 \sqrt{\frac{P}{\ell m}} t + 0.7594 \cos 1.5848 \sqrt{\frac{P}{\ell m}} t) \quad (14)$$

$$x_3(t) = x_{20} (0.1987 \cos 0.5626 \sqrt{\frac{P}{\ell m}} t - 0.06157 \cos 0.9158 \sqrt{\frac{P}{\ell m}} t - 0.1372 \cos 1.5848 \sqrt{\frac{P}{\ell m}} t) \quad (15)$$

6.69

From solution of Problem 6.46, we obtain

$$\omega_1 = 0.3376 \sqrt{\frac{k}{m}} ; \omega_2 = 1.4142 \sqrt{\frac{k}{m}} ; \omega_3 = 2.0943 \sqrt{\frac{k}{m}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.4430 \\ 1.6286 \end{Bmatrix} ; \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0.5 \\ -0.5 \end{Bmatrix} ; \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -0.693 \\ 0.2047 \end{Bmatrix}$$

Initial conditions:

$$x_i(0) = 0 ; i = 1, 2, 3 ; \dot{x}_1(0) = \dot{x}_{10} , \dot{x}_2(0) = \dot{x}_3(0) = 0$$

Equations (6.98) and (6.99) can be expressed as:

$$A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3 = 0 \quad (1)$$

$$1.4430 A_1 \cos \phi_1 + 0.5 A_2 \cos \phi_2 - 0.6930 A_3 \cos \phi_3 = 0 \quad (2)$$

$$1.6286 A_1 \cos \phi_1 - 0.5 A_2 \cos \phi_2 + 0.2047 A_3 \cos \phi_3 = 0 \quad (3)$$

$$-0.3376 \sqrt{\frac{k}{m}} A_1 \sin \phi_1 - 1.4142 \sqrt{\frac{k}{m}} A_2 \sin \phi_2 - 2.0943 \sqrt{\frac{k}{m}} A_3 \sin \phi_3 = \dot{x}_{10} \quad (4)$$

$$-0.4872 \sqrt{\frac{k}{m}} A_1 \sin \phi_1 - 0.7071 \sqrt{\frac{k}{m}} A_2 \sin \phi_2 + 1.4513 \sqrt{\frac{k}{m}} A_3 \sin \phi_3 = 0 \quad (5)$$

$$-0.5498 \sqrt{\frac{k}{m}} A_1 \sin \phi_1 + 0.7071 \sqrt{\frac{k}{m}} A_2 \sin \phi_2 - 0.4287 \sqrt{\frac{k}{m}} A_3 \sin \phi_3 = 0 \quad (6)$$

Equations (1) to (3) give:

$$\phi_i = \frac{\pi}{2} ; i = 1, 2, 3$$

Treating $\sqrt{\frac{k}{m}} A_i \sin \phi_i$ ($i = 1, 2, 3$) as unknowns and noting that all $\phi_i = \frac{\pi}{2}$, Eqs. (4) to (6) can be solved to obtain (using Program 12):

$$\begin{aligned}\sqrt{\frac{k}{m}} A_1 &= -0.2257 \dot{x}_{10} ; A_1 = -0.2257 \sqrt{\frac{m}{k}} \dot{x}_{10} \\ \sqrt{\frac{k}{m}} A_2 &= -0.3143 \dot{x}_{10} ; A_2 = -0.3143 \sqrt{\frac{m}{k}} \dot{x}_{10} \\ \sqrt{\frac{k}{m}} A_3 &= -0.2289 \dot{x}_{10} ; A_3 = -0.2289 \sqrt{\frac{m}{k}} \dot{x}_{10}\end{aligned}$$

The free vibration solution of the system can be expressed, using Eq. (6.96), as

$$\begin{aligned}x_1(t) = \dot{x}_{10} \sqrt{\frac{m}{k}} & \left(-0.2257 \cos \left(0.3376 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) - 0.3143 \cos \left(1.4142 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) \right. \\ & \left. - 0.2289 \cos \left(2.0943 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) \right) \quad (7)\end{aligned}$$

$$\begin{aligned}x_2(t) = \dot{x}_{10} \sqrt{\frac{m}{k}} & \left(-0.3257 \cos \left(0.3376 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) - 0.1571 \cos \left(1.4142 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) \right. \\ & \left. + 0.1586 \cos \left(2.0943 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) \right) \quad (8)\end{aligned}$$

$$\begin{aligned}x_3(t) = \dot{x}_{10} \sqrt{\frac{m}{k}} & \left(-0.3676 \cos \left(0.3376 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) + 0.1571 \cos \left(1.4142 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) \right. \\ & \left. - 0.0469 \cos \left(2.0943 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) \right) \quad (9)\end{aligned}$$

6.70

From solution of Problem 6.54, we find:

$$\omega_1 = 0.3863 \sqrt{\frac{k}{m}} ; \omega_2 = 1.4142 \sqrt{\frac{k}{m}} ; \omega_3 = 1.8305 \sqrt{\frac{k}{m}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.8508 \\ 2 \end{Bmatrix} ; \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -0.5 \end{Bmatrix} ; \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -1.3508 \\ 2 \end{Bmatrix}$$

Initial conditions:

$$x_i(0) = 0, i = 1, 2, 3 ; \dot{x}_1(0) = \dot{x}_2(0) = 0, \dot{x}_3(0) = \dot{x}_{30}$$

Equations (6.98) and (6.99) can be expressed as:

$$A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3 = 0 \quad (1)$$

$$1.8508 A_1 \cos \phi_1 - 1.3508 A_3 \cos \phi_3 = 0 \quad (2)$$

$$2 A_1 \cos \phi_1 - 0.5 A_2 \cos \phi_2 + 2 A_3 \cos \phi_3 = 0 \quad (3)$$

$$\begin{aligned}-0.3863 \sqrt{\frac{k}{m}} A_1 \sin \phi_1 & - 1.4142 \sqrt{\frac{k}{m}} A_2 \sin \phi_2 \\ & - 1.8305 \sqrt{\frac{k}{m}} A_3 \sin \phi_3 = 0 \quad (4)\end{aligned}$$

$$-0.7150 \sqrt{\frac{k}{m}} A_1 \sin \phi_1 + 2.4726 \sqrt{\frac{k}{m}} A_3 \sin \phi_3 = 0 \quad (5)$$

$$\begin{aligned} -0.7726 \sqrt{\frac{k}{m}} A_1 \sin \phi_1 + 0.7071 \sqrt{\frac{k}{m}} A_2 \sin \phi_2 \\ - 3.6610 \sqrt{\frac{k}{m}} A_3 \sin \phi_3 = \dot{x}_{30} \end{aligned} \quad (6)$$

Equations (1) to (3) yield:

$$\phi_i = \frac{\pi}{2} ; i = 1, 2, 3$$

and $\sqrt{\frac{k}{m}} A_i \sin \phi_i$ ($i = 1, 2, 3$) are the unknowns in Eqs. (4) to (6). The solution of Eqs. (4) to (6) gives, using Program 12 with $\phi_i = \frac{\pi}{2}$:

$$\begin{aligned} \sqrt{\frac{k}{m}} A_1 &= -0.4369 \dot{x}_{30} ; A_1 = -0.4369 \dot{x}_{30} \sqrt{\frac{m}{k}} \\ \sqrt{\frac{k}{m}} A_2 &= 0.2828 \dot{x}_{30} ; A_2 = 0.2828 \dot{x}_{30} \sqrt{\frac{m}{k}} \\ \sqrt{\frac{k}{m}} A_3 &= -0.1263 \dot{x}_{30} ; A_3 = -0.1263 \dot{x}_{30} \sqrt{\frac{m}{k}} \end{aligned}$$

Thus the free vibration response of the system can be expressed as (see Eq. (6.96)):

$$\begin{aligned} x_1(t) &= \dot{x}_{30} \sqrt{\frac{m}{k}} \left(-0.4369 \cos \left(0.3863 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) + 0.2828 \cos \left(1.4142 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) \right. \\ &\quad \left. - 0.1263 \cos \left(1.8305 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) \right) \\ x_2(t) &= \dot{x}_{30} \sqrt{\frac{m}{k}} \left\{ -0.8086 \cos \left(0.3863 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) - 0.1706 \cos \left(1.8305 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) \right\} \\ x_3(t) &= \dot{x}_{30} \sqrt{\frac{m}{k}} \left(-0.8738 \cos \left(0.3863 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) - 0.1414 \cos \left(1.4142 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) \right. \\ &\quad \left. - 0.2526 \cos \left(1.8305 \sqrt{\frac{k}{m}} t + \frac{\pi}{2} \right) \right) \end{aligned}$$

6.71 From Example 6.13, we obtain:

$$[m] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; [k] = k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

The natural frequencies and mode shapes are given by:

$$\omega_1 = 0 ; \omega_2 = \sqrt{\frac{k}{m}} ; \omega_3 = \sqrt{\frac{3k}{m}}$$

$$\vec{X}^{(1)} = a \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} ; \vec{X}^{(2)} = b \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} ; \vec{X}^{(3)} = c \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$$

Orthonormalization of mode shapes:

$$\vec{X}^{(1)T} [m] \vec{X}^{(1)} = a^2 (1 \ 1 \ 1) m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 3 a^2 m = 1 \quad \text{or} \quad a = \sqrt{\frac{1}{3m}}$$

$$\vec{X}^{(2)T} [m] \vec{X}^{(2)} = b^2 m (1 \ 0 \ -1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = 2 b^2 m = 1 \quad \text{or} \quad b = \sqrt{\frac{1}{2m}}$$

$$\vec{X}^{(3)T} [m] \vec{X}^{(3)} = c^2 m (1 \ -2 \ 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix} = 6 c^2 m = 1 \quad \text{or} \quad c = \sqrt{\frac{1}{6m}}$$

$$\text{Modal matrix: } [X] = [\vec{X}^{(1)} \ \vec{X}^{(2)} \ \vec{X}^{(3)}] = \frac{1}{\sqrt{m}} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \quad (1)$$

Solution is given by Eq. (6.104):

$$\vec{x}(t) = [X] \vec{q}(t) \quad (2)$$

where $\vec{q}(t)$ is given by Eqs. (6.113):

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = Q_i(t) ; \quad i = 1, 2, 3 \quad (3)$$

where

$$\vec{Q}(t) = [X]^T \vec{F}(t) = \vec{0} \quad (\text{no external forces})$$

For the rigid body mode, $\omega_1^2 = 0$ and Eq. (3) reduces to:

$$\ddot{q}_1(t) = 0 \quad (4)$$

whose solution can be written as

$$q_1(t) = c_1 + c_2 t \quad (5)$$

where c_1 and c_2 are constants given by

$$c_1 = q_1(t=0) ; \quad c_2 = \dot{q}_1(t=0) \quad (6)$$

Initial conditions of the problem:

$$\vec{x}(0) = \begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (7)$$

$$\text{and } \dot{\vec{x}}(0) = \begin{Bmatrix} \dot{x}_0 \\ 0 \\ 0 \end{Bmatrix} \quad (8)$$

Using Eqs. (6.115) and (6.116), we find

$$\vec{q}(0) = [X]^T [m] \vec{x}(0) = \vec{0} \quad (9)$$

$$\dot{\vec{q}}(0) = [X]^T [m] \dot{\vec{x}}(0) = \sqrt{m} \dot{x}_0 \begin{Bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{Bmatrix} \quad (10)$$

From Eqs. (6), (9) and (10), we obtain

$$c_1 = q_1(0) = 0 ; \quad c_2 = \dot{q}_1(0) = \sqrt{\frac{m}{3}} \dot{x}_0 \quad (11)$$

and hence, from Eq. (5), we find

$$q_1(t) = \sqrt{\frac{m}{3}} \dot{x}_0 t \quad (12)$$

Solution of Eqs. (3) for $q_2(t)$ and $q_3(t)$ can be expressed as

$$q_2(t) = q_2(0) \cos \omega_2 t + \frac{\dot{q}_2(0)}{\omega_2} \sin \omega_2 t = \frac{m \dot{x}_0}{\sqrt{2} k} \sin \sqrt{\frac{k}{m}} t \quad (13)$$

$$q_3(t) = q_3(0) \cos \omega_3 t + \frac{\dot{q}_3(0)}{\omega_3} \sin \omega_3 t = \frac{m \dot{x}_0}{3 \sqrt{2} k} \sin \sqrt{\frac{3 k}{m}} t \quad (14)$$

The free vibration of the system is given by Eq. (2):

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{Bmatrix} \sqrt{\frac{m}{3}} \dot{x}_0 t \\ \frac{m \dot{x}_0}{\sqrt{2} k} \sin \sqrt{\frac{k}{m}} t \\ \frac{m \dot{x}_0}{3 \sqrt{2} k} \sin \sqrt{\frac{3 k}{m}} t \end{Bmatrix} \quad (15)$$

6.72

Equations of motion are

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 3k & -k & -k \\ -k & k & 0 \\ -k & 0 & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F(t) \end{Bmatrix}$$

where $m = 1 \text{ kg}$, $k = 1000 \text{ N/m}$, $F(t) = 5 \sin 10t \text{ N}$.Eigenvalue analysis:

Frequency equation is

$$\left| -\omega^2 m \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + k \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\text{or } \begin{vmatrix} -2\lambda + 3 & -1 & -1 \\ -1 & -\lambda + 1 & 0 \\ -1 & 0 & -\lambda + 1 \end{vmatrix} = 0 \quad \text{where } \lambda = \frac{\omega^2 m}{k}$$

$$\text{or } 2\lambda^3 - 7\lambda^2 + 6\lambda - 1 = 0$$

Roots are

$$\lambda_1 = 0.219220,$$

$$\lambda_2 = 1.0,$$

$$\lambda_3 = 2.28078$$

$$\omega_1 = 0.4682094 \sqrt{\frac{k}{m}}, \quad \omega_2 = 1.0 \sqrt{\frac{k}{m}},$$

$$\omega_3 = 1.5102251 \sqrt{\frac{k}{m}}$$

Mode shapes are given by

$$\begin{bmatrix} -2\lambda_i + 3 & -1 & -1 \\ -1 & -\lambda_i + 1 & 0 \\ -1 & 0 & -\lambda_i + 1 \end{bmatrix} \begin{Bmatrix} x_1^{(i)} \\ x_2^{(i)} \\ x_3^{(i)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{or } x_1^{(i)} = (-\lambda_i + 1) x_2^{(i)}, \quad x_3^{(i)} = (-2\lambda_i + 3) x_1^{(i)} - x_2^{(i)}$$

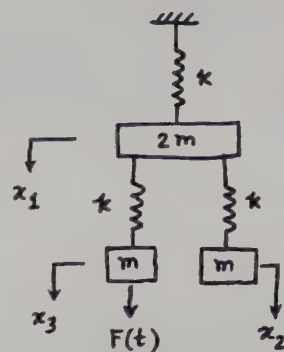
$$\vec{x}^{(i)} = \begin{Bmatrix} -\lambda_i + 1 \\ 1 \\ (-2\lambda_i + 3)(-\lambda_i + 1) - 1 \end{Bmatrix} x_2^{(i)}$$

$$\vec{x}^{(1)} = \begin{Bmatrix} 0.78078 \\ 1 \\ 1 \end{Bmatrix} x_2^{(1)}, \quad \vec{x}^{(2)} = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} x_2^{(2)}, \quad \vec{x}^{(3)} = \begin{Bmatrix} -1.28078 \\ 1 \\ 1 \end{Bmatrix} x_2^{(3)}$$

Normalization of mode shapes with respect to $[m]$:

$$\vec{x}^{(1)T} [m] \vec{x}^{(1)} = (0.78078 \quad 1 \quad 1) \begin{Bmatrix} 1.56156 \\ 1 \\ 1 \end{Bmatrix} (x_2^{(1)})^2 = 3.21923 (x_2^{(1)})^2 = 1$$

$$x_2^{(1)} = 0.55734; \quad \vec{x}^{(1)} = \begin{Bmatrix} 0.43516 \\ 0.55734 \\ 0.55734 \end{Bmatrix}$$



$$\vec{x}^{(2)T} [m] \vec{x}^{(2)} = (0 \quad 1 \quad -1) \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} (x_2^{(2)})^2 = 2 (x_2^{(2)})^2 = 1$$

$$x_2^{(2)} = 0.70711; \quad \vec{x}^{(2)} = \begin{Bmatrix} 0 \\ 0.70711 \\ -0.70711 \end{Bmatrix}$$

$$\vec{x}^{(3)T} [m] \vec{x}^{(3)} = (-1.28078 \quad 1 \quad 1) \begin{Bmatrix} -2.56156 \\ 1 \\ 1 \end{Bmatrix} (x_2^{(3)})^2 = 5.28079 (x_2^{(3)})^2 = 1$$

$$x_2^{(3)} = 0.43516; \quad \vec{x}^{(3)} = \begin{Bmatrix} -0.55734 \\ 0.43516 \\ 0.43516 \end{Bmatrix}$$

$\vec{Q} = [X]^T \vec{F}(t)$ = vector of generalized forces

$$= \begin{bmatrix} 0.43516 & 0.55734 & 0.55734 \\ 0 & 0.70711 & -0.70711 \\ -0.55734 & 0.43516 & 0.43516 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ F_0 \sin \omega t \end{Bmatrix} = \begin{Bmatrix} 2.7867 \\ -3.5356 \\ 2.1758 \end{Bmatrix} \sin 10t$$

Uncoupled equations of motion are

$$\left. \begin{aligned} \ddot{q}_1 + 219.22 q_1 &= 2.7867 \sin 10t \\ \ddot{q}_2 + 1000.00 q_2 &= -3.5356 \sin 10t \\ \ddot{q}_3 + 2280.78 q_3 &= 2.1758 \sin 10t \end{aligned} \right\} \quad (E.1)$$

Particular solutions of (E.1) are

$$q_1(t) = \left(\frac{2.7867}{219.22 - 100} \right) \sin 10t = 0.0233744 \sin 10t$$

$$q_2(t) = \left(\frac{-3.5356}{1000 - 100} \right) \sin 10t = -0.0039284 \sin 10t$$

$$q_3(t) = \left(\frac{2.1758}{2280.78 - 100} \right) \sin 10t = 0.0009977 \sin 10t$$

Since $\vec{x} = [X] \vec{q} = \begin{bmatrix} 0.43516 & 0 & -0.55734 \\ 0.55734 & 0.70711 & 0.43516 \\ 0.55734 & -0.70711 & 0.43516 \end{bmatrix} \vec{q}$, we get

$$x_1(t) = 0.0096155 \sin 10t \quad m$$

$$x_2(t) = 0.0095333 \sin 10t \quad m$$

$$x_3(t) = 0.0162395 \sin 10t \quad m.$$

6.73 (a) From Example 6.8, for $k_{t1} = k_{t2} = k_{t3} = k_t$ and $J_1 = J_2 = J_3 = J_0$,

$$[m] = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [k] = k_t \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Eigenvalue problem becomes

$$\begin{bmatrix} -\lambda + 2 & -1 & 0 \\ -1 & -\lambda + 2 & -1 \\ 0 & -1 & -\lambda + 1 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{where } \lambda = \frac{\omega^2 J_0}{k_t}.$$

--- (E.1)

Frequency equation is

$$\lambda^3 - 5\lambda^2 + 6\lambda - 1 = 0$$

Roots are

$$\lambda_1 = 0.198,$$

$$\lambda_2 = 1.555,$$

$$\lambda_3 = 3.247$$

$$\omega_1 = 0.44497 \sqrt{\frac{k_t}{J_0}}, \quad \omega_2 = 1.24700 \sqrt{\frac{k_t}{J_0}}, \quad \omega_3 = 1.80194 \sqrt{\frac{k_t}{J_0}}$$

$$(E.1) \text{ gives } \Phi_2^{(j)} = (-\lambda_j + 2) \Phi_1^{(j)}, \quad \Phi_3^{(j)} = \left(\frac{-\lambda_j + 2}{-\lambda_j + 1} \right) \Phi_1^{(j)}$$

$$\vec{\Phi}^{(j)} = \begin{Bmatrix} 1 \\ -\lambda_j + 2 \\ \left(\frac{-\lambda_j + 2}{-\lambda_j + 1} \right) \end{Bmatrix} \Phi_1^{(j)}$$

$$\vec{\Phi}^{(1)} = \begin{Bmatrix} 1 \\ 1.802 \\ 2.247 \end{Bmatrix} \Phi_1^{(1)}, \quad \vec{\Phi}^{(2)} = \begin{Bmatrix} 1 \\ 0.445 \\ -0.802 \end{Bmatrix} \Phi_1^{(2)}, \quad \vec{\Phi}^{(3)} = \begin{Bmatrix} 1 \\ -1.247 \\ 0.555 \end{Bmatrix} \Phi_1^{(3)}$$

Normalization:

$$\vec{\Phi}^{(1)T} [m] \vec{\Phi}^{(1)} = (1 \quad 1.802 \quad 2.247) \begin{Bmatrix} 1 \\ 1.802 \\ 2.247 \end{Bmatrix} J_0 \left(\Phi_1^{(1)} \right)^2 = 9.2962 \left(\Phi_1^{(1)} \right)^2 J_0 = 1$$

$$\Phi_1^{(1)} = 0.328 / \sqrt{J_0}$$

$$\vec{\Phi}^{(2)T} [m] \vec{\Phi}^{(2)} = (1 \quad 0.445 \quad -0.802) \begin{Bmatrix} 1 \\ 0.445 \\ -0.802 \end{Bmatrix} J_0 \left(\Phi_1^{(2)} \right)^2 = 1.8412 \left(\Phi_1^{(2)} \right)^2 J_0 = 1$$

$$\Phi_1^{(2)} = 0.737 / \sqrt{J_0}$$

$$\vec{\Phi}^{(3)T} [m] \vec{\Phi}^{(3)} = (1 \quad -1.247 \quad 0.555) \begin{Bmatrix} 1 \\ -1.247 \\ 0.555 \end{Bmatrix} J_0 \left(\Phi_1^{(3)} \right)^2 = 2.863 \left(\Phi_1^{(3)} \right)^2 J_0 = 1$$

$$\Phi_1^{(3)} = 0.591 / \sqrt{J_0}$$

[m] - orthonormal modal matrix is

$$[X] = \frac{1}{\sqrt{J_0}} \begin{bmatrix} 0.328 & 0.737 & 0.591 \\ 0.591 & 0.328 & -0.737 \\ 0.737 & -0.591 & 0.328 \end{bmatrix}$$

For given data,

$$\omega_1 = 4.4497 \text{ rad/s}, \quad \omega_2 = 12.4700 \text{ rad/s}, \quad \omega_3 = 18.0194 \text{ rad/s}$$

$$[X] = \begin{bmatrix} 0.328 & 0.737 & 0.591 \\ 0.591 & 0.328 & -0.737 \\ 0.737 & -0.591 & 0.328 \end{bmatrix}$$

(b) According to modal analysis, uncoupled equations of motion are

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = Q_i(t); \quad i = 1, 2, 3 \quad (E.2)$$

where $\vec{Q}(t) = [X]^T \vec{M}_t(t) = [X]^T \begin{Bmatrix} 0 \\ 0 \\ m_{t0} \cos \omega t \end{Bmatrix}$

$$= \begin{Bmatrix} Q_{10} \\ Q_{20} \\ Q_{30} \end{Bmatrix} \cos 100t \equiv \begin{Bmatrix} 368.5 \\ -295.5 \\ 164.0 \end{Bmatrix} \cos 100t$$

Steady state solution of (E-2) is $q_i(t) = \left(\frac{Q_{i0}}{\omega_i^2 - \omega^2} \right) \cos \omega t$

$$\therefore q_1(t) = -0.03692 \cos 100t$$

$$q_2(t) = 0.03002 \cos 100t$$

$$q_3(t) = -0.01695 \cos 100t$$

Angular deflections are

$$\vec{\Theta}(t) = [X] \vec{q}(t) = \begin{Bmatrix} -0.0000025 \\ 0.0005190 \\ -0.0505115 \end{Bmatrix} \cos 100t \quad \text{radian}$$

6.74

From problem 6.51, $\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} X_1^{(1)}$, $\vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} X_1^{(2)}$, $\vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix} X_1^{(3)}$

$$[m] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Normalization gives

$$\vec{X}^{(1)T} [m] \vec{X}^{(1)} = (1 \quad \sqrt{2} \quad 1) \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} (X_1^{(1)})^2 m = 1 \Rightarrow X_1^{(1)} = \frac{1}{2\sqrt{m}}$$

$$\vec{X}^{(2)T} [m] \vec{X}^{(2)} = (1 \quad 0 \quad -1) \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} (X_1^{(2)})^2 m = 1 \Rightarrow X_1^{(2)} = \frac{1}{\sqrt{2m}}$$

$$\vec{X}^{(3)T} [m] \vec{X}^{(3)} = (1 \quad -\sqrt{2} \quad 1) \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix} (X_1^{(3)})^2 m = 1 \Rightarrow X_1^{(3)} = \frac{1}{2\sqrt{m}}$$

Modal matrix is

$$[X] = \frac{1}{\sqrt{m}} \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{bmatrix}$$

original coupled equations are $[m] \ddot{\vec{x}} + [k] \vec{x} = \vec{0}$

uncoupled equations with $\vec{x} = [X] \vec{q}$ are

$$\ddot{\vec{q}} + [\omega^2] \vec{q} = \vec{0}$$

$$\text{or } \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{with } \omega_1^2 = 0.585786 \frac{k}{m}, \quad \omega_2^2 = 2 \frac{k}{m}, \quad \omega_3^2 = 3.414214 \frac{k}{m}.$$

$$6.75 \quad [k]^{-1} \vec{F}(t) = \frac{1}{k} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} F_0 \cos \omega t = \frac{1}{k} \begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} F_0 \cos \omega t$$

From Example 6.9,

$$\begin{aligned} \varrho_{10} &= \frac{Q_{10}}{\omega_1^2} \frac{1}{\left| 1 - \left(\frac{\omega}{\omega_1} \right)^2 \right|} = \frac{1.6561 \frac{F_0}{\sqrt{m}}}{\left(0.19806 \frac{k}{m} \right)} \frac{1}{\left| 1 - \left(\frac{1.75}{0.44504} \right)^2 \right|} \\ &= 0.57816 F_0 \sqrt{m}/k \end{aligned}$$

$$\text{and } \varrho_1(t) = \varrho_{10} \cos \omega t ; \quad \ddot{\varrho}_1 = -\varrho_{10} \omega^2 \cos \omega t$$

$$\begin{aligned} -\frac{1}{\omega_1^2} \vec{x}^{(1)} \ddot{\varrho}_1(t) &= \varrho_{10} \left(\frac{\omega^2}{\omega_1^2} \right) \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.3280 \\ 0.5911 \\ 0.7370 \end{Bmatrix} \cos \omega t \\ &= 8.93979 \frac{F_0}{k} \begin{Bmatrix} 0.3280 \\ 0.5911 \\ 0.7370 \end{Bmatrix} \cos \omega t \end{aligned}$$

Eg. (E.3) in problem statement gives

$$\begin{aligned} \vec{x}(t) &= \begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} \frac{F_0}{k} \cos \omega t + \begin{Bmatrix} 2.93225 \\ 5.28431 \\ 6.58863 \end{Bmatrix} \frac{F_0}{k} \cos \omega t \\ &= \begin{Bmatrix} 5.93225 \\ 10.28431 \\ 12.58863 \end{Bmatrix} \frac{F_0}{k} \cos \omega t \end{aligned}$$

6.76

Equations of motion for free vibration

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \ddot{\vec{x}} + k \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix} \vec{x} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

From problem 6.46,

$$\omega_1 = 0.337627, \quad \omega_2 = 1.414221, \quad \omega_3 = 2.094278$$

$$\vec{x}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.443004 \\ 1.628659 \end{Bmatrix} x_1^{(1)}, \quad \vec{x}^{(2)} = \begin{Bmatrix} 1.0 \\ 0.49999 \\ -0.49998 \end{Bmatrix} x_1^{(2)}, \quad \vec{x}^{(3)} = \begin{Bmatrix} 1.0 \\ -0.693 \\ 0.204666 \end{Bmatrix} x_1^{(3)}$$

Free vibratory motion is given by (see section 5.3)

$$\vec{x}(t) = \vec{x}^{(1)}(t) + \vec{x}^{(2)}(t) + \vec{x}^{(3)}(t)$$

$$= \begin{Bmatrix} x_1^{(1)} \cos(\omega_1 t + \phi_1) + x_1^{(2)} \cos(\omega_2 t + \phi_2) + x_1^{(3)} \cos(\omega_3 t + \phi_3) \\ 1.443 x_1^{(1)} \cos(\omega_1 t + \phi_1) + 0.5 x_1^{(2)} \cos(\omega_2 t + \phi_2) - 0.693 x_1^{(3)} \cos(\omega_3 t + \phi_3) \\ 1.6287 x_1^{(1)} \cos(\omega_1 t + \phi_1) - 0.5 x_1^{(2)} \cos(\omega_2 t + \phi_2) + 0.2047 x_1^{(3)} \cos(\omega_3 t + \phi_3) \end{Bmatrix}$$

known initial conditions give

--- (E.1)

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1.443 & 0.5 & -0.693 \\ 1.6287 & -0.5 & 0.2047 \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \cos \phi_1 \\ X_1^{(2)} \cos \phi_2 \\ X_1^{(3)} \cos \phi_3 \end{Bmatrix} \quad \text{--- (E.2)}$$

$$\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} = \begin{bmatrix} -\omega_1 & -\omega_2 & -\omega_3 \\ -1.443 \omega_1 & -0.5 \omega_2 & 0.693 \omega_3 \\ -1.6287 \omega_1 & 0.5 \omega_2 & -0.2047 \omega_3 \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \sin \phi_1 \\ X_1^{(2)} \sin \phi_2 \\ X_1^{(3)} \sin \phi_3 \end{Bmatrix} \quad \text{--- (E.3)}$$

Solution of (E.2) and (E.3) gives (using Program # 12)

$$X_1^{(1)} \cos \phi_1 = 0.8884, \quad X_1^{(1)} \sin \phi_1 = 0.4514$$

$$X_1^{(2)} \cos \phi_2 = 0.6667, \quad X_1^{(2)} \sin \phi_2 = -0.7857$$

$$X_1^{(3)} \cos \phi_3 = -0.5551, \quad X_1^{(3)} \sin \phi_3 = 0.4578$$

$$\text{Hence } X_1^{(1)} = 0.9965, \quad X_1^{(2)} = 1.0304, \quad X_1^{(3)} = 0.7195$$

$$\phi_1 = 26.9353^\circ, \quad \phi_2 = -49.6840^\circ, \quad \phi_3 = 140.4870^\circ$$

and the final solution is given by (E.1).

6.77 $\vec{x}(t) = [X] \vec{q}(t) ; \quad \dot{\vec{x}}(t) = [X] \dot{\vec{q}}(t) \quad (1)$

If $\vec{x}(0)$ and $\dot{\vec{x}}(0)$ are the known initial conditions in terms of physical coordinates, to find the corresponding values of $\vec{q}(0)$ and $\dot{\vec{q}}(0)$, we premultiply both sides of Eq. (1) by $[X]^T [m]$ to obtain at $t = 0$:

$$[X]^T [m] \vec{x}(0) = [X]^T [m] [X] \vec{q}(0) \quad (2)$$

$$[X]^T [m] \dot{\vec{x}}(0) = [X]^T [m] [X] \dot{\vec{q}}(0) \quad (3)$$

Since the normal modes are normalized with respect to the mass matrix as

$$[X]^T [m] [X] = [I] \quad (4)$$

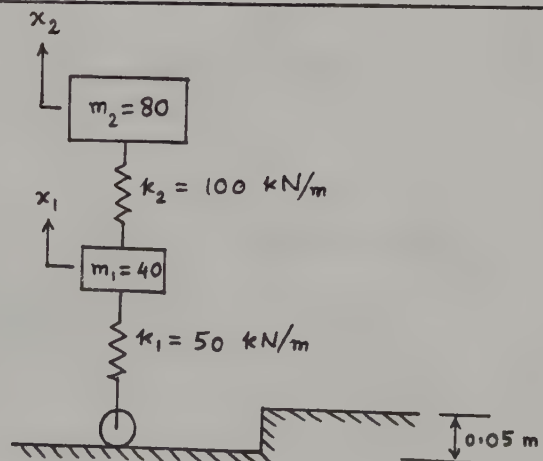
Eqs. (2) and (3) reduce to:

$$\vec{q}(0) = [X]^T [m] \vec{x}(0) ; \quad \dot{\vec{q}}(0) = [X]^T [m] \dot{\vec{x}}(0) \quad (5)$$

6.78

$$[k] = \begin{bmatrix} 150 & -100 \\ -100 & 100 \end{bmatrix} (10^3) \text{ N/m}$$

$$[m] = \begin{bmatrix} 40 & 0 \\ 0 & 80 \end{bmatrix} \text{ kg}$$



Natural frequencies are given by (see Eq. (3) in the solution of Problem 5.1):

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2 m_1} + \frac{k_2}{2 m_2} \pm \left\{ \frac{1}{4} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2} \right\}^{\frac{1}{2}}$$

$$= \left[\frac{150}{80} + \frac{100}{160} \pm \left\{ \frac{1}{4} \left(\frac{150}{40} + \frac{100}{80} \right)^2 - \frac{(100)(50)}{(40)(80)} \right\}^{\frac{1}{2}} \right] (10^3)$$

$$= 334.9365, 4665.1$$

$$\omega_1 = 18.3013 \text{ rad/sec} ; \omega_2 = 68.3015 \text{ rad/sec}$$

Mode shapes are defined by Eqs. (4) and (5) in the solution of Problem 5.1:

$$\frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-m_2 \omega_1^2 + k_2} = \frac{100 (10^3)}{-(80) (334.9365) + (100) (10^3)} = 1.3660$$

$$\vec{X}^{(1)} = a \begin{Bmatrix} 1.0 \\ 1.366 \end{Bmatrix}$$

where a is a constant.

$$\frac{X_2^{(2)}}{X_1^{(2)}} = \frac{k_2}{-m_2 \omega_2^2 + k_2} = \frac{(100) (10^3)}{-(80) (4665.1) + (100) (10^3)} = -0.3660$$

$$\vec{X}^{(2)} = b \begin{Bmatrix} 1.0 \\ -0.366 \end{Bmatrix}$$

where b is a constant.

Orthogonalization of modes:

$$\vec{X}^{(1)T} [m] \vec{X}^{(1)} = a^2 (1.0 \quad 1.366) \begin{bmatrix} 40 & 0 \\ 0 & 80 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 1.366 \end{Bmatrix} = 189.2765 a^2 = 1$$

$$a = 0.07269$$

$$\vec{X}^{(2)T} [m] \vec{X}^{(2)} = b^2 (1.0 \quad -0.366) \begin{bmatrix} 40 & 0 \\ 0 & 80 \end{bmatrix} \begin{Bmatrix} 1.0 \\ -0.366 \end{Bmatrix} = 50.7165 b^2 = 1$$

$$b = 0.14042$$

Modal matrix:

$$[X] = \begin{bmatrix} 0.07269 & 0.14042 \\ 0.09929 & -0.05139 \end{bmatrix}$$

Due to the elevation of 0.05 m, spring k_1 and hence m_1 will be subjected to additional compression of $k_1 (0.05) = 2500$ N.

$$\vec{F}(t) = \begin{Bmatrix} 2500 \\ 0 \end{Bmatrix} \text{ N}$$

Equation (6.111) gives:

$$\vec{Q}(t) = [X]^T \vec{F}(t) = \begin{Bmatrix} 181.725 \\ 351.05 \end{Bmatrix}$$

Solution is given by (without initial conditions) Eq. (6.114):

$$q_i(t) = \frac{1}{\omega_i} \int_0^t Q_i(\tau) \sin \omega_i (t - \tau) d\tau ; i = 1, 2$$

$$\text{Since } \int_{\tau=0}^t \sin \Omega (t - \tau) d\tau = - \int_{\tau'=t-\tau=0}^{\tau'=t-\tau=t} \sin \Omega \tau' d\tau' = \frac{1}{\Omega} (1 - \cos \Omega t)$$

we find

$$q_1(t) = \frac{181.725}{(18.3013^2)} (1 - \cos 18.3013 t) = 0.5426 (1 - \cos 18.3013 t)$$

$$q_2(t) = \frac{351.05}{(68.3015^2)} (1 - \cos 68.3015 t) = 0.07525 (1 - \cos 68.3015 t)$$

Response of the masses can be found from Eq. (6.104):

$$\vec{x}(t) = [X] \vec{q}(t) = \begin{bmatrix} 0.07269 & 0.14042 \\ 0.09929 & -0.05139 \end{bmatrix} \begin{Bmatrix} 0.5426 (1 - \cos 18.3013 t) \\ 0.07525 (1 - \cos 68.3015 t) \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.03944 (1 - \cos 18.3013 t) + 0.01057 (1 - \cos 68.3015 t) \\ 0.05387 (1 - \cos 18.3013 t) - 0.00387 (1 - \cos 68.3015 t) \end{Bmatrix}$$

Note:

This problem can also be solved by specifying the initial conditions as

$$\vec{x}(0) = \begin{Bmatrix} 0.05 \\ 0.05 \end{Bmatrix} \text{ m} , ; \quad \dot{\vec{x}}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

and solving the free vibration problem.

6.79 $\ell_i = 0.5 \text{ m} ; m_i = 1 \text{ kg} (i = 1, 2, 3)$. Assume $m_1 = m_2 = m_3 = m = 1$;
 $\ell_1 = \ell_2 = \ell_3 = \ell = 0.5$.

From solution of Problem 6.37, we obtain:

$$[m] = m \ell^2 \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 0.25 \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$[k] = m g \ell \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 4.905 \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From solution of Problem 6.53, the natural frequencies and mode shapes can be found as

$$\omega_1 = 0.644798 \sqrt{\frac{g}{\ell}} ; \omega_2 = 1.514698 \sqrt{\frac{g}{\ell}} ; \omega_3 = 2.507977 \sqrt{\frac{g}{\ell}}$$

Since $\sqrt{\frac{g}{\ell}} = \sqrt{\frac{9.81}{0.5}} = 4.4294$, we find

$$\omega_1 = 2.8561 \text{ rad/sec} ; \omega_2 = 6.7092 \text{ rad/sec} ; \omega_3 = 11.1088 \text{ rad/sec}$$

$$\vec{X}^{(1)} = a \begin{bmatrix} 1.0 \\ 1.2922 \\ 1.6312 \end{bmatrix} ; \vec{X}^{(2)} = b \begin{bmatrix} 1.0 \\ 0.3527 \\ -2.3978 \end{bmatrix} ; \vec{X}^{(3)} = c \begin{bmatrix} 1.0 \\ -1.6450 \\ 0.7669 \end{bmatrix}$$

Orthonormalization of mode shapes:

$$\begin{aligned} \vec{X}^{(1)T} [m] \vec{X}^{(1)} &= 5.4118 a^2 = 1 \quad \text{or} \quad a = 0.4299 \\ \vec{X}^{(2)T} [m] \vec{X}^{(2)} &= 0.9805 b^2 = 1 \quad \text{or} \quad b = 1.0099 \\ \vec{X}^{(3)T} [m] \vec{X}^{(3)} &= 0.3577 c^2 = 1 \quad \text{or} \quad c = 1.6720 \end{aligned}$$

Modal matrix:

$$[X] = \begin{bmatrix} \vec{X}^{(1)} & \vec{X}^{(2)} & \vec{X}^{(3)} \end{bmatrix} = \begin{bmatrix} 0.4299 & 1.0099 & 1.6720 \\ 0.5555 & 0.3562 & -2.7504 \\ 0.7012 & -2.4215 & 1.2822 \end{bmatrix} \quad (1)$$

$$\vec{F}(t) = \begin{bmatrix} 0 \\ 0 \\ M_{t3}(t) \end{bmatrix} \quad (2)$$

$$\vec{Q}(t) = [X]^T \vec{F}(t) = \begin{bmatrix} 0.7012 M_{t3}(t) \\ -2.4215 M_{t3}(t) \\ 1.2822 M_{t3}(t) \end{bmatrix} \quad (3)$$

Solution of $q_i(t)$ without initial conditions:

$$q_i(t) = \frac{1}{\omega_i} \int_0^t Q_i(\tau) \sin \omega_i (t - \tau) d\tau ; i = 1, 2, 3 \quad (4)$$

We denote $M_{t3}(t)$ as

$$M_{t3}(t) = M_0 \left[u(t) - u(t - t_0) \right] \quad (5)$$

where $M_0 = 0.1 \text{ N-m}$, $t_0 = 0.1 \text{ sec}$ and $u(t)$ and $u(t - t_0)$ are the unit step functions:

$$u(t) = \begin{cases} 0 & ; t < 0 \\ 1 & ; t > 0 \end{cases}$$

$$u(t - t_0) = \begin{cases} 0 & ; t < t_0 \\ 1 & ; t > t_0 \end{cases}$$

Thus we obtain

$$q_1(t) = \frac{1}{\omega_1} \int_0^t Q_1(\tau) \sin \omega_1 (t - \tau) d\tau$$

$$= \frac{1}{2.8561} \int_0^t 0.7012 (0.1) \left\{ u(\tau) - u(\tau - 0.1) \right\} \sin 2.8561 (t - \tau) d\tau$$

$$= 0.02455 \left\{ \int_0^t u(\tau) \sin 2.8561 (t - \tau) d\tau - \int_0^t u(\tau - 0.1) \sin 2.8561 (t - \tau) d\tau \right\}$$

By noting that

$$\int_0^t u(\tau) \sin \Omega (t - \tau) d\tau = \frac{1}{\Omega} (1 - \cos \Omega t) \quad (6)$$

$$\text{and } \int_0^t u(\tau - t_0) \sin \Omega (t - \tau) d\tau = \frac{1}{\Omega} [1 - \cos \Omega (t - t_0)] \quad (7)$$

we can derive

$$q_1(t) = 0.008596 [1 - \cos 2.8561 t] \quad ; \quad 0 < t < 0.1 \text{ sec}$$

$$= 0.008596 [\cos 2.8561 (t - 0.1) - \cos 2.8561 t] \quad ; \quad t \geq 0.1 \text{ sec} \quad (8)$$

Similarly, we can derive

$$q_2(t) = -0.005379 [1 - \cos 6.7092 t] \quad ; \quad 0 < t < 0.1 \text{ sec}$$

$$= -0.005379 [\cos 6.7092 (t - 0.1) - \cos 6.7092 t] \quad ; \quad t \geq 0.1 \text{ sec} \quad (9)$$

$$q_3(t) = 0.001039 [1 - \cos 11.1088 t] \quad ; \quad 0 < t < 0.1 \text{ sec}$$

$$= 0.001039 [\cos 11.1088 (t - 0.1) - \cos 11.1088 t] \quad ; \quad t \geq 0.1 \text{ sec} \quad (10)$$

Thus the angular (physical) displacements of the pendulum can be expressed as:

$$\vec{x}(t) = [X] \vec{q}(t) \quad (11)$$

where $[X]$ is given by Eq. (1) and $\vec{q}(t)$ is given by Eqs. (8) to (10). Hence

$$x_1(t) = 0.003675 (1 - \cos 2.8561 t) - 0.005432 (1 - \cos 6.7092 t)$$

$$+ 0.001737 (1 - \cos 11.1088 t) \quad 0 < t < 0.1 \text{ sec}$$

$$= 0.003695 [\cos 2.8561 (t - 0.1) - \cos 2.8561 t]$$

$$- 0.005432 [\cos 6.7092 (t - 0.1) - \cos 6.7092 t]$$

$$+ 0.001737 [\cos 11.1088 (t - 0.1) - \cos 11.1088 t] \quad t \geq 0.1 \text{ sec}$$

$$\begin{aligned}
x_2(t) &= 0.004775 (1 - \cos 2.8561 t) - 0.001916 (1 - \cos 6.7092 t) \\
&\quad - 0.002858 (1 - \cos 11.1088 t) \quad 0 < t < 0.1 \text{ sec} \\
&= 0.004775 \left[\cos 2.8561 (t - 0.1) - \cos 2.8561 t \right] \\
&\quad - 0.001916 \left[\cos 6.7092 (t - 0.1) - \cos 6.7092 t \right] \\
&\quad - 0.002858 \left[\cos 11.1088 (t - 0.1) - \cos 11.1088 t \right] \quad t \geq 0.1 \text{ sec} \\
x_3(t) &= 0.006027 (1 - \cos 2.8561 t) + 0.013025 (1 - \cos 6.7092 t) \\
&\quad + 0.001332 (1 - \cos 11.1088 t) \quad 0 < t < 0.1 \text{ sec} \\
&= 0.006027 \left[\cos 2.8561 (t - 0.1) - \cos 2.8561 t \right] \\
&\quad + 0.013025 \left[\cos 6.7092 (t - 0.1) - \cos 6.7092 t \right] \\
&\quad + 0.001332 \left[\cos 11.1088 (t - 0.1) - \cos 11.1088 t \right] \quad t \geq 0.1 \text{ sec}
\end{aligned}$$

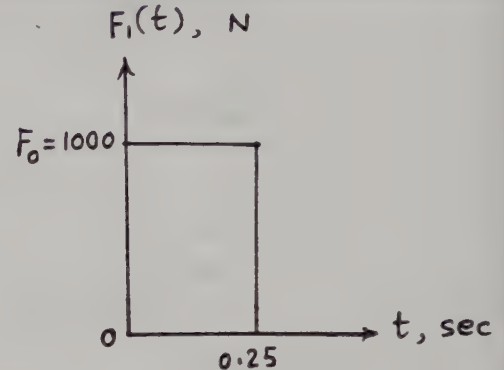
6.80

$$m = 2 \text{ kg}, k = 10,000 \text{ N/m},$$

$$\sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{2}} = 70.7107.$$

$$[m] = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$[k] = 10^4 \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$



From the solution of Problem 6.46, we find the natural frequencies and mode shapes as:

$$\begin{aligned}
\omega_1 &= 0.337627 \sqrt{\frac{k}{m}}; \quad \omega_2 = 1.414221 \sqrt{\frac{k}{m}}; \quad \omega_3 = 2.094278 \sqrt{\frac{k}{m}} \\
\text{or } \omega_1 &= 23.8738 \text{ rad/sec}; \quad \omega_2 = 100.0006 \text{ rad/sec}; \quad \omega_3 = 148.0879 \text{ rad/sec}
\end{aligned}$$

$$\vec{X}^{(1)} = a \begin{bmatrix} 1.0 \\ 1.4430 \\ 1.6286 \end{bmatrix}; \quad \vec{X}^{(2)} = b \begin{bmatrix} 1.0 \\ 0.5 \\ -0.5 \end{bmatrix}; \quad \vec{X}^{(3)} = c \begin{bmatrix} 1.0 \\ -0.6930 \\ 0.2047 \end{bmatrix}$$

Orthonormalization of modes:

$$\vec{X}^{(1)T} [m] \vec{X}^{(1)} = 26.2430 a^2 = 1 \quad \text{or} \quad a = 0.1952$$

$$\vec{X}^{(2)T} [m] \vec{X}^{(2)} = 4.5 b^2 = 1 \quad \text{or} \quad b = 0.4714$$

$$\vec{X}^{(3)T} [m] \vec{X}^{(3)} = 4.1724 c^2 = 1 \quad \text{or} \quad c = 0.4896$$

Modal matrix:

$$[X] = \begin{bmatrix} \vec{X}^{(1)} & \vec{X}^{(2)} & \vec{X}^{(3)} \end{bmatrix} = \begin{bmatrix} 0.1952 & 0.4714 & 0.4896 \\ 0.2817 & 0.2357 & -0.3393 \\ 0.3179 & -0.2357 & 0.1002 \end{bmatrix} \quad (1)$$

$$\vec{F}(t) = \begin{Bmatrix} F_1(t) \\ 0 \\ 0 \end{Bmatrix} \quad (2)$$

$$\vec{Q}(t) = [X]^T \vec{F}(t) = \begin{Bmatrix} 0.1952 F_1(t) \\ 0.4714 F_1(t) \\ 0.4896 F_1(t) \end{Bmatrix} \quad (3)$$

Solution of $q_i(t)$ without initial conditions:

$$q_i(t) = \frac{1}{\omega_i} \int_0^t Q_i(\tau) \sin \omega_i (t - \tau) d\tau ; i = 1, 2, 3 \quad (4)$$

$$F_1(t) = F_0 \left\{ u(t) - u(t - t_0) \right\}$$

where $F_0 = 1000$ N, $t_0 = 0.25$ sec, and $u(t)$ and $u(t - t_0)$ are unit step functions:

$$u(t) = \begin{cases} 0 & ; t < 0 \\ 1 & ; t > t_0 \end{cases}$$

$$u(t - t_0) = \begin{cases} 0 & ; t < t_0 \\ 1 & ; t > t_0 \end{cases}$$

Equations (3) give:

$$\vec{Q}(t) = \begin{Bmatrix} Q_1(t) \\ Q_2(t) \\ Q_3(t) \end{Bmatrix} = \begin{Bmatrix} 195.2 \left\{ u(t) - u(t - 0.25) \right\} \\ 471.4 \left\{ u(t) - u(t - 0.25) \right\} \\ 489.6 \left\{ u(t) - u(t - 0.25) \right\} \end{Bmatrix} \quad (5)$$

and Eqs. (4) yield:

$$q_1(t) = 0.3425 (1 - \cos 23.8738 t) ; 0 < t < 0.25 \text{ sec}$$

$$= 0.3425 \left[\cos 23.8738 (t - 0.25) - \cos 23.8738 t \right] ; t \geq 0.25 \text{ sec}$$

$$q_2(t) = 0.04714 (1 - \cos 100.0006 t) ; 0 < t < 0.25 \text{ sec}$$

$$= 0.04714 \left[\cos 100.0006 (t - 0.25) - \cos 100.0006 t \right] ; t \geq 0.25 \text{ sec}$$

$$q_3(t) = 0.02232 (1 - \cos 148.0879 t) ; 0 < t < 0.25 \text{ sec}$$

$$= 0.02232 \left[\cos 148.0879 (t - 0.25) - \cos 148.0879 t \right] ; t \geq 0.25 \text{ sec}$$

The physical displacements of the masses are given by:

$$\vec{x}(t) = [X] \vec{q}(t)$$

which can be explicitly expressed as:

$$\begin{aligned} x_1(t) &= 0.06686 (1 - \cos 23.8738 t) + 0.02222 (1 - \cos 100.0006 t) \\ &\quad + 0.01093 (1 - \cos 148.0879 t) ; 0 < t < 0.25 \text{ sec} \\ &= 0.06686 \left[\cos 23.8738 (t - 0.25) - \cos 23.8738 t \right] \\ &\quad + 0.02222 \left[\cos 100.0006 (t - 0.25) - \cos 100.0006 t \right] \\ &\quad + 0.01093 \left[\cos 148.0879 (t - 0.25) - \cos 148.0879 t \right] ; t \geq 0.25 \text{ sec} \\ x_2(t) &= 0.09648 (1 - \cos 23.8738 t) + 0.01111 (1 - \cos 100.0006 t) \\ &\quad - 0.007573 (1 - \cos 148.0879 t) ; 0 < t < 0.25 \text{ sec} \\ &= 0.09648 \left[\cos 23.8738 (t - 0.25) - \cos 23.8738 t \right] \\ &\quad + 0.01111 \left[\cos 100.0006 (t - 0.25) - \cos 100.0006 t \right] \\ &\quad - 0.007573 \left[\cos 148.0879 (t - 0.25) - \cos 148.0879 t \right] ; t \geq 0.25 \text{ sec} \\ x_3(t) &= 0.1089 (1 - \cos 23.8738 t) - 0.01111 (1 - \cos 100.0006 t) \\ &\quad + 0.002236 (1 - \cos 148.0879 t) ; 0 < t < 0.25 \text{ sec} \\ &= 0.1089 \left[\cos 23.8738 (t - 0.25) - \cos 23.8738 t \right] \\ &\quad - 0.01111 \left[\cos 100.0006 (t - 0.25) - \cos 100.0006 t \right] \\ &\quad + 0.002236 \left[\cos 148.0879 (t - 0.25) - \cos 148.0879 t \right] ; t \geq 0.25 \text{ sec} \end{aligned}$$

6.81 Equations of motion $[m] \ddot{\vec{x}} + [c] \dot{\vec{x}} + [k] \vec{x} = \vec{F} \quad \dots (E.1)$

where

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, [c] = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix},$$

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}, \vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \vec{F} = \begin{Bmatrix} F_1 = F_0 \cos \omega t \\ F_2 = 0 \\ F_3 = 0 \end{Bmatrix}$$

Since $F_j(t) = \text{Re} [F_{j0} e^{i\omega t}]$ with $F_{10} = F_0$, and $F_{20} = F_{30} = 0$, we assume $x_j(t) = X_j e^{i\omega t}$; $j=1,2,3$. Then (E.1) becomes

$$[Z_{rs}(i\omega)] \vec{X} = \vec{F}_0 \quad \dots (E.2)$$

where $Z_{11}(i\omega) = -m_1 \omega^2 + (c_1 + c_2) i\omega + (k_1 + k_2) = -\omega^2 + 2i\omega + 200$

$$Z_{12}(i\omega) = Z_{21}(i\omega) = -c_2 i\omega - k_2 = -i\omega - 100$$

$$Z_{13}(i\omega) = Z_{31}(i\omega) = 0$$

$$Z_{22}(i\omega) = -m_2 \omega^2 + (c_2 + c_3) i\omega + (k_2 + k_3) = -\omega^2 + 2i\omega + 200 \dots (E.3)$$

$$Z_{23}(i\omega) = Z_{32}(i\omega) = -c_3 i\omega - k_3 = -i\omega - 100$$

$$Z_{33}(i\omega) = -m_3 \omega^2 + (c_3 + c_4) i\omega + (k_3 + k_4) = -\omega^2 + 2i\omega + 200$$

(E.2) becomes

$$\begin{aligned} (2i+199) X_1 - (i+100) X_2 + (0) X_3 &= 10 \\ -(i+100) X_1 + (2i+199) X_2 - (i+100) X_3 &= 0 \\ (0) X_1 - (i+100) X_2 + (2i+199) X_3 &= 0 \end{aligned} \quad \dots (E.4)$$

Solution of (E.4) can be expressed as

$$\text{where } X_j = \frac{\Delta_j}{\Delta} \quad ; \quad j=1, 2, 3 \quad \dots (E.5)$$

$$\Delta_1 = \begin{vmatrix} 10 & -(i+100) & 0 \\ 0 & (2i+199) & -(i+100) \\ 0 & -(i+100) & (2i+199) \end{vmatrix} = 295980 + 5960i$$

$$\Delta_2 = \begin{vmatrix} (2i+199) & 10 & 0 \\ -(i+100) & 0 & -(i+100) \\ 0 & 0 & (2i+199) \end{vmatrix} = 198980 + 3990i$$

$$\Delta_3 = \begin{vmatrix} (2i+199) & -(i+100) & 10 \\ -(i+100) & (2i+199) & 0 \\ 0 & -(i+100) & 0 \end{vmatrix} = 99990 + 2000i$$

$$\Delta = \begin{vmatrix} (2i+199) & -(i+100) & 0 \\ -(i+100) & (2i+199) & -(i+100) \\ 0 & -(i+100) & (2i+199) \end{vmatrix} = 118002i + 3899409$$

Using (E.5), we get

$$X_1 = \frac{87639.682}{1154850.392 + 11685.754i} ; \quad \text{Amplitude} = 0.0758845 \text{ m} \\ \text{Phase angle} = 0.5798^\circ$$

$$X_2 = \frac{39608.961}{776375.228 + 7921.396i} ; \quad \text{Amplitude} = 0.0510152 \text{ m} \\ \text{Phase angle} = 0.5845^\circ$$

$$X_3 = \frac{10002.000}{390137.914 + 4000.202i} ; \quad \text{Amplitude} = 0.0256357 \text{ m} \\ \text{Phase angle} = 0.5874^\circ$$

Thus the steady state responses are:

$$x_1(t) = 0.0758845 \cos(\omega t + 0.5798^\circ) \text{ m}$$

$$x_2(t) = 0.0510152 \cos(\omega t + 0.5845^\circ) \text{ m}$$

$$x_3(t) = 0.0256357 \cos(\omega t + 0.5874^\circ) \text{ m}$$

6.82

Modal matrix can be expressed as $[X] = [\vec{X}^{(1)} \quad \vec{X}^{(2)} \quad \vec{X}^{(3)}]$

$$\text{Using Eq. (6.122), } \underset{12 \times 1}{\vec{x}(t)} = \underset{12 \times 3}{[X]} \underset{3 \times 1}{\vec{q}(t)} \quad \text{where } \vec{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix}$$

equations of motion can be written as

$$[m][\ddot{x}] + [c][\dot{x}] + [k][x] = -[m]\ddot{u}_1 - \ddot{x}_o(t)$$

Premultiplication by $[x]^T$ gives

$$[x]^T[m][\ddot{x}] + [x]^T[c][\dot{x}] + [x]^T[k][x] = -[x]^T[m]\ddot{u}_1 - \ddot{x}_o(t) \quad \text{--- (E.1)}$$

Assuming that the mass matrix is diagonal and the damping matrix is proportional, (E.1) can be expressed in scalar form as

$$m_{ii}\ddot{v}_i + c_{ii}\dot{v}_i + k_{ii}v_i = -\ddot{x}_o(t) \sum_{j=1}^{12} m_j x_j^{(i)}; \quad i = 1, 2, 3 \quad \text{--- (E.2)}$$

where m_{ii} , c_{ii} and k_{ii} are generalized mass, generalized damping, and generalized stiffness, m_j is the mass at the j^{th} d.o.f. and $x_j^{(i)}$ is the j^{th} component of the vector $\vec{x}^{(i)}$.

$$\text{Here } m_{ii} = \sum_{j=1}^{12} m_j (x_j^{(i)})^2 = m \sum_{j=1}^{12} (x_j^{(i)})^2$$

$$c_{ii} = 2\zeta_i \omega_i \quad \text{and} \quad \frac{k_{ii}}{m_{ii}} = \omega_i^2; \quad i = 1, 2, 3$$

where m = mass at each d.o.f.

Eq. (E.2) can be written, noting that there is no damping in the system, as

$$\ddot{v}_i + \omega_i^2 v_i = - \frac{\ddot{x}_o(t) \left\{ \sum_{j=1}^{12} m_j x_j^{(i)} \right\}}{\left\{ \sum_{j=1}^{12} m_j (x_j^{(i)})^2 \right\}} = -\ddot{x}_o(t) \frac{\sum_{j=1}^{12} x_j^{(i)}}{\sum_{j=1}^{12} (x_j^{(i)})^2} \quad ; \quad i = 1, 2, 3 \quad \text{--- (E.3)}$$

By noting that

$$\sum_{j=1}^{12} x_j^{(i)} = 7.964 \text{ for } i=1, -2.67 \text{ for } i=2, 1.618 \text{ for } i=3,$$

$$\sum_{j=1}^{12} (x_j^{(i)})^2 = 6.275658 \text{ for } i=1, 6.48612 \text{ for } i=2, 6.90962 \text{ for } i=3,$$

Eqs. (E.3) can be reduced to

$$\ddot{v}_1(t) + 50625.0 v_1(t) = -1.26903 \ddot{x}_o(t)$$

$$\ddot{v}_2(t) + 435600.0 v_2(t) = 0.411648 \ddot{x}_o(t)$$

--- (E.4)

$$\ddot{v}_3(t) + 1210000.0 v_3(t) = -0.234166 \ddot{x}_o(t)$$

6.83

C =====

C

C SOLUTION OF PROBLEM 6.83

C

C =====

C


```

C N      = NUMBER OF DEGREES OF FREEDOM OF THE SYSTEM
C NM     = N-1
C XK(N,N) = STIFFNESS MATRIX
C XM(N,N) = MASS MATRIX
C OM(N)   = VECTOR OF NATURAL FREQUENCIES
C X(N,N)  = MATRIX OF EIGENVECTORS, J TH EIGENVECTOR IS STORED IN THE
C          J TH COLUMN OF THE MATRIX X
C DIMENSIONS OF OTHER MAIRICES: A(NM,NM),B(NM),LA(NM),LB(NM,2),S(NM)
C

```

```

        DIMENSION XK(3,3),XM(3,3),OM(3),X(3,3),A(2,2),B(2),LA(2),LB(2,2),
2 S(2)

```

```

        DATA XK/2.,-1.,0.,-1.,2.,-1.,0.,-1.,2./

```

```

        DATA XM/2.,0.,0.,0.,3.,0.,0.,0.,2./

```

```

        DATA OM/.482087,1.,1.197605/

```

```

        V=3

```

```

        NM=N-1

```

```

        IND=1

```

```

        DO 10 I=1,N

```

```

        DO 20 J=1,NM

```

```

        DO 20 K=1,NM

```

```

        A(J,K)=XK(J,K+1)-(OM(I)**2)*XM(J,K+1)

```

```

20      B(J)=-XK(J,1)+(OM(I)**2)*XM(J,1)

```

```

        CALL SIMUL (A,B,NM,IND,LA,LB,S)

```

```

        X(1,I)=1.0

```

```

        DO 30 J=1,NM

```

```

30      X(J+1,I)=B(J)

```

```

        *RITE (44,110) OM(I),(X(J,1),J=1,N)

```

```

110     FORMAT (//,2X,19H NATURAL FREQUENCY=,E15.8,/,2X,13H EIGENVECTOR=,
2 /,(2X,4E15.8))

```

```

10     CONTINUE

```

```

        STOP

```

```

        END

```

```

NATURAL FREQUENCY= 0.48208699E+00

```

```

EIGENVECTOR=

```

```

0.10000000E+01 0.15351844E+01 0.10000019E+01

```

```

NATURAL FREQUENCY= 0.10000000E+01

```

```

EIGENVECTOR=

```

```

0.10000000E+01 0.00000000E+00-0.10000000E+01

```

```

NATURAL FREQUENCY= 0.11976050E+01

```

```

EIGENVECTOR=

```

```

0.10000000E+01-0.86851549E+00 0.99999440E+00

```

6.84

```

C
C PROBLEM 6.84
C

```

```

C =====
C N      = NUMBER OF DEGREES OF FREEDOM OF THE SYSTEM
C XM(N,N)= MASS MATRIX

```

```

C X(N,N) = MATRIX CONTAINING THE ORIGINAL I TH NORMAL MODE IN I TH
C COLUMN
C XN(N,N)= MATRIX CONTAINING THE [M]-ORTHONORMAL I TH NORMAL MODE
C IN I TH COLUMN
C DIMENSIONS OF OTHER VECTORS: A(N),B(N)
  DIMENSION XM(3,3),X(3,3),XN(3,3),A(3),B(3)
  N=3
  DATA XM /1.0,0.0,0.0,0.0,2.0,0.0,0.0,0.0,1.0/
  DATA X /1.0,-1.0,1.0,1.0,1.0,1.0,0.0,1.0,2.0/
  DO 10 I=1,N
  DO 20 J=1,N
20  A(J)=X(J,I)
  CALL MULT (XM,A,B,N)
  SUM=0.0
  DO 30 K=1,N
30  SUM=SUM+X(K,I)*B(K)
  SUM=SQRT(1.0/SUM)
  DO 40 K=1,N
40  XN(K,I)=X(K,I)*SUM
  PRINT 50, I,(XN(J,I),J=1,N)
50  FORMAT (/,2X,13H EIGENVECTOR:,15,/,2X,(4E15.8))
10  CONTINUE
  STOP
  END

```

```

C =====
C
C SUBROUTINE MULT
C
C =====

```

```

  SUBROUTINE MULT (XM,A,B,N)
  DIMENSION XM(N,N),A(N),B(N)
  DO 10 I=1,N
  B(I)=0.0
  DO 20 J=1,N
20  B(I)=B(I)+XM(I,J)*A(J)
10  CONTINUE
  RETURN
  END

```

```

EIGENVECTOR:      1
0.50000000E+00-0.50000000E+00 0.50000000E+00

```

```

EIGENVECTOR:      2
0.50000000E+00 0.50000000E+00 0.50000000E+00

```

```

EIGENVECTOR:      3
0.00000000E+00 0.40824831E+00 0.81649661E+00

```

6.85 The main program and results are given below.

```

C =====
C
C MAIN PROGRAM WHICH CALLS POLCOF
C
C =====
C FOLLOWING 4 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION A(3,3),H(3,3),C(3,3),P(3),S(3),PCF(4)
  N=3
  NP=4
  DATA A/5.0,3.0,1.0,3.0,6.0,2.0,2.0,4.0,6.0/

```

```

C END OF PROBLEM-DEPENDENT DATA
  CALL POLCOF (A,B,C,P,S,PCF,N,NP)
  WRITE (22,10)
10  FORMAT (//,49H POLYNOMIAL EXPANSION OF A DETERMINANTAL EQUATION)
  WRITE (22,20)
20  FORMAT (//,21H DATA: DETERMINANT A:,//)
  DO 30 I=1,N
30  WRITE (22,40) (A(I,J),J=1,N)
40  FORMAT (4E15.6)
  WRITE (22,50)
50  FORMAT (/,35H RESULT: POLYNOMIAL COEFFICIENTS IN,//
2    53H PCF(NP)*(X**N)+PCF(N)*(X**N-1)+...+PCF(2)*X+PCF(1)=0,/)
  WRITE (22,60) (PCF(I),I=1,NP)
60  FORMAT (4E15.6)
  STOP
  END

```

POLYNOMIAL EXPANSION OF A DETERMINANTAL EQUATION

DATA: DETERMINANT A:

0.500000E+01	0.300000E+01	0.200000E+01
0.300000E+01	0.600000E+01	0.400000E+01
0.100000E+01	0.200000E+01	0.600000E+01

RESULT: POLYNOMIAL COEFFICIENTS IN
 $PCF(NP) \cdot (X^{**N}) + PCF(N) \cdot (X^{**N-1}) + \dots + PCF(2) \cdot X + PCF(1) = 0$

-0.980000E+02	0.770000E+02	-0.170000E+02	0.100000E+01
---------------	--------------	---------------	--------------

6.86 The main program and output are given.

```

C =====
C
C MAIN PROGRAM FOR CALLING CROOTS
C
C =====
C FOLLOWING 6 LINES CONTAIN PROBLEM-DEPENDENT DATA
C A(NP1) DENOTES THE COEFFICIENT OF (X**N) IN THE POLYNOMIAL
  COMPLEX A(4),X(3)
  DATA N,NP1,IMAX,EPS/3,4,50,1.0E-06/
  A(4)=(1.0,0.0)
  A(3)=(-17.0,0.0)
  A(2)=(77.0,0.0)
  A(1)=(-98.0,0.0)
C END OF PROBLEM-DEPENDENT DATA
  CALL CROOTS (A,N,NP1,EPS,IMAX,X,J,ID)
  WRITE (23,10)
10  FORMAT (//,28H COMPLEX ROOTS OF POLYNOMIAL,/)
  DO 20 I=1,N
20  WRITE (23,30) I,X(I)
30  FORMAT ((I5,2E15.6))
  STOP
  END

COMPLEX ROOTS OF POLYNOMIAL

```

1	0.221400E+01	0.471845E-13
2	0.416924E+01	-0.614934E-13
3	0.106168E+02	0.143089E-13

6.87

```

=====
C
C PROBLEM 6.87
C
C =====
C ND = DEGREES OF FREEDOM OF THE SYSTEM
C DIMENSIONS OF MATRICES K,M,BK,U,UI,UTI,BMU,UMU,B,C,XX ARE (ND,ND)
C DIMENSION OF VECTORS P,S,CM,X IS (ND)
C DIMENSION OF MATRIX AA IS (ND-1,ND-1)
C DIMENSION OF MATRIX LBB IS (ND-1,2)
C DIMENSION OF VECTORS BB,SS,LAA IS (ND-1)
C DIMENSION OF VECTORS PCF,A IS (ND+1)
C K = STIFFNESS MATRIX, M = MASS MATRIX
  REAL K(3,3),M(3,3)
  DIMENSION BK(3,3),U(3,3),UI(3,3),UTI(3,3),BMU(3,3),UMU(3,3),B(3,3)
  2 ,C(3,3),P(3),S(3),PCF(4),CM(3),AA(2,2),BB(2),LAA(2),LBB(2,2),
  3 SS(2),XX(3,3)
  COMPLEX A(4),X(3)
C INPUT DATA
  DATA K/11.772,0.0,0.0,0.0,14.715,0.0,0.0,0.0,11.772/
  DATA M/0.24,0.30,0.24,0.30,0.45,0.36,0.24,0.36,0.48/
  ND=3
C END OF INPUT DATA
C PART (I)
  DO 10 I=1,ND
  DO 10 J=1,ND
10   BK(I,J)=K(I,J)
  CALL DECOMP (BK,U,ND)
  PRINT 100
100  FORMAT (//,29H UPPER TRIANGULAR MATRIX [U]:,/)
  DO 110 I=1,ND
110  PRINT 120,(U(I,J),J=1,ND)
120  FORMAT (4(1X,E15.6))
  DO 130 I=1,ND
  DO 130 J=1,ND
130  UI(I,J)=0.0
  DO 140 I=1,ND
140  UI(I,I)=1.0/U(I,I)
  DO 160 J=1,ND
  DO 160 II=1,ND
  I=ND-II+1
  IF (I .GE. J) GO TO 160
  IP=I+1
  SUM=0.0
  DO 150 KK=IP,J
150  SUM=SUM+U(I,KK)*UI(KK,J)
  UI(I,J)=-SUM/U(I,I)
160  CONTINUE
  DO 190 I=1,ND
  DO 190 J=1,ND
190  UTI(I,J)=UI(J,I)
  CALL MATMUL (BMU,M,UI,ND,ND,ND)
  CALL MATMUL (UMU,UTI,BMU,ND,ND,ND)
  PRINT 200

```



```

200  FORMAT (/ ,29H MATRIX [UMU] = [UT1][M][UI]:,/ )
      DO 210 I=1,ND
210  PPINT 120, (UMU(I,J),J=1,ND)
C PART (II)
      NP=ND+1
      CALL POLCOF (UMU,H,C,P,S,PCF,ND,NP)
      PRINT 300
300  FORMAT (// ,49H POLYNOMIAL EXPANSION OF A DETERMINANTAL EQUATION)
      PRINT 310
310  FORMAT (/ ,35H RESULT: POLYNOMIAL COEFFICIENTS IN,/ ,
      2 53H PCF(NP)*(X**N)+PCF(N)*(X**N-1)+...+PCF(2)*X+PCF(1)=0,/ )
      PRINT 320, (PCF(I),I=1,NP)
320  FORMAT (4E15.6)
C PART (III)
      DO 325 I=1,NP
325  A(I)=CMPLX(PCF(NP-I+1),0.0)
      EPS=1.0E-06
      IMAX=50
      CALL CROOTS (A,ND,NP,EPS,IMAX,X,J,ID)
      PRINT 400
400  FORMAT (// ,28H COMPLEX ROOTS OF POLYNOMIAL,/ )
      DO 420 I=1,ND
420  PRINT 430,I,X(I)
430  FORMAT ((15,2E15.6))
C PART (IV)
      DO 500 I=1,ND
500  OM(I)=SQRT(REAL(X(I)))
      NM=ND-1
      IND=1
      DO 510 I=1,ND
      DO 520 J=1,NM
      DO 520 KK=1,NM
      AA(J,KK)=K(J,KK+1)-(OM(I)**2)*M(J,KK+1)
520  BB(J)=-K(J,1)+(OM(I)**2)*M(J,1)
      CALL SIMUL (AA,BB,NM,IND,LAA,LBB,SS)
      XX(1,I)=1.0
      DO 530 J=1,NM
530  XX(J+1,I)=BB(J)
      PRINT 610, OM(I),(XX(J,I),J=1,ND)
610  FORMAT (// ,2X,19H NATURAL FREQUENCY=,E15.8,/ ,2X,13H EIGENVECTOR=,
      2 / , (2X,4E15.8))
510  CONTINUE
      PRINT 620
620  FORMAT (// ,23H ORIGINAL EIGENVECTORS:,// )
      CALL MATMUL (B,UI,XX,ND,ND,ND)
      DO 630 I=1,ND
630  PRINT 640, (B(J,I),J=1,ND)
640  FORMAT (2X,3E15.6)
      STOP
      END

```

UPPER TRIANGULAR MATRIX [U]:

0.343103E+01	0.000000E+00	0.000000E+00
0.000000E+00	0.383601E+01	0.000000E+00
0.000000E+00	0.000000E+00	0.343103E+01

MATRIX [UM0] = [UTI][M][UI]:

0.203874E-01	0.227938E-01	0.203874E-01
0.227938E-01	0.305810E-01	0.273525E-01
0.203874E-01	0.273525E-01	0.407747E-01

POLYNOMIAL EXPANSION OF A DETERMINANTAL EQUATION

RESULT: POLYNOMIAL COEFFICIENTS IN

PCF(NP)*(X**N)+PCF(N)*(X**N-1)+...+PCF(2)*X+PCF(1)=0

-0.169475E-05 0.101833E-02 -0.917431E-01 0.100000E+01

COMPLEX ROOTS OF POLYNOMIAL

1	0.126346E+02	0.000000E+00
2	0.946085E+02	0.000000E+00
3	0.493631E+03	0.000000E+00

NATURAL FREQUENCY= 0.35545251E+01

EIGENVECTOR=

0.10000000E+01 0.11484827E+01 0.14465784E+01

NATURAL FREQUENCY= 0.97266874E+01

EIGENVECTOR=

0.10000000E+01 0.81423664E+00 -0.14993432E+01

NATURAL FREQUENCY= 0.22217810E+02

EIGENVECTOR=

0.10000000E+01 -0.81276625E+00 0.11532355E+00

ORIGINAL EIGENVECTORS:

0.291457E+00	0.299395E+00	0.421616E+00
0.291457E+00	0.212261E+00	-0.436994E+00
0.291457E+00	-0.211878E+00	0.336119E-01

6.88

The main program and results are shown below.

```
C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE MODAL
C
C =====
C
      DIMENSION XM(3,3),OM(3),Z(3),X0(3),XD0(3),Y0(3),YD0(3),EV(3,3),
2     EVT(3,3),XMX(3,3),XIMX(3,3),T(40),F(3,40),X(3,40),U(3,40),
3     V(3,40),Q(3,40)
```

```

DATA N,NVEC,NSTEP,DELT/3,3,40,0.1/
DATA X4/2.0,0.0,0.0,0.0,2.0,0.0,0.0,0.0,2.0/
OMF=5.0
DATA Q4/1.530734,2.828428,3.695518/
DATA Z/0.0,0.0,0.0/
DATA XO/0.0,0.0,0.0/
DATA XDO/0.0,0.0,0.0/
DATA (EV(I,1),I=1,3)/1.0,1.414214,1.0/
DATA (EV(I,2),I=1,3)/1.0,0.0,-1.0/
DATA (EV(I,3),I=1,3)/1.0,-1.414214,1.0/
DO 5 I=1,NSTEP
TIME=REAL(I)*DELT
5 F(1,I)=10.0*SIN(5.0*TIME)
DO 10 I=1,NSTEP
10 F(2,I)=0.0
F(3,I)=0.0
DO 20 I=1,NVEC
DO 20 J=1,N
20 EVT(I,J)=EV(J,I)
CALL MODAL (XM,OM,OMF,I,Z,XO,XDO,YO,YDO,Q,F,DELT,EV,EVT,XX,
2 XTX,X,U,V,NSTEP,N,NVEC)
WRITE (29,30)
30 FORMAT (//,40H RESPONSE OF SYSTEM USING MODAL ANALYSIS,/)
DO 40 I=1,N
40 WRITE (29,50) I,(X(I,J),J=1,NSTEP)
50 FORMAT (/,11H COORDINATE,15,/,/(1X,5E14.6))
STOP
END

```

RESPONSE OF SYSTEM USING MODAL ANALYSIS

COORDINATE 1

0.119060E-01	0.556720E-01	0.140716E+00	0.262090E+00	0.400585E+00
0.526842E+00	0.608511E+00	0.618689E+00	0.543493E+00	0.386804E+00
0.170858E+00	-0.676210E-01	-0.284971E+00	-0.440109E+00	-0.503791E+00
-0.465406E+00	-0.335646E+00	-0.144346E+00	0.660445E-01	0.249893E+00
0.368518E+00	0.398672E+00	0.337414E+00	0.202231E+00	0.264541E-01
-0.148937E+00	-0.235117E+00	-0.354495E+00	-0.346914E+00	-0.271541E+00
-0.154034E+00	-0.297635E-01	0.652391E-01	0.103368E+00	0.723514E-01
-0.216811E-01	-0.155688E+00	-0.295363E+00	-0.403527E+00	-0.449197E+00

COORDINATE 2

0.397433E-04	0.655754E-03	0.356972E-02	0.119145E-01	0.297943E-01
0.612223E-01	0.108686E+00	0.171738E+00	0.246047E+00	0.323279E+00
0.391991E+00	0.439459E+00	0.454134E+00	0.428213E+00	0.359732E+00
0.253658E+00	0.121640E+00	-0.196366E-01	-0.151233E+00	-0.255357E+00
-0.313689E+00	-0.334940E+00	-0.306091E+00	-0.242021E+00	-0.158585E+00
-0.745270E-01	-0.784475E-02	0.276338E-01	0.247827E-01	-0.156785E-01
-0.856190E-01	-0.171361E+00	-0.256531E+00	-0.325272E+00	-0.365137E+00
-0.369097E+00	-0.336327E+00	-0.271711E+00	-0.184302E+00	-0.851769E-01

COORDINATE 3

0.526197E-07	0.338722E-05	0.398979E-04	0.232946E-03	0.912614E-03
0.274891E-02	0.635317E-02	0.147839E-01	0.284018E-01	0.495493E-01
0.795763E-01	0.118784E+00	0.165901E+00	0.217749E+00	0.269161E+00

```

0.313352E+00 0.342632E+00 0.349486E+00 0.327819E+00 0.274185E+00
0.188751E+00 0.758134E-01 -0.563065E-01 -0.195996E+00 -0.329661E+00
-0.443429E+00 -0.524977E+00 -0.565224E+00 -0.559583E+00 -0.508616E+00
-0.417951E+00 -0.297528E+00 -0.160267E+00 -0.204086E-01 0.108245E+00
0.213908E+00 0.288030E+00 0.326005E+00 0.327354E+00 0.295404E+00

```

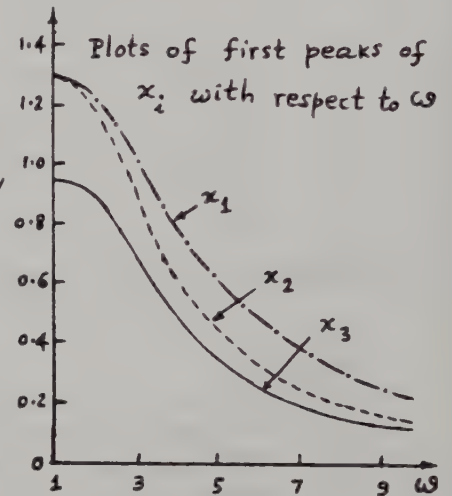
6.89

The main program used to generate the numerical results and the graph are given.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE MODAL
C
C =====
C
      DIMENSION XM(3,3),OM(3),Z(3),X0(3),XDO(3),Y0(3),YDO(3),EV(3,3),
2     EVT(3,3),XMX(3,3),XTMX(3,3),T(40),F(3,40),X(3,40),U(3,40),
3     V(3,40),Q(3,40)
      DATA N,NVEC,NSTEP,DELT/3,3,40,0.1/
      DATA XM/2.0,0.0,0.0,0.0,2.0,0.0,0.0,0.0,2.0/
      DATA OM/1.530734,2.828428,3.695518/
      DATA Z/0.0,0.0,0.0/
      DATA X0/0.0,0.0,0.0/
      DATA XDO/0.0,0.0,0.0/
      DATA (EV(I,1),I=1,3)/1.0,1.414214,1.0/
      DATA (EV(I,2),I=1,3)/1.0,0.0,-1.0/
      DATA (EV(I,3),I=1,3)/1.0,-1.414214,1.0/
      OMF=0.0
      DO 55 IJK=1,10
      OMF=OMF+1.0
      DO 5 I=1,NSTEP
      TIME=REAL(I)*DELT
5     F(1,I)=10.0*SIN(OMF*TIME)
      DO 10 I=1,NSTEP
      F(2,I)=0.0
10    F(3,I)=0.0
      DO 20 I=1,NVEC
      DO 20 J=1,N
20    EVT(I,J)=EV(J,I)
      CALL MODAL (XM,OM,OMF,T,Z,X0,XDO,Y0,YDO,O,F,DELT,EV,EVT,XMX,
2     XTMX,X,U,V,NSTEP,N,NVEC)
      WRITE (29,30)
30    FORMAT (//,40H RESPONSE OF SYSTEM USING MODAL ANALYSIS,/)
      DO 40 I=1,N
40    WRITE (29,50) I,(X(I,J),J=1,NSTEP)
50    FORMAT (/,11H COORDINATE,I5,/, (1X,5E14.6))
55    CONTINUE
      STOP
      END

```



6.90

(a) Equation of motion :

$$\begin{bmatrix} m_f & 0 & 0 \\ 0 & m_b & 0 \\ 0 & 0 & m_h \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_3(t) \end{Bmatrix}$$

Forcing function can be written as

$$F_3(t) = \text{Re}(F_0 e^{i\omega t}) = F_{30} \cos \omega t \equiv 1000 \cos 60t \text{ lb.} \quad (E_1)$$

steady-state solution can be assumed as

$$x_j(t) = X_j e^{i\omega t}, \quad j = 1, 2, 3 \quad (E_2)$$

Equations of motion become

$$\begin{bmatrix} -m_f \omega^2 + (c_1 + c_2)i\omega + k_1 + k_2 & -c_2 i\omega - k_2 & 0 \\ -c_2 i\omega - k_2 & -m_b \omega^2 + (c_2 + c_3)i\omega + k_2 + k_3 & 0 \\ 0 & -c_3 i\omega - k_3 & -m_h \omega^2 + c_3 i\omega + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_{30} \end{Bmatrix} \quad (E_3)$$

For known data, Eq. (E3) becomes

$$[Z_{ij}] \vec{X} = \vec{F}_0 \quad (E_4)$$

where

$$Z_{11} = -50 \omega^2 + i 20 \omega + 5500 = -174500 + i 1200$$

$$Z_{12} = Z_{21} = -i 10 \omega - 500 = -500 - i 600$$

$$Z_{13} = Z_{31} = 0 \quad (E_5)$$

$$Z_{22} = -10 \omega^2 + i 20 \omega + 2500 = -33500 + i 1200$$

$$Z_{23} = Z_{32} = -i 10 \omega - 2000 = -2000 - i 600$$

$$Z_{33} = -2 \omega^2 + i 10 \omega + 2000 = -5200 + i 600$$

Solution of Eq. (E4) can be expressed as

$$\vec{X} = [Z_{ij}]^{-1} \vec{F}_0 \quad (E_6)$$

Using Cramer's rule, we get

$$X_1 = \begin{vmatrix} 0 & Z_{12} & Z_{13} \\ 0 & Z_{22} & Z_{23} \\ F_{30} & Z_{32} & Z_{33} \end{vmatrix} \cdot \frac{1}{\det[Z_{ij}]} = (-0.1230 \times 10^{-4} - i 0.5284 \times 10^{-4})$$

$$X_2 = \begin{vmatrix} Z_{11} & 0 & Z_{13} \\ Z_{21} & 0 & Z_{23} \\ Z_{31} & F_{30} & Z_{33} \end{vmatrix} \cdot \frac{1}{\det[Z_{ij}]} = (0.1087 \times 10^{-1} + i 0.5373 \times 10^{-2})$$

$$X_3 = \begin{vmatrix} Z_{11} & Z_{12} & 0 \\ Z_{21} & Z_{22} & 0 \\ Z_{31} & Z_{32} & F_{30} \end{vmatrix} \cdot \frac{1}{\det[Z_{ij}]} = (-0.1929 - i 0.02558)$$

$$\therefore x_1(t) = X_1 (\cos 60t + i \sin 60t)$$

$$= (-0.1230 \times 10^{-4} \cos 60t + 0.5284 \times 10^{-4} \sin 60t) \\ - i (0.5284 \times 10^{-4} \cos 60t + 0.1230 \times 10^{-4} \sin 60t)$$

$$\text{Actual response of } m_f = \text{Real}[x_1(t)]$$

$$= (-0.1230 \times 10^{-4} \cos 60t + 0.5284 \times 10^{-4} \sin 60t) \text{ in.}$$

Similarly, we find:

$$\text{Actual response of } m_b = \text{Real}[x_2(t)]$$

$$= (0.01087 \cos 60t - 0.005373 \sin 60t) \text{ in.}$$

$$\text{Actual response of } m_h = \text{Real}[x_3(t)]$$

$$= (-0.1929 \cos 60t + 0.02558 \sin 60t) \text{ in.}$$

(b) Change the stiffness k_2 from 100 lb/in in increments of 100 lb/in and find the response of the tool head (x_3) using the procedure outlined in part (a).

Find the value of k_2 for which the maximum response of x_3 is 25% lower than the value found in part (a).

(c) Change the values of c_1, c_2, c_3, k_1 and k_3 individually and find whether any of these quantities can be used to achieve the goal of part (b).

Chapter 7

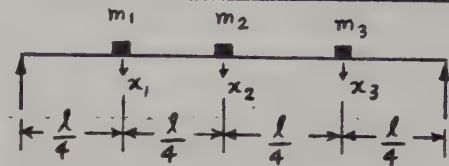
Determination of Natural Frequencies and Mode Shapes

7.1

From Example 6.6,

$$a_{11} = a_{33} = \frac{3}{256} \frac{l^3}{EI}$$

$$a_{22} = \frac{1}{48} \frac{l^3}{EI}$$



(a) Eq. (7.6) gives

$$\frac{1}{\omega_1^2} \approx \frac{ml^3}{EI} \left(\frac{3 \times 5}{256} + \frac{1}{48} \times 1 + \frac{3 \times 5}{256} \right) = \frac{106}{768} \frac{ml^3}{EI} = 0.13802 \frac{ml^3}{EI}$$

$$\omega_1 \approx 2.6917 \sqrt{\frac{EI}{ml^3}}$$

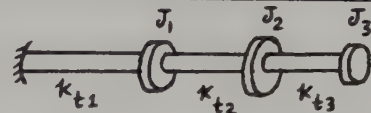
$$(b) \frac{1}{\omega_1^2} \approx \frac{ml^3}{EI} \left(\frac{3}{256} + \frac{1 \times 5}{48} + \frac{3}{256} \right) = \frac{98}{768} \frac{ml^3}{EI} = 0.12760 \frac{ml^3}{EI}$$

$$\omega_1 \approx 2.7994 \sqrt{\frac{EI}{ml^3}}$$

7.2

Flexibility influence coefficients:

a_{11} = rotation of J_1 when a unit torque is applied to $J_1 = 1/k_{t1}$



a_{22} = rotation of J_2 when a unit torque is applied to J_2

$$= \frac{1}{k_{teq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}}$$

a_{33} = rotation of J_3 when a unit torque is applied to J_3

$$= \frac{1}{k_{teq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}}$$

$$(a) \text{ Eq. (7.6) gives } \frac{1}{\omega_1^2} \approx a_{11} J_1 + a_{22} J_2 + a_{33} J_3 = \frac{J_0}{k_t} (1+2+3)$$

$$\omega_1 \approx 0.4082 \sqrt{k_t/J_0}$$

$$(b) \frac{1}{\omega_1^2} \approx \frac{J_1}{k_{t1}} + J_2 \left(\frac{1}{k_{t1}} + \frac{1}{k_{t2}} \right) + J_3 \left(\frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}} \right)$$

$$\approx \frac{J_0}{k_t} + 2 J_0 \left(\frac{1}{k_t} + \frac{1}{2k_t} \right) + 3 J_0 \left(\frac{1}{k_t} + \frac{1}{2k_t} + \frac{1}{3k_t} \right) = \frac{9.5 J_0}{k_t}$$

$$\omega_1 \approx 0.3244 \sqrt{k_t/J_0}$$

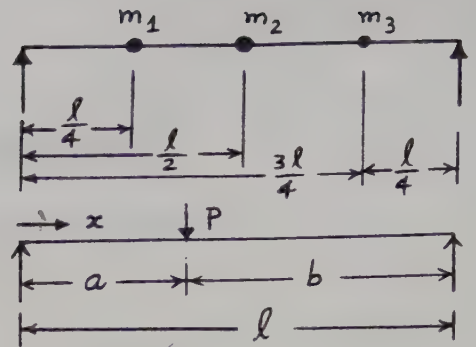
7.3

$$m_1 = m, m_2 = 2m, m_3 = 3m;$$

$$l_1 = l_2 = l_3 = l_4 = \frac{l}{4}$$

$$w(x) = \frac{Pbx}{6EI l} (l^2 - b^2 - x^2); 0 \leq x \leq a$$

$$= -\frac{Pa(l-x)}{6EI l} (a^2 + x^2 - 2lx); a \leq x \leq l$$



Deflection due to weight of m_1 : ($P = mg$)

At location of m_1 ($x = \frac{l}{4}$, $b = \frac{3l}{4}$, $l = l$)

$$w_1' = \frac{(mg)(\frac{3l}{4})(\frac{l}{4})}{6EI l} \left\{ l^2 - \frac{9}{16} l^2 - \frac{1}{16} l^2 \right\} = \frac{3mg l^3}{256 EI}$$

At location of m_2 ($a = \frac{l}{4}$, $x = \frac{l}{2}$, $l = l$)

$$w_2' = -\frac{(mg)(\frac{l}{4})(l - \frac{l}{2})}{6EI l} \left(\frac{l^2}{16} + \frac{l^2}{4} - l^2 \right) = \frac{11mg l^3}{768 EI}$$

At location of m_3 ($a = \frac{l}{4}$, $x = \frac{3l}{4}$, $l = l$)

$$w_3' = -\frac{(mg)(\frac{l}{4})(l - \frac{3l}{4})}{6EI l} \left(\frac{l^2}{16} + \frac{9}{16} l^2 - \frac{6}{4} l^2 \right) = \frac{7mg l^3}{768 EI}$$

Deflection due to weight of m_2 : ($P = 2mg$)

At location of m_1 ($x = \frac{l}{4}$, $b = \frac{l}{2}$, $l = l$)

$$w_1'' = \frac{(2mg)(\frac{l}{2})(\frac{l}{4})}{6EI l} \left(l^2 - \frac{l^2}{4} - \frac{1}{16} l^2 \right) = \frac{11mg l^3}{384 EI}$$

At location of m_2 ($x = \frac{l}{2}$, $b = \frac{l}{2}$, $l = l$)

$$w_2'' = \frac{(2mg)(\frac{l}{2})(\frac{l}{2})}{6EI l} \left(l^2 - \frac{l^2}{4} - \frac{l^2}{4} \right) = \frac{mg l^3}{24 EI}$$

At location of m_3 ($x = \frac{3l}{4}$, $a = \frac{l}{2}$, $b = \frac{l}{2}$, $l = l$)

$$w_3'' = -\frac{(2mg)(\frac{l}{2})(l - \frac{3l}{4})}{6EI l} \left(\frac{l^2}{4} + \frac{9}{16} l^2 - \frac{6}{4} l^2 \right) = \frac{11mg l^3}{1024 EI}$$

Deflection due to weight of m_3 : ($P = 3mg$)

At location of m_1 ($x = \frac{l}{4}$, $b = \frac{l}{4}$, $l = l$)

$$w_1''' = \frac{(3mg)(\frac{l}{4})(\frac{l}{4})}{6EI l} \left(l^2 - \frac{l^2}{16} - \frac{l^2}{16} \right) = \frac{7mg l^3}{256 EI}$$

At location of m_2 ($x = \frac{l}{2}$, $b = \frac{l}{4}$, $l = l$)

$$w_2''' = \frac{(3mg)\left(\frac{l}{4}\right)\left(\frac{l}{2}\right)}{6EI l} \left(l^2 - \frac{l^2}{16} - \frac{l^2}{4}\right) = \frac{11 mg l^3}{256 EI}$$

At location of m_3 ($x = \frac{3l}{4}$, $b = \frac{l}{4}$, $l = l$)

$$w_3''' = \frac{(3mg)\left(\frac{l}{4}\right)\left(\frac{3l}{4}\right)}{6EI l} \left(l^2 - \frac{1}{16} l^2 - \frac{9}{16} l^2\right) = \frac{9 mg l^3}{256 EI}$$

Total deflection of masses:

$$w_1 = w_1' + w_1'' + w_1''' = \frac{mg l^3}{EI} \left(\frac{3}{256} + \frac{11}{384} + \frac{7}{256}\right) = \frac{13}{192} \frac{mg l^3}{EI}$$

$$w_2 = w_2' + w_2'' + w_2''' = \frac{mg l^3}{EI} \left(\frac{11}{768} + \frac{1}{24} + \frac{11}{256}\right) = \frac{19}{192} \frac{mg l^3}{EI}$$

$$w_3 = w_3' + w_3'' + w_3''' = \frac{mg l^3}{EI} \left(\frac{7}{768} + \frac{11}{1024} + \frac{9}{256}\right) = \frac{169}{3072} \frac{mg l^3}{EI}$$

$$\omega = \left\{ \frac{g \frac{m^2 g l^3}{EI} \left(\frac{13}{192} + 2 \times \frac{19}{192} + 3 \times \frac{169}{3072}\right)}{\frac{m^2 g^2 l^6 m}{E^2 I^2} \left\{ \left(\frac{13}{192}\right)^2 + 2 \left(\frac{19}{192}\right)^2 + 3 \left(\frac{169}{3072}\right)^2 \right\}} \right\}^{1/2}$$

$$= 3.5987 \sqrt{\frac{EI}{m l^3}}$$

7.4

ω_{11} = natural frequency of wing itself = 20 Hz = 125.664 $\frac{\text{rad}}{\text{sec}}$

ω_{22} = natural frequency of weapon attached at the tip of the wing (neglecting the effect of mass of wing)

$$= \sqrt{\frac{k}{m}} = \sqrt{\frac{50000}{12} \left(\frac{386.4}{2000}\right)} = 28.3725 \frac{\text{rad}}{\text{sec}}$$

New frequency of vibration of the wing with weapon is given by

$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} = \frac{1}{125.664^2} + \frac{1}{28.3725^2}$$

$$= 130.5563 \times 10^{-5}$$

$$\therefore \omega_1 = 27.6759 \frac{\text{rad}}{\text{sec}} = 4.4047 \text{ Hz}$$

- 7.5 For a simply supported beam, natural frequency is given by (assuming its mass to be concentrated at the middle)

$$\omega_{11} = \sqrt{\frac{k}{m}} \quad \text{where } k = \frac{48EI}{l^3}$$

Neglecting the mass of beam, if trolley is placed at the middle of girder, its natural frequency is given by

$$\omega_{22} = \sqrt{\frac{k}{10m}}$$

Fundamental natural frequency of the combined system is given by

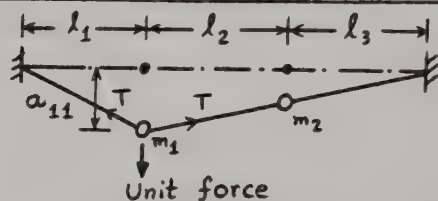
$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} = \frac{m}{k} + \frac{10m}{k} = \frac{11m}{k}$$

$$\therefore \omega_1 = 0.3015 \sqrt{\frac{k}{m}} = 30.15\% \text{ of the natural frequency of the girder (without the trolley)}$$

- 7.6 Flexibility coefficients:

Let T = tension in string

a_{11} = deflection of m_1 due to unit force applied to m_1



Unit force = sum of components of tension in vertical direction

$$\text{i.e. } 1 = T \left(\frac{a_{11}}{l_1} \right) + T \left(\frac{a_{11}}{l_2 + l_3} \right) = T a_{11} \left(\frac{1}{l} + \frac{1}{2l} \right) = \frac{3Ta_{11}}{2l}$$

$$a_{11} = \frac{2l}{3T} = a_{22} \text{ (by symmetry)}$$

$$\frac{1}{\omega_1^2} \approx a_{11} m_1 + a_{22} m_2 = \frac{2l}{3T} (m + m) = \frac{4ml}{3T}$$

$$\omega_1 \approx 0.866 \sqrt{\frac{T}{ml}} ; \text{ Exact solution is } \omega_1 = \sqrt{\frac{T}{ml}} \text{ (Problem 5.18).}$$

- 7.7 From Example 6.5,

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = k \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Assuming the mode shape as $\vec{x} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$, Rayleigh's quotient becomes

$$R(\vec{x}) = \omega^2 = \frac{\vec{x}^T [k] \vec{x}}{\vec{x}^T [m] \vec{x}}$$

$$= \frac{(1 \ 2 \ 3) k \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1 \ 2 \ 3) m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} = \frac{1}{6} \frac{k}{m}$$

$$\omega_1 = 0.4082 \sqrt{\frac{k}{m}}$$

Exact value is $\omega_1 = 0.3376 \sqrt{\frac{k}{m}}$ (Problem 6.46).

7.8

Stiffness matrix:

Give each disc a unit angular displacement holding other discs with zero rotation. Torque required will give stiffness coefficients.

$$\theta_1 = 1, \theta_2 = \theta_3 = 0: M_{t1} = k_{t1} + k_{t2}, M_{t2} = -k_{t2}, M_{t3} = 0$$

$$\theta_2 = 1, \theta_1 = \theta_3 = 0: M_{t2} = k_{t2} + k_{t3}, M_{t1} = -k_{t2}, M_{t3} = -k_{t3}$$

$$\theta_3 = 1, \theta_1 = \theta_2 = 0: M_{t3} = k_{t3}, M_{t2} = -k_{t3}, M_{t1} = 0$$

$$[k] = \begin{bmatrix} k_{t1} + k_{t2} & -k_{t2} & 0 \\ -k_{t2} & k_{t2} + k_{t3} & -k_{t3} \\ 0 & -k_{t3} & k_{t3} \end{bmatrix} = k_t \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[m] = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Assume the mode shape as $\vec{\theta} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

$$R(\vec{\theta}) = \omega^2 = \frac{(1 \ 2 \ 3) k_t \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1 \ 2 \ 3) J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} = \frac{1}{12} \frac{k_t}{J_0}$$

$$\omega_1 \approx 0.2887 \sqrt{\frac{k_t}{J_0}}$$

7.9

Stiffness matrix:

When $x_1 = 1, x_2 = 0$:

$$F_1 = k_{11} = \frac{T}{l_1} + \frac{T}{l_2}$$

$$F_2 = k_{21} = -\frac{T}{l_2}$$

When $x_1 = 0, x_2 = 1$:

$$F_2 = \frac{T}{l_2} + \frac{T}{l_3}$$

$$F_1 = k_{12} = -\frac{T}{l_2}$$

$$[k] = T \begin{bmatrix} \left(\frac{1}{l_1} + \frac{1}{l_2}\right) & -\frac{1}{l_2} \\ -\frac{1}{l_2} & \left(\frac{1}{l_2} + \frac{1}{l_3}\right) \end{bmatrix} = \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Assume the mode shape as $\vec{X} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

$$R(\vec{X}) = \omega^2 = \frac{(1 \ 2) \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}}{(1 \ 2) m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}} = \frac{6}{5} \frac{T}{lm}$$

$$\omega_1 \approx 1.0954 \sqrt{\frac{T}{lm}}$$

Exact value is $\omega_1 = \sqrt{\frac{T}{lm}}$ (Problem 5.18).

7.10

From problem 7.9, for $l_1 = l_2 = l_3 = l$,

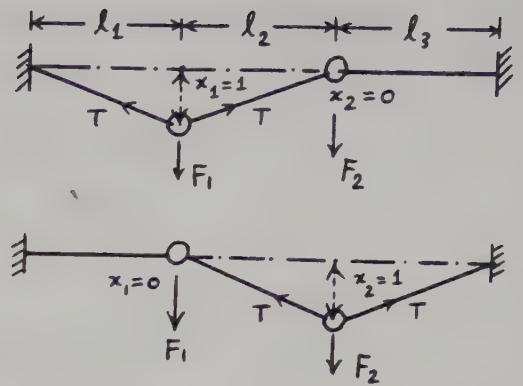
$$[k] = \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\text{and for } m_1 = m, m_2 = 5m, \quad [m] = m \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

Let $\vec{X} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

$$R(\vec{X}) = \omega^2 = \frac{(1 \ 2) \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}}{(1 \ 2) m \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}} = \frac{2}{7} \frac{T}{ml}$$

$$\omega_1 \approx 0.5345 \sqrt{\frac{T}{ml}}$$



7.11

Apply a unit load to masses m_1 and m_2 along x_1 and x_2 , respectively:

$$a_{11} = \frac{1}{2k_1} ; \quad a_{22} = \frac{1}{2k_1} + \frac{1}{2k_2} = \frac{k_2 + k_1}{2k_1 k_2}$$

Dunkerley's equation gives

$$\frac{1}{\omega_1^2} = a_{11} m_1 + a_{22} m_2 = \frac{m_1}{2k_1} + m_2 \left(\frac{k_2 + k_1}{2k_1 k_2} \right) \quad (E_1)$$

Since $k_1 = k_2 = 3EI/h^3 \equiv k$, $m_1 = 2m$, $m_2 = m$ and hence (E₁) gives

$$\omega_1 = \sqrt{\frac{3EI}{2mh^3}}$$

7.12

Eg. (7.21) gives, for $r = n$,

$$R(\vec{x}) = \frac{c_n^2 \omega_n^2 + c_n^2 \sum_{i=1}^{n-1} \left(\frac{c_i}{c_n} \right)^2 \omega_i^2}{c_n^2 + c_n^2 \sum_{i=1}^{n-1} \left(\frac{c_i}{c_n} \right)^2} \quad (E_1)$$

Let $\left| \frac{c_i}{c_n} \right| = \epsilon_i \ll 1$. Then Eg. (E₁) becomes

$$\begin{aligned} R(\vec{x}) &= \frac{\omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2}{1 + \sum_{i=1}^{n-1} \epsilon_i^2} \approx \left(\omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2 \right) \left(1 - \sum_{i=1}^{n-1} \epsilon_i^2 \right) \\ &\approx \omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2 - \omega_n^2 \sum_{i=1}^{n-1} \epsilon_i^2 \\ &\approx \omega_n^2 + \sum_{i=1}^{n-1} (\omega_i^2 - \omega_n^2) \epsilon_i^2 \quad (E_2) \end{aligned}$$

Since $\omega_i^2 < \omega_n^2$, in general, (E₂) shows that

$$R(\vec{x}) \leq \omega_n^2$$

7.13

Equations of motion

$$m_1 \ddot{x}_1 + k_1 (x_1 - x_2) = 0 \quad \dots (E.1)$$

$$m_2 \ddot{x}_2 + k_1 (x_2 - x_1) + k_2 (x_2 - x_3) = 0 \quad \dots (E.2)$$

$$m_3 \ddot{x}_3 + k_2 (x_3 - x_2) = 0 \quad \dots (E.3)$$

With $x_i(t) = X_i \cos \omega t$, Eqs. (E.1) and (E.2) give

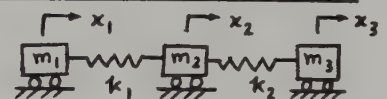
$$m_1 \omega^2 X_1 = k_1 (X_1 - X_2) ; \quad m_2 \omega^2 X_2 = -\omega^2 m_1 X_1 + k_2 (X_2 - X_3)$$

$$\text{or } X_2 = X_1 - \frac{\omega^2 m_1 X_1}{k_1} ; \quad X_3 = X_2 - \frac{\omega^2}{k_2} (m_1 X_1 + m_2 X_2)$$

since the system is free-free, the resultant force applied to mass 3,

$$F = \sum_{i=1}^3 \omega^2 m_i X_i, \text{ must be zero.}$$

The computer program, with the subroutine FUN and the output, are given below.



```

C =====
C
C  HOLTZ METHOD
C  THIS PROGRAM REQUIRES USER SUPPLIED SUBROUTINE FUN
C
C  =====
C  NT = NUMBER OF TIMES INCREMENT OF OM CHANGED
C  FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
C  NR = NUMBER OF ROOTS REQUIRED
C  NR=2  <-----
C  END OF PROBLEM-DEPENDENT DATA
C  NFUN=1
C  OM=0.1 <-----
C  CALL FUN (OM,F)
C  PRINT 60,NFUN,OM,F
C  FB=F
C  INR=0
C  OM=0.0
100  DEL=10.0 <-----
C  NT=0
200  CONTINUE
C  F1=FB
10   OM=OM+DEL
C  CALL FUN (OM,F)
C  NFUN=NFUN+1
C  PRINT 60,NFUN,OM,F
C  F2=F1+F
C  IF (F2 .LT. 0.0) GO TO 20
C  F1=F
C  GO TO 10
20   NT=NT+1
C  PRINT 30, OM,F,DEL,NFUN
30   FORMAT (//,31H CHANGE OF SIGN DETECTED AT OM=,E15.8,/,3H F=,
2   E15.8,/,5H DEL=,E15.8,/,6H NFUN=,15,/)
C  IF (NT .EQ. 6) GO TO 40
C  IF (NT .EQ. 1) OMB=OM
C  OM=OM-DEL
C  DEL=DEL/10.0
C  GO TO 200
40   INR=INR+1
C  IF (INR .EQ. NR) GO TO 50
C  OMB=OMB
C  CALL FUN (OM,F)
C  NFUN=NFUN+1
C  PRINT 60,NFUN,OM,F
C  FB=F
C  GO TO 100
50   CONTINUE
60   FORMAT (2X,6H NFUN=,I4,2X,4H OM=,E15.8,2X,3H F=,E15.8)
C  STOP
C  END
C
C  =====
C
C  SUBROUTINE FUN
C
C  =====

```

SUBROUTINE FUN (OM,F)

XM1=100.0

XM2=20.0

XM3=200.0

AK1=8000.0

AK2=4000.0

OMS=OM**2

X1=1.0

X2=(1.0-(OMS*XM1/AK1))*X1

X3=X2-(OMS/AK2)*(XM1*X1+XM2*X2)

F=OMS*(XM1*X1+XM2*X2+XM3*X3)

RETURN

END

$\omega_1 = 0$

NFUN= 1 OM= 0.10000000E+00 F= 0.31991253E+01
NFUN= 2 OM= 0.10000000E+02 F=-0.43000000E+05

CHANGE OF SIGN DETECTED AT OM= 0.10000000E+02

F=-0.43000000E+05

DEL= 0.10000000E+02

NFUN= 2

NFUN= 3 OM= 0.10000000E+01 F= 0.31126251E+03
NFUN= 4 OM= 0.20000000E+01 F= 0.11408000E+04
NFUN= 5 OM= 0.30000000E+01 F= 0.21803625E+04
NFUN= 6 OM= 0.40000000E+01 F= 0.29312000E+04
NFUN= 7 OM= 0.50000000E+01 F= 0.27265625E+04
NFUN= 8 OM= 0.60000000E+01 F= 0.76320044E+03
NFUN= 9 OM= 0.70000000E+01 F=-0.38581377E+04

CHANGE OF SIGN DETECTED AT OM= 0.70000000E+01

F=-0.38581377E+04

DEL= 0.10000000E+01

NFUN= 9

:

CHANGE OF SIGN DETECTED AT OM= 0.62220016E+01

← $\omega_2 = 6.2220016$

F=-0.28649628E+00

DEL= 0.99999990E-01

NFUN= 27

NFUN= 28 OM= 0.10000000E+02 F=-0.43000000E+05
NFUN= 29 OM= 0.20000000E+02 F=-0.47200000E+06
NFUN= 30 OM= 0.30000000E+02 F= 0.23130000E+07

CHANGE OF SIGN DETECTED AT OM= 0.30000000E+02

F= 0.23130000E+07

DEL= 0.10000000E+02

NFUN= 30

NFUN= 31 OM= 0.21000000E+02 F=-0.48851225E+06
NFUN= 32 OM= 0.22000000E+02 F=-0.47761113E+06
NFUN= 33 OM= 0.23000000E+02 F=-0.42888006E+06

CHANGE OF SIGN DETECTED AT OM= 0.25715595E+02
 F= 0.21467506E+02
 DEL= 0.99999990E-04
 NFUN= 58

$$\leftarrow \omega_3 = 25.715595$$

7.14 Eigenvalue problem is

$$\begin{bmatrix} (\lambda-2) & 1 & 0 \\ 1 & (\lambda-2) & 1 \\ 0 & 1 & (2\lambda-3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (E_1)$$

where $\lambda = \frac{m\omega^2}{k}$

If $x_1 = 1, (E_1)$ gives $x_2 = -(\lambda-2)x_1, x_3 = -x_1 - (\lambda-2)x_2,$

$E = x_2 + (2\lambda-3)x_3$ (should be zero)

The computer program and results are given.

```

C =====
C
C HOLZER METHOD
C THIS PROGRAM REQUIRES USER SUPPLIED SUBROUTINE FUN
C
C =====
C NT = NUMBER OF TIMES INCREMENT OF OM CHANGED
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
C NR = NUMBER OF ROOTS REQUIRED
      NR=3<-----
C END OF PROBLEM-DEPENDENT DATA
      NFUN=1
      OM=0.01<-----
      CALL FUN (OM,F)
      WRITE(13,60) NFUN,OM,F
      FB=F
      INR=0
      OM=0.0
100  DEL=0.25<-----
      NT=0
200  CONTINUE
      F1=FB
      10  OM=OM+DEL
      CALL FUN (OM,F)
      NFUN=NFUN+1
      WRITE(13,60) NFUN,OM,F
      F2=F1+F
      IF (F2 .LT. 0.0) GO TO 20
      F1=F
      GO TO 10
20   NT=NT+1
      WRITE(13,30) OM,F,DEL,NFUN
      30  FORMAT (//,31H CHANGE OF SIGN DETECTED AT OM=,E15.8,/,3H F=,
2  E15.8,/,5H DEL=,E15.8,/,6H NFUN=,I5,/)
      IF (NT .EQ. 6) GO TO 40
      IF (NT .EQ. 1) OMB=OM
      OM=OM-DEL
      DEL=DEL/10.0
      GO TO 200
40   INR=INR+1
  
```


IF (INR .EQ. NR) GO TO 50

OM=OMB

CALL FUN (OM,F)

NFUN=NFUN+1

WRITE(13,60) NFUN,OM,F

FB=F

GO TO 100

50 CONTINUE

60 FORMAT (2X,6H NFUN=,I4,2X,4H OM=,E15.8,2X,3H F=,E15.8)

STOP

END

C =====

C

C SUBROUTINE FUN

C

C =====

SUBROUTINE FUN (OM,F)

X1=1.0

X2=-(OM-2.0)*X1

X3=-X1-(OM-2.0)*X2

F=X2+(2.0*OM-3.0)*X3

RETURN

END

NFUN=	1	OM=	0.99999998E-02	F=-0.68310986E+01
NFUN=	2	OM=	0.25000000E+00	F=-0.34062500E+01
NFUN=	3	OM=	0.50000000E+00	F=-0.10000000E+01
NFUN=	4	OM=	0.75000000E+00	F= 0.40625000E+00

CHANGE OF SIGN DETECTED AT OM= 0.75000000E+00

F= 0.40625000E+00

DEL= 0.25000000E+00

NFUN= 4

NFUN=	5	OM=	0.52499998E+00	F=-0.81746900E+00
NFUN=	6	OM=	0.54999995E+00	F=-0.64475036E+00
NFUN=	7	OM=	0.57499993E+00	F=-0.48165655E+00
NFUN=	8	OM=	0.59999990E+00	F=-0.32800055E+00
NFUN=	9	OM=	0.62499988E+00	F=-0.18359447E+00
NFUN=	10	OM=	0.64999986E+00	F=-0.48250675E-01
NFUN=	11	OM=	0.67499983E+00	F= 0.78217864E-01

:

NFUN=	28	OM=	0.65932715E+00	F=-0.38385391E-04
NFUN=	29	OM=	0.65932965E+00	F=-0.25749207E-04
NFUN=	30	OM=	0.65933216E+00	F=-0.12755394E-04
NFUN=	31	OM=	0.65933466E+00	F=-0.11920929E-06
NFUN=	32	OM=	0.65933716E+00	F= 0.12636185E-04

CHANGE OF SIGN DETECTED AT OM= 0.65933716E+00

F= 0.12636185E-04

DEL= 0.24999997E-05

NFUN= 32

← $\lambda_1 = 0.65933716$

NFUN=	33	OM=	0.75000000E+00	F= 0.40625000E+00
-------	----	-----	----------------	-------------------

NFUN=	34	OM=	0.10000000E+01	F=	0.10000000E+01
NFUN=	35	OM=	0.12500000E+01	F=	0.96875000E+00
NFUN=	36	OM=	0.15000000E+01	F=	0.50000000E+00
NFUN=	37	OM=	0.17500000E+01	F=	-0.21875000E+00
:					
NFUN=	63	OM=	0.16789526E+01	F=	0.32067299E-04
NFUN=	64	OM=	0.16789551E+01	F=	0.24497509E-04
NFUN=	65	OM=	0.16789576E+01	F=	0.16927719E-04
NFUN=	66	OM=	0.16789601E+01	F=	0.93579292E-05
NFUN=	67	OM=	0.16789626E+01	F=	0.17881393E-05
NFUN=	68	OM=	0.16789651E+01	F=	-0.57816505E-05

CHANGE OF SIGN DETECTED AT OM= 0.16789651E+01

← $\lambda_2 = 1.6789651$

F=-0.57816505E-05

DEL= 0.24999997E-05

NFUN= 68

NFUN=	69	OM=	0.17500000E+01	F=	-0.21875000E+00
NFUN=	70	OM=	0.20000000E+01	F=	-0.10000000E+01
NFUN=	71	OM=	0.22500000E+01	F=	-0.16562500E+01
NFUN=	72	OM=	0.25000000E+01	F=	-0.20000000E+01
NFUN=	73	OM=	0.27500000E+01	F=	-0.18437500E+01
NFUN=	74	OM=	0.30000000E+01	F=	-0.10000000E+01
NFUN=	75	OM=	0.32500000E+01	F=	0.71875000E+00

:

NFUN=	95	OM=	0.31615264E+01	F=	-0.13036728E-02
NFUN=	96	OM=	0.31615515E+01	F=	-0.11179447E-02
NFUN=	97	OM=	0.31615765E+01	F=	-0.93221664E-03
NFUN=	98	OM=	0.31616015E+01	F=	-0.74636936E-03
NFUN=	99	OM=	0.31616266E+01	F=	-0.56064129E-03
NFUN=	100	OM=	0.31616516E+01	F=	-0.37479401E-03
NFUN=	101	OM=	0.31616766E+01	F=	-0.18906593E-03
NFUN=	102	OM=	0.31617017E+01	F=	-0.32186508E-05
NFUN=	103	OM=	0.31617267E+01	F=	0.18215179E-03

CHANGE OF SIGN DETECTED AT OM= 0.31617267E+01

← $\lambda_3 = 3.1617267$

F= 0.18215179E-03

DEL= 0.24999998E-04

NFUN= 103

7.15 The program listed in Problems 7.13 and 7.14 is used with
NR= 1, initial value of OM= 0.01, DEL= 0.25 and

```

=====
C
C SUBROUTINE FUN
C
C =====
      SUBROUTINE FUN (OM,F)
      X1=1.0
      X2=(2.0-OM)*X1
      X3=-X1+(2.0-OM)*X2
      F=-X2+(1.0-OM)*X3
      @1=1
      @2=(-lambda+2)*@1
      @3=-@1+(-lambda+2)*@2
      E=-@2+(-lambda+1)*@3
      ; lambda = (J_0 omega^2) / k_t

```

RETURN

END

The output of the program is given below.

NFUN=	1	DM=	0.99999998E-02	F=	0.94049907E+00
NFUN=	2	DM=	0.25000000E+00	F=	-0.20312500E+00

CHANGE OF SIGN DETECTED AT DM= 0.25000000E+00

F=-0.20312500E+00

DEL= 0.25000000E+00

NFUN= 2

NFUN=	3	DM=	0.25000000E-01	F=	0.85310948E+00
NFUN=	4	DM=	0.50000001E-01	F=	0.71237516E+00
NFUN=	5	DM=	0.75000003E-01	F=	0.57770300E+00
NFUN=	6	DM=	0.10000000E+00	F=	0.44899976E+00
NFUN=	7	DM=	0.12500000E+00	F=	0.32617188E+00
NFUN=	8	DM=	0.15000001E+00	F=	0.20912516E+00
NFUN=	9	DM=	0.17500001E+00	F=	0.97765565E-01
NFUN=	10	DM=	0.20000002E+00	F=	-0.80001354E-02

CHANGE OF SIGN DETECTED AT DM= 0.20000002E+00

F=-0.80001354E-02

DEL= 0.25000000E-01

NFUN= 10

NFUN=	11	DM=	0.17750001E+00	F=	0.86938858E-01
NFUN=	12	DM=	0.18000001E+00	F=	0.76168060E-01
NFUN=	13	DM=	0.18250000E+00	F=	0.65452933E-01
NFUN=	14	DM=	0.18500000E+00	F=	0.54793477E-01
NFUN=	15	DM=	0.18750000E+00	F=	0.44189453E-01
NFUN=	16	DM=	0.19000000E+00	F=	0.33641100E-01
NFUN=	17	DM=	0.19250000E+00	F=	0.23147941E-01
NFUN=	18	DM=	0.19499999E+00	F=	0.12710214E-01
NFUN=	19	DM=	0.19749999E+00	F=	0.23275614E-02
NFUN=	20	DM=	0.19999999E+00	F=	-0.79998970E-02

CHANGE OF SIGN DETECTED AT DM= 0.19999999E+00

F=-0.79998970E-02

DEL= 0.24999999E-02

NFUN= 20

NFUN=	21	DM=	0.19774999E+00	F=	0.12923479E-02
NFUN=	22	DM=	0.19799998E+00	F=	0.25773048E-03
NFUN=	23	DM=	0.19824998E+00	F=	-0.77641010E-03

CHANGE OF SIGN DETECTED AT DM= 0.19824998E+00

F=-0.77641010E-03

DEL= 0.24999998E-03

NFUN= 23

NFUN= 24	DM= 0.19802499E+00	F= 0.15425682E-03
NFUN= 25	DM= 0.19804999E+00	F= 0.50902367E-04
NFUN= 26	DM= 0.19807500E+00	F=-0.52571297E-04

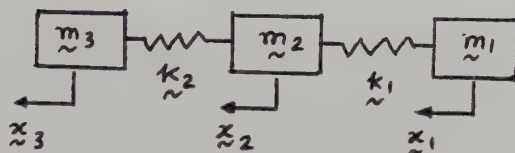
CHANGE OF SIGN DETECTED AT DM= 0.19807500E+00
 F=-0.52571297E-04
 DEL= 0.24999998E-04
 NFUN= 26

NFUN= 27	DM= 0.19805250E+00	F= 0.40531158E-04
NFUN= 28	DM= 0.19805500E+00	F= 0.30040741E-04
NFUN= 29	DM= 0.19805750E+00	F= 0.19669533E-04
NFUN= 30	DM= 0.19806001E+00	F= 0.92983246E-05
NFUN= 31	DM= 0.19806251E+00	F=-0.10728836E-05

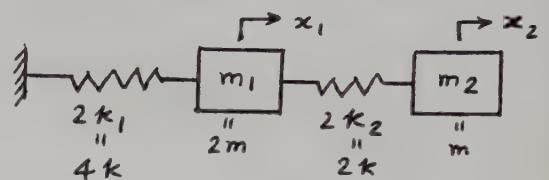
CHANGE OF SIGN DETECTED AT DM= 0.19806251E+00
 F=-0.10728836E-05
 DEL= 0.24999997E-05
 NFUN= 31

← $\lambda_1 = 0.19806251$

7.16 The system can be modeled as shown.
 The system can be redrawn as follows:



Here $\tilde{m}_1 = m$, $\tilde{k}_1 = 2k$, $\tilde{m}_2 = 2m$,
 $\tilde{k}_2 = 4k$, $\tilde{x}_1 = x_2$, $\tilde{x}_2 = x_1$, $\tilde{x}_3 = 0$,
 $\tilde{m}_3 = \text{any value}$.



Let $\tilde{x}_1 = 1$

$$\tilde{x}_2 = \tilde{x}_1 - \frac{\omega^2 \tilde{m}_1 \tilde{x}_1}{\tilde{k}_1}$$

$$= \tilde{x}_1 \left(1 - \frac{m \omega^2}{2k} \right)$$

$$\tilde{x}_3 = \tilde{x}_2 - \frac{\omega^2}{\tilde{k}_2} (\tilde{m}_1 \tilde{x}_1 + \tilde{m}_2 \tilde{x}_2)$$

$$= \tilde{x}_2 \left(1 - \frac{m \omega^2}{2k} \right) - \tilde{x}_1 \left(\frac{m \omega^2}{4k} \right)$$

$$= \tilde{x}_1 \left(1 + \frac{m^2 \omega^4}{4k^2} - \frac{5m \omega^2}{4k} \right)$$

Holzer's procedure involves assuming different values for ω^2 and finding, out of those values, the correct frequency ω as the one which gives $\tilde{x}_3 = 0$ (boundary condition to be satisfied).

From the expression for $\tilde{x}_3$, we can find the correct

frequency (with out trial and error) by setting

$$\omega^4 \left(\frac{m^2}{4k^2} \right) - \omega^2 \left(\frac{5m}{4k} \right) + 1 = 0$$

$$\text{or } \omega^2 = \frac{\left(\frac{5m}{4k} \right) \pm \sqrt{\frac{25m^2}{16k^2} - \frac{m^2}{k^2}}}{\left(\frac{m^2}{2k^2} \right)} = \frac{\frac{5m}{4k} \pm \frac{3m}{4k}}{\left(\frac{m^2}{2k^2} \right)}$$

Thus the first natural frequency is given by

$$\omega_1^2 = \left(\frac{5m}{4k} - \frac{3m}{4k} \right) / \left(\frac{m^2}{2k^2} \right) \quad \text{or} \quad \omega_1 = \sqrt{\frac{k}{m}}$$

7.17

Eqs. (7.32) to (7.35) give

$$\Theta_2 = \Theta_1 - \frac{\omega^2 J_1}{k_{t1}} \Theta_1$$

$$\Theta_3 = \Theta_2 - \frac{\omega^2}{k_{t2}} (J_1 \Theta_1 + J_2 \Theta_2)$$

$$E = \sum_{i=1}^3 \omega^2 J_i \Theta_i = \text{sum of inertia torques (should be zero)}$$

The computer program of Problems 7.13 and 7.14 is used with

NR=3, OM=0.01 and DEL=10.0.

The subroutine FUN and results are given.

```
C =====
C
C SUBROUTINE FUN
C
C =====
```

```
      SUBROUTINE FUN (OM,F)
```

```
      XJ1=10.0
```

```
      XJ2=5.0
```

```
      XJ3=1.0
```

```
      XKT1=1.0E+06
```

```
      XKT2=1.0E+06
```

```
      X1=1.0
```

```
      OMS=OM**2
```

```
      X2=X1-(OMS*XJ1/XKT1)*X1
```

```
      X3=X2-(OMS/XKT2)*(XJ1*X1+XJ2*X2)
```

```
      F=OMS*(XJ1*X1+XJ2*X2+XJ3*X3)
```

```
      RETURN
```

```
      END
```

$\omega_1 = 0$

NFUN=	1	OM=	0.99999998E-02	F=	0.16000000E-02
NFUN=	2	OM=	0.10000000E+02	F=	0.15992500E+04
NFUN=	3	OM=	0.20000000E+02	F=	0.63880034E+04
NFUN=	4	OM=	0.30000000E+02	F=	0.14339287E+05
NFUN=	5	OM=	0.40000000E+02	F=	0.25408205E+05
NFUN=	6	OM=	0.50000000E+02	F=	0.39532031E+05
NFUN=	7	OM=	0.60000000E+02	F=	0.56630336E+05
NFUN=	8	OM=	0.70000000E+02	F=	0.76605133E+05
NFUN=	9	OM=	0.80000000E+02	F=	0.99341109E+05
NFUN=	10	OM=	0.90000000E+02	F=	0.12470582E+06

⋮

NFUN= 50	DM= 0.49000000E+03	F= 0.21006358E+06
NFUN= 51	DM= 0.50000000E+03	F= 0.93750000E+05
NFUN= 52	DM= 0.51000000E+03	F=-0.32486393E+05

CHANGE OF SIGN DETECTED AT DM= 0.51000000E+03

F=-0.32486393E+05

DEL= 0.10000000E+02

NFUN= 52

⋮

NFUN= 69	DM= 0.50750012E+03	F= 0.93337460E+01
NFUN= 70	DM= 0.50750021E+03	F= 0.79214058E+01
NFUN= 71	DM= 0.50750031E+03	F= 0.66932836E+01
NFUN= 72	DM= 0.50750040E+03	F= 0.57721915E+01
NFUN= 73	DM= 0.50750049E+03	F= 0.45440674E+01
NFUN= 74	DM= 0.50750058E+03	F= 0.33159423E+01
NFUN= 75	DM= 0.50750067E+03	F= 0.20878162E+01
NFUN= 76	DM= 0.50750076E+03	F= 0.92109573E+00
NFUN= 77	DM= 0.50750085E+03	F=-0.30703202E+00

CHANGE OF SIGN DETECTED AT DM= 0.50750085E+03

F=-0.30703202E+00

DEL= 0.99999990E-04

NFUN= 77

← $\omega_2 = 507.50085$

NFUN= 78	DM= 0.51000000E+03	F=-0.32486393E+05
NFUN= 79	DM= 0.52000000E+03	F=-0.16878158E+06
NFUN= 80	DM= 0.53000000E+03	F=-0.31524272E+06

⋮

NFUN= 137	DM= 0.11000000E+04	F=-0.18694431E+07
NFUN= 138	DM= 0.11100000E+04	F=-0.62095219E+06
NFUN= 139	DM= 0.11200000E+04	F= 0.74758494E+06

CHANGE OF SIGN DETECTED AT DM= 0.11200000E+04

F= 0.74758494E+06

DEL= 0.10000000E+02

NFUN= 139

⋮

NFUN= 168	DM= 0.11146489E+04	F=-0.37916328E+02
NFUN= 169	DM= 0.11146490E+04	F=-0.23697710E+02
NFUN= 170	DM= 0.11146492E+04	F=-0.94790859E+01
NFUN= 171	DM= 0.11146493E+04	F= 0.47395439E+01

CHANGE OF SIGN DETECTED AT DM= 0.11146493E+04

F= 0.47395439E+01

DEL= 0.99999990E-04

NFUN= 171

← $\omega_3 = 1114.6493$

7.18

Eigenvector $\vec{X}^{(1)}$ corresponding to $\lambda_1 = 10.38068 (= \frac{1}{\omega_1^2})$ is given by

$$\begin{bmatrix} (2.5 - \lambda_1) & -1 & 0 \\ -1 & (5 - \lambda_1) & -\sqrt{2} \\ 0 & -\sqrt{2} & (10 - \lambda_1) \end{bmatrix} \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

i.e. $x_2^{(1)} = -7.88068 x_1^{(1)}$ and $x_3^{(1)} = -3.71497 x_2^{(1)} = 29.27649 x_1^{(1)}$

$$\vec{x}^{(1)} = \alpha \begin{Bmatrix} 1 \\ -7.88068 \\ 29.27649 \end{Bmatrix} \quad \text{where } \alpha = \text{value of } x_1^{(1)}$$

When $\vec{x}^{(1)}$ is normalized as $\vec{x}^{(1)T} [m] \vec{x}^{(1)} = 1$, $\alpha = 0.03296$

$$\therefore \vec{x}^{(1)} = \begin{Bmatrix} 0.03296 \\ -0.25975 \\ 0.96495 \end{Bmatrix}$$

$$[D_2] = [D_1] - \lambda_1 \vec{x}^{(1)} \vec{x}^{(1)T} [m]$$

$$= \begin{bmatrix} 2.5 & -1.0 & 0.0 \\ -1.9 & 5.0 & -1.4142 \\ 0.0 & -1.4142 & 10.0 \end{bmatrix} - (10.38068) \begin{Bmatrix} 0.03296 \\ -0.25975 \\ 0.96495 \end{Bmatrix} \begin{Bmatrix} 0.03296 & -0.25975 & 0.96495 \end{Bmatrix}$$

$$= \begin{bmatrix} 2.4887228 & -0.9111273 & -0.3301549 \\ -0.9111273 & 4.2996149 & 1.1876738 \\ -0.3301549 & 1.1876738 & 0.3342530 \end{bmatrix}$$

If $\vec{x}_0 = \begin{Bmatrix} 1.0 \\ -7.88068 \\ 29.27649 \end{Bmatrix}$ is used, the iterative procedure $\vec{x}_{i+1} = [D_2] \vec{x}_i$

gives the following results (with $x_{1,i+1} = 1$):

Iter. No. (i)	$\vec{x}_{i+1}$ as a row vector (with $x_{1,i+1} = 1$)
ITER= 0	0.10000000E+01-0.78806801E+01 0.29276489E+02
ITER= 1	0.10000000E+01-0.10666667E+02 0.29333334E+02
ITER= 2	0.10000000E+01-0.47272644E+01-0.13066854E+01
ITER= 3	0.10000000E+01-0.31531227E+01-0.88291872E+00
ITER= 4	0.10000000E+01-0.27448514E+01-0.77301961E+00
ITER= 5	0.10000000E+01-0.25989382E+01-0.73374254E+00
ITER= 6	0.10000000E+01-0.25411220E+01-0.71817952E+00
ITER= 7	0.10000000E+01-0.25172873E+01-0.71176374E+00
ITER= 8	0.10000000E+01-0.25073018E+01-0.70907575E+00
ITER= 9	0.10000000E+01-0.25030899E+01-0.70794201E+00
ITER= 10	0.10000000E+01-0.25013082E+01-0.70746231E+00
ITER= 11	0.10000000E+01-0.25005538E+01-0.70725930E+00
ITER= 12	0.10000000E+01-0.25002341E+01-0.70717323E+00
ITER= 13	0.10000000E+01-0.25000987E+01-0.70713681E+00
ITER= 14	0.10000000E+01-0.25000415E+01-0.70712131E+00
ITER= 15	0.10000000E+01-0.25000172E+01-0.70711482E+00

converged value of $\lambda_2 = 5.00004216$ (or $\omega_2 = 0.44721171$)

By using a similar procedure, $[D_3]$ is found. with the same $\vec{x}_0$, the results obtained from $\vec{x}_{i+1} = [D_3] \vec{x}_i$ are given below.

Iter. no. (i)	$\vec{X}_{i+1}$ as a row vector (with $X_{1,i+1} = 1$)		
ITER= 0	0.10000000E+01	-0.78806801E+01	0.29276489E+02
ITER= 1	0.10000000E+01	0.19600000E+02	-0.72824997E+02
ITER= 2	0.10000000E+01	0.38067123E+00	0.68348452E-01
ITER= 3	0.10000000E+01	0.38067168E+00	0.68312615E-01
ITER= 4	0.10000000E+01	0.38067159E+00	0.68312593E-01

Converged value of $\lambda_3 = 2.11932243$ ($\omega_3 = 0.68691260$).

(7.19) $[k]^{-1} = \frac{1}{k} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2.5 \end{bmatrix}$, $[D] = [k]^{-1} [m] = \frac{m}{k} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix}$

Using $\vec{X}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained:

Iteration number (i)	$\vec{X}_{i+1} = [D] \vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)		
ITER= 0	0.10000000E+01	0.10000000E+01	0.10000000E+01
ITER= 1	0.10000000E+01	0.17500000E+01	0.20000000E+01
ITER= 2	0.10000000E+01	0.18518518E+01	0.21481481E+01
ITER= 3	0.10000000E+01	0.18601036E+01	0.21606219E+01
ITER= 4	0.10000000E+01	0.18607503E+01	0.21616161E+01
ITER= 5	0.10000000E+01	0.18608015E+01	0.21616952E+01
ITER= 6	0.10000000E+01	0.18608055E+01	0.21617017E+01 $\leftarrow \vec{X}^{(1)}$

Converged frequency = $0.373088 = \omega_1 \sqrt{\frac{m}{k}}$

Repetition of the procedure with $[D_2]$ and $[D_3]$ gives the following results:

$$\omega_2 \sqrt{\frac{m}{k}} = 1.321324$$

$$\vec{X}^{(2)} = \begin{Bmatrix} 1.000000 & 0.254098 & -0.340706 \end{Bmatrix}$$

$$\omega_3 \sqrt{\frac{m}{k}} = 2.028523$$

$$\vec{X}^{(3)} = \begin{Bmatrix} 1.000000 & -2.114801 & 0.678847 \end{Bmatrix}$$

$$(7.20) [k]^{-1} = \frac{1}{k} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}, [k]^{-1} [m] = \frac{m}{k} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained (Program 14):

Iteration number (i)	$\vec{X}_{i+1} = [D] \vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)		
0	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	0.13333334E+01	0.14443334E+01
2	0.10000000E+01	0.13676430E+01	0.14949605E+01
3	0.10000000E+01	0.13705537E+01	0.14994360E+01
4	0.10000000E+01	0.13708007E+01	0.14998223E+01
5	0.10000000E+01	0.13708220E+01	0.14998555E+01
6	0.10000000E+01	0.13708237E+01	0.14998584E+01 ← $\vec{X}^{(1)}$

converged frequency = 0.50828409 = ω_1

0	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	-0.56849640E-01	-0.61459929E+00
2	0.10000000E+01	-0.22656711E-01	-0.64602280E+00
3	0.10000000E+01	-0.91092410E-02	-0.65840477E+00
4	0.10000000E+01	-0.38191630E-02	-0.66323972E+00
5	0.10000000E+01	-0.17635337E-02	-0.66511846E+00
6	0.10000000E+01	-0.96627331E-03	-0.66584712E+00
7	0.10000000E+01	-0.65728393E-03	-0.66612953E+00
8	0.10000000E+01	-0.53756207E-03	-0.66623896E+00
9	0.10000000E+01	-0.49118389E-03	-0.66628140E+00
10	0.10000000E+01	-0.47320157E-03	-0.66629785E+00
11	0.10000000E+01	-0.46624581E-03	-0.66630417E+00
12	0.10000000E+01	-0.46355074E-03	-0.66630661E+00
13	0.10000000E+01	-0.46252197E-03	-0.66630769E+00
14	0.10000000E+01	-0.46209697E-03	-0.66630799E+00
15	0.10000000E+01	-0.46194042E-03	-0.66630810E+00
16	0.10000000E+01	-0.46188448E-03	-0.66630810E+00
17	0.10000000E+01	-0.46187337E-03	-0.66630816E+00
18	0.10000000E+01	-0.46186216E-03	-0.66630822E+00
19	0.10000000E+01	-0.46185096E-03	-0.66630816E+00
20	0.10000000E+01	-0.46185098E-03	-0.66630822E+00 ← $\vec{X}^{(2)}$

converged frequency = 1.7323176 = ω_2

0	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	-0.23654659E+01	0.15024464E+01
2	0.10000000E+01	-0.23733659E+01	0.15024524E+01
3	0.10000000E+01	-0.23733664E+01	0.15024524E+01 ← $\vec{X}^{(3)}$

converged frequency = 2.783294 = ω_3

$$(7.21) \text{ From problem 6.18, } [k] = \frac{GJ}{l} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, [J_d] = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 0.75 & 0.50 & 0.25 \\ 0.50 & 1.00 & 0.50 \\ 0.25 & 0.50 & 0.75 \end{bmatrix}$$

With $\vec{X}_0 = \left\{ \frac{1}{3} \right\}$, the following results are obtained (Program 14):

Iteration number (i)	$\vec{X}_{i+1} = [D] \vec{X}_i$ with $x_{1,i} = 1$ (given as row vector)		
0	0.10000000E+01	0.20000000E+01	0.30000000E+01
1	0.10000000E+01	0.16000000E+01	0.14000000E+01
2	0.10000000E+01	0.14736842E+01	0.11052632E+01
3	0.10000000E+01	0.14328358E+01	0.10298507E+01
4	0.10000000E+01	0.14199134E+01	0.10086581E+01
5	0.10000000E+01	0.14159292E+01	0.10025285E+01
6	0.10000000E+01	0.14147242E+01	0.10007399E+01
7	0.10000000E+01	0.14143645E+01	0.10002166E+01
8	0.10000000E+01	0.14142580E+01	0.10000634E+01
9	0.10000000E+01	0.14142267E+01	0.10000186E+01
10	0.10000000E+01	0.14142175E+01	0.10000055E+01
11	0.10000000E+01	0.14142147E+01	0.10000015E+01 ← $\vec{X}^{(1)}$

converged frequency = 0.76536608 = ω_1

0	0.10000000E+01	0.20000000E+01	0.30000000E+01
1	0.10000000E+01	0.29291141E+00	-0.14141805E+01
2	0.10000000E+01	0.15801974E+00	-0.12234681E+01
3	0.10000000E+01	0.88471927E-01	-0.11251128E+01
4	0.10000000E+01	0.50517395E-01	-0.10714371E+01
5	0.10000000E+01	0.29161688E-01	-0.10412357E+01
⋮			
27	0.10000000E+01	0.19669531E-05	-0.99999768E+00
28	0.10000000E+01	0.18924472E-05	-0.99999750E+00
29	0.10000000E+01	0.18179417E-05	-0.99999750E+00
30	0.10000000E+01	0.18030403E-05	-0.99999738E+00
31	0.10000000E+01	0.17732382E-05	-0.99999744E+00
32	0.10000000E+01	0.17732380E-05	-0.99999726E+00 ← $\vec{X}^{(2)}$

converged frequency = 1.4142134 = ω_2

0	0.10000000E+01	0.20000000E+01	0.30000000E+01
1	0.10000000E+01	-0.14143940E+01	0.10000018E+01
2	0.10000000E+01	-0.14142135E+01	0.10000004E+01
3	0.10000000E+01	-0.14142135E+01	0.10000004E+01 ← $\vec{X}^{(3)}$

converged frequency = 1.8477590 = ω_3

7.22 From Problem 7.9, $[k] = \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $[m] = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $[k]^{-1} = \frac{l}{3T} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$[D] = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

With $\vec{X}_0 = \left\{ \frac{1}{2} \right\}$, the following results are obtained (Program 14):

Iteration number (i)	$\vec{X}_{i+1} = [D] \vec{X}_i$ with $x_{1,i} = 1$ (given as row vector)	
0	0.10000000E+01	0.20000000E+01
1	0.10000000E+01	0.12500000E+01
2	0.10000000E+01	0.10769231E+01
3	0.10000000E+01	0.10250001E+01
4	0.10000000E+01	0.10082645E+01
5	0.10000000E+01	0.10027474E+01
6	0.10000000E+01	0.10009151E+01
7	0.10000000E+01	0.10003049E+01

8 0.10000000E+01 0.10001017E+01
 9 0.10000000E+01 0.10000340E+01
 10 0.10000000E+01 0.10000113E+01
 11 0.10000000E+01 0.10000038E+01 $\leftarrow \vec{X}^{(1)}$
 converged frequency = 0.99999809 = $\omega_1 \sqrt{ml/T}$
 0 0.10000000E+01 0.20000000E+01
 1 0.10000000E+01 -0.99991989E+00
 2 0.10000000E+01 -0.99998856E+00
 3 0.10000000E+01 -0.99998856E+00 $\leftarrow \vec{X}^{(2)}$
 converged frequency = 1.7320508 = $\omega_2 \sqrt{ml/T}$

7.23 $[k]^{-1} = \frac{1}{k} \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1.0 & 1.0 & 1.0 \\ 0.5 & 1.0 & 2.0 & 2.0 \\ 0.5 & 1.0 & 2.0 & 3.0 \end{bmatrix}$, $[D] = \frac{m}{k} \begin{bmatrix} 1.5 & 1.0 & 0.5 & 0.5 \\ 1.5 & 2.0 & 1.0 & 1.0 \\ 1.5 & 2.0 & 2.0 & 2.0 \\ 1.5 & 2.0 & 2.0 & 3.0 \end{bmatrix}$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained for $\vec{X}^{(1)}$ and ω_1 .

Iteration number (i)	$\vec{X}_{i+1} = [D] \vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)			
0	0.10000000E+01	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	0.15714285E+01	0.21428571E+01	0.24285715E+01
2	0.10000000E+01	0.17200000E+01	0.25733335E+01	0.30266669E+01
3	0.10000000E+01	0.17508308E+01	0.26810634E+01	0.31838319E+01
4	0.10000000E+01	0.17574103E+01	0.27059193E+01	0.32208292E+01
5	0.10000000E+01	0.17588729E+01	0.27116063E+01	0.32293591E+01
6	0.10000000E+01	0.17592046E+01	0.27129092E+01	0.32313194E+01
7	0.10000000E+01	0.17592804E+01	0.27132087E+01	0.32317696E+01
8	0.10000000E+01	0.17592978E+01	0.27132771E+01	0.32318730E+01
9	0.10000000E+01	0.17593019E+01	0.27132931E+01	0.32318966E+01

Converged frequency = 0.40058133 = $\omega_1 \sqrt{m/k}$

7.24 The problem-dependent data for Program 13 and output are given.

```

C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION D(3,3),E(3,3)
  DATA N,ITMAX,EPS/3,200,1.0E-05/
  DATA D/3.0,-2.0,0.0,-2.0,5.0,-3.0,0.0,-3.0,3.0/
C END OF PROBLEM-DEPENDENT DATA
EIGENVALUE SOLUTION BY JACOBI METHOD

```

GIVEN MATRIX

0.300000E+01	-0.200000E+01	0.000000E+00
-0.200000E+01	0.500000E+01	-0.300000E+01
0.000000E+00	-0.300000E+01	0.300000E+01

EIGEN VALUES ARE

0.774166E+01	0.300000E+01	0.258343E+00
--------------	--------------	--------------

EIGEN VECTORS

FIRST	SECOND	THIRD
0.335734E+00	0.832048E+00	0.441564E+00
-0.796032E+00	-0.343903E-05	0.605254E+00
0.503602E+00	-0.554704E+00	0.662336E+00

7.25

The problem-dependent data for Program 13 and output are given.

```

C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
  DIMENSION D(3,3),E(3,3)
  DATA N,ITMAX,EPS/3,200,1.0E-05/
  DATA D/3.0,2.0,1.0,2.0,2.0,1.0,1.0,1.0,1.0/
C END OF PROBLEM-DEPENDENT DATA
-----
EIGENVALUE SOLUTION BY JACOBI METHOD

```

GIVEN MATRIX

0.300000E+01	0.200000E+01	0.100000E+01
0.200000E+01	0.200000E+01	0.100000E+01
0.100000E+01	0.100000E+01	0.100000E+01

EIGEN VALUES ARE

0.504892E+01	0.643104E+00	0.307979E+00
--------------	--------------	--------------

EIGEN VECTORS

FIRST	SECOND	THIRD
0.736978E+00	-0.591004E+00	0.327991E+00
0.591007E+00	0.327977E+00	-0.736982E+00
0.327985E+00	0.736984E+00	0.590999E+00

7.26

The main program which calls DECOMP and results are given.

```

C =====
C
C PROGRAM 15
C MAIN PROGRAM WHICH CALLS DECOMP
C
C =====
  DIMENSION A(4,4),U(4,4)
  DATA A/4.0,-2.0,6.0,4.0,-2.0,2.0,-1.0,3.0,6.0,-1.0,22.0,13.0,
2    4.0,3.0,13.0,46.0/
  N=4
  CALL DECOMP (A,U,N)
  WRITE (17,10)
10  FORMAT (/,25H UPPER TRIANGULAR MATRIX:,:)
  DO 30 I=1,N
    WRITE (17,20) (U(I,J),J=1,N)
20  FORMAT (3E15.8)
30  CONTINUE
    STOP
  END
-----
UPPER TRIANGULAR MATRIX:

0.20000000E+01-0.10000000E+01 0.30000000E+01
0.20000000E+01
0.00000000E+00 0.10000000E+01 0.20000000E+01
0.00000000E+00
0.00000000E+00 0.00000000E+00 0.30000000E+01
0.23333333E+01
0.00000000E+00 0.00000000E+00 0.00000000E+00
0.60461192E+01

```


7.27 From Eq. (7.84), $u_{11} = \sqrt{5} = 2.236068$, $u_{12} = \frac{a_{12}}{u_{11}} = \frac{-1}{u_{11}} = -0.44721359$,

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{1}{u_{11}} = 0.44721359$$

$$u_{22} = (a_{22} - u_{12}^2)^{1/2} = 2.408319, \quad u_{23} = \frac{1}{u_{22}} (a_{23} - u_{12} u_{13}) = -1.5778641$$

$$u_{33} = (a_{33} - u_{13}^2 - u_{23}^2)^{1/2} = 0.55708611$$

$$[U] = \begin{bmatrix} 2.236068 & -0.44721359 & 0.44721359 \\ 0 & 2.408319 & -1.5778641 \\ 0 & 0 & 0.55708611 \end{bmatrix}$$

$$\alpha_{11} = \frac{1}{u_{11}} = 0.44721359 \quad \alpha_{12} = -\frac{1}{u_{11}} (u_{12} \alpha_{22}) = 0.083045475$$

$$\alpha_{22} = \frac{1}{u_{22}} = 0.41522738 \quad \alpha_{23} = -\frac{1}{u_{22}} (u_{23} \alpha_{33}) = 1.1760702$$

$$\alpha_{33} = \frac{1}{u_{33}} = 1.7950547 \quad \alpha_{13} = -\frac{1}{u_{11}} (u_{12} \alpha_{23} + u_{13} \alpha_{33}) = -0.12379687$$

$$[U]^{-1} = [\alpha_{ij}] = \begin{bmatrix} 0.44721359 & 0.083045475 & -0.12379687 \\ 0 & 0.41522738 & 1.1760702 \\ 0 & 0 & 1.7950547 \end{bmatrix}$$

7.28 From Eqs. (7.84) and (7.86),

$$u_{11} = \sqrt{2} = 1.4142135, \quad u_{12} = \frac{a_{12}}{u_{11}} = \frac{5}{u_{11}} = 3.5355339, \quad u_{13} = \frac{a_{13}}{u_{11}} = 5.6568542$$

$$u_{22} = (a_{22} - u_{12}^2)^{1/2} = 1.8708287, \quad u_{23} = \frac{1}{u_{22}} (a_{23} - u_{12} u_{13}) = 4.2761798$$

$$u_{33} = (a_{33} - u_{13}^2 - u_{23}^2)^{1/2} = 1.9272485$$

$$[U] = \begin{bmatrix} 1.4142135 & 3.5355339 & 5.6568542 \\ 0 & 1.8708287 & 4.2761798 \\ 0 & 0 & 1.9272485 \end{bmatrix}$$

$$\alpha_{11} = \frac{1}{u_{11}} = 0.70710677 \quad \alpha_{12} = -\frac{1}{u_{11}} (u_{12} \alpha_{22}) = -1.3363062$$

$$\alpha_{22} = \frac{1}{u_{22}} = 0.53452247 \quad \alpha_{23} = -\frac{1}{u_{22}} (u_{23} \alpha_{33}) = -1.1859988$$

$$\alpha_{33} = \frac{1}{u_{33}} = 0.51887447 \quad \alpha_{13} = -\frac{1}{u_{11}} (u_{12} \alpha_{23} + u_{13} \alpha_{33}) = 0.88949931$$

$$[U]^{-1} = [\alpha_{ij}] = \begin{bmatrix} 0.70710677 & -1.3363062 & 0.88949931 \\ 0 & 0.53452247 & -1.1859988 \\ 0 & 0 & 0.51887447 \end{bmatrix}$$

7.29 The problem-dependent data to be used in Program 13 and results are given.

THE FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA

DIMENSION D(4,4), E(4,4)

DATA N, ITMAX, EPS/4, 200, 1.0E-05/

DATA D/4., -2., 6., 4., -2., 2., -1., 3., 6., -1., 22., 13., 4., 3., 13., 46./

C END OF PROBLEM-DEPENDENT DATA

EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX

```

0.400000E+01  -0.200000E+01  0.600000E+01
0.400000E+01  -0.200000E+01  0.600000E+01
-0.200000E+01  0.200000E+01  -0.100000E+01
0.300000E+01  0.200000E+01  0.220000E+02
0.600000E+01  -0.100000E+01  0.220000E+02
0.130000E+02  0.300000E+01  0.130000E+02
0.400000E+01  0.300000E+01  0.130000E+02
0.460000E+02
    
```

EIGEN VALUES ARE

```

0.525424E+02  0.178109E+02  0.346931E+01  0.177413E+00
    
```

EIGEN VECTORS

FIRST	SECOND	THIRD
0.123182E+00	-0.274316E+00	0.718788E+00
0.626834E+00	0.407039E-01	0.167114E+00
-0.614583E+00	0.769873E+00	0.407591E+00
-0.851753E+00	-0.316805E+00	-0.895647E-01
0.903902E+00	0.413933E+00	0.725754E-01
-0.797051E-01		

$$(7.30) \quad [k] = k \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad [m] = m \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Eq. (7.84) gives, for $[A] = [k]$,

$$[U] = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 1.41421 & -0.707107 & 0 \\ 0 & 0 & 1.22474 & -0.816497 \\ 0 & 0 & 0 & 0.577350 \end{bmatrix} \sqrt{k}$$

Eq. (7.86) gives

$$[U]^{-1} = \begin{bmatrix} 0.5 & 0.35355338 & 0.20412414 & 0.28867510 \\ 0 & 0.70710677 & 0.40824828 & 0.57735020 \\ 0 & 0 & 0.81649655 & 1.1547004 \\ 0 & 0 & 0 & 1.7320507 \end{bmatrix} \frac{1}{\sqrt{k}}$$

standard eigenproblem is $[D] \vec{Y} = \lambda \vec{Y}$

where

$$[D] = ([U]^{-1})^T [m] [U]^{-1}$$

$$= \frac{m}{k} \begin{bmatrix} 0.75 & 0.53033006 & 0.30618620 & 0.43301266 \\ 0.53033006 & 1.3749999 & 0.79385662 & 1.1226827 \\ 0.30618620 & 0.79385662 & 1.125 & 1.5909899 \\ 0.43301266 & 1.1226827 & 1.5909899 & 5.2499990 \end{bmatrix}$$

$$\lambda = \frac{1}{\omega^2}$$

$$\text{and } \vec{Y} = [U] \vec{X}$$

7.31

From Eq. (7.84),

$$u_{11} = \sqrt{a_{11}} = \sqrt{16} = 4, \quad u_{12} = \frac{a_{12}}{u_{11}} = \frac{-20}{4} = -5, \quad u_{13} = \frac{a_{13}}{u_{11}} = \frac{-24}{4} = -6$$

$$u_{22} = (a_{22} - u_{12}^2)^{1/2} = (89 - 25)^{1/2} = 8$$

$$u_{23} = \frac{1}{u_{22}} (a_{23} - u_{12} u_{13}) = \frac{1}{8} (-50 - (-5)(-6)) = -10$$

$$u_{33} = (a_{33} - u_{13}^2 - u_{23}^2)^{1/2} = (280 - 36 - 100)^{1/2} = 12$$

$$[A] = [U]^T [U]$$

with $[U] = \begin{bmatrix} 4 & -5 & -6 \\ 0 & 8 & -10 \\ 0 & 0 & 12 \end{bmatrix}$

7.32

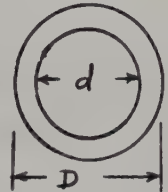
From Example 7.3, $\omega = 6.5591 \times 10^{-4} \sqrt{EI}$ Since $E = 2.07 \times 10^{11} \text{ N/m}^2$, $\omega = 298.421 \sqrt{I} \text{ rad/sec}$ In order to have $\omega = 0.5 \text{ Hz} = 3.1416 \text{ rad/sec}$,

$$298.421 \sqrt{I} = 3.1416 \Rightarrow I = 0.00011083 \text{ m}^4$$

For a tubular section,

$$I = \frac{\pi}{64} (D^4 - d^4) = 0.00011083$$

$$\text{i.e. } D^4 - d^4 = 0.0022578 \text{ m}^4$$

To minimize weight, we need to minimize $(D^2 - d^2)$.

Problem is: Find D and d to minimize $(D^2 - d^2)$
 subject to $D^4 - d^4 = 0.0022578$

or Find d and $r = D/d$

$$\text{to minimize } f = d^2(r^2 - 1) \quad (E_1)$$

$$\text{subject to } d^4(r^4 - 1) = 0.0022578 \quad (E_2)$$

$$\text{Eq. (E}_2\text{) gives } d^2 = \frac{0.047516}{\sqrt{r^4 - 1}}$$

$$\text{Eq. (E}_1\text{) becomes } f = \frac{0.047516 (r^2 - 1)}{\sqrt{r^4 - 1}}$$

For minimum of f ,

$$\frac{df}{dr} = \frac{d}{dr} \left(\frac{r^2 - 1}{\sqrt{r^4 - 1}} \right) = 0$$

$$\text{i.e., } \frac{(r^2 - 1) \cdot \frac{1}{2} (r^4 - 1)^{-1/2} (4r^3) - \sqrt{r^4 - 1} (2r)}{r^4 - 1} = 0$$

$$\text{i.e., } r^2(r^2-1) = (r^2+1)(r^2-1)$$

$$\text{i.e., } r^2=1 \text{ or } r^2=r^2+1$$

i.e., $r=1$ is the only feasible solution.

i.e., A solid circular section.

$$\text{Since } I = \frac{\pi D^4}{64} = 0.00011083, \quad D = 0.21798 \text{ m}$$

7.33 The problem-dependent data for Program 13 and the output are given below.

C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA

DIMENSION D(3,3),E(3,3)

DATA N,ITMAX,EPS/3,200,1.0E-05/

DATA D/2.5,-1.0,0.0,-1.0,5.0,-1.41421356,0.0,-1.41421356,10.0/

C END OF PROBLEM-DEPENDENT DATA

EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX

0.250000E+01	-0.100000E+01	0.000000E+00
-0.100000E+01	0.500000E+01	-0.141421E+01
0.000000E+00	-0.141421E+01	0.100000E+02

EIGEN VALUES ARE

0.103807E+02	0.500000E+01	0.211932E+01
--------------	--------------	--------------

EIGEN VECTORS

FIRST	SECOND	THIRD
0.329706E-01	0.359223E+00	0.932669E+00
-0.259784E+00	-0.898022E+00	0.355062E+00
0.965104E+00	-0.253999E+00	0.637121E-01

7.34 The problem-dependent data for Program 16 and output are given.

C FOLLOWING 5 LINES CONTAIN PROBLEM-DEPENDENT DATA

DIMENSION BK(3,3),BM(3,3),U(3,3),UI(3,3),UT1(3,3),BMU(3,3),

2- UMU(3,3),XF(3,3),EV(3,3)

DATA BK/2.,-1.,0.,-1.,3.,-2.,0.,-2.,2./

DATA BM/1.,0.,0.,0.,1.,0.,0.,0.,2./

ND=3

C END OF PROBLEM-DEPENDENT DATA

UPPER TRIANGULAR MATRIX [U]:

0.141421E+01	-0.707107E+00	0.000000E+00
0.000000E+00	0.158114E+01	-0.126491E+01
0.000000E+00	0.000000E+00	0.632456E+00

INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

0.707107E+00	0.316228E+00	0.632456E+00
0.000000E+00	0.632456E+00	0.126491E+01
0.000000E+00	0.000000E+00	0.158114E+01

MATRIX [UMU] = [UTI][M][UI]:

0.500000E+00	0.223607E+00	0.447214E+00
0.223607E+00	0.500000E+00	0.100000E+01
0.447214E+00	0.100000E+01	0.700000E+01

EIGENVALUES:

0.718421E+01	0.572771E+00	0.243019E+00
--------------	--------------	--------------

EIGENVECTORS (COLUMNWISE):

0.721309E+00	-0.664609E+00	0.194954E+00
0.134220E+01	-0.168876E+00	-0.412289E+00
0.155923E+01	0.226436E+00	0.132344E+00

7.35

The problem-dependent data and results are given (Program 16).

C FOLLOWING 5 LINES CONTAIN PROBLEM-DEPENDENT DATA

```

DIMENSION BK(4,4),BM(4,4),U(4,4),UI(4,4),UTI(4,4),BMU(4,4),
2  UMU(4,4),XF(4,4),EV(4,4)
DATA BK/4.,-2.,0.,0.,-2.,3.,-1.,0.,0.,-1.,2.,-1.,0.,0.,-1.,1./
DATA BM/3.,0.,0.,0.,0.,2.,0.,0.,0.,0.,1.,0.,0.,0.,0.,1./
ND=4

```

C END OF PROBLEM-DEPENDENT DATA

UPPER TRIANGULAR MATRIX [U]:

0.200000E+01	-0.100000E+01	0.000000E+00	0.000000E+00
0.000000E+00	0.141421E+01	-0.707107E+00	0.000000E+00
0.000000E+00	0.000000E+00	0.122474E+01	-0.816497E+00
0.000000E+00	0.000000E+00	0.000000E+00	0.577350E+00

INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

0.500000E+00	0.353553E+00	0.204124E+00	0.288675E+00
0.000000E+00	0.707107E+00	0.408248E+00	0.577350E+00
0.000000E+00	0.000000E+00	0.816497E+00	0.115470E+01
0.000000E+00	0.000000E+00	0.000000E+00	0.173205E+01

MATRIX [UMU] = [UTI][M][UI]:

0.750000E+00	0.530330E+00	0.306186E+00	0.433013E+00
0.530330E+00	0.137500E+01	0.793857E+00	0.112268E+01
0.306186E+00	0.793857E+00	0.112500E+01	0.159099E+01
0.433013E+00	0.112268E+01	0.159099E+01	0.525000E+01

EIGENVALUES:

0.623190E+01	0.143190E+01	0.500000E+00	0.336191E+00
--------------	--------------	--------------	--------------

EIGENVECTORS (COLUMNWISE):

0.480456E+00	-0.437028E+00	0.267261E+00	-0.821001E-01
0.845263E+00	-0.416241E+00	-0.267258E+00	0.202104E+00
0.130360E+01	0.206727E+00	-0.267263E+00	-0.431800E+00
0.155276E+01	0.685329E+00	0.267266E+00	0.218693E+00

7.36

The problem-dependent data and output are given (Program 16).

C FOLLOWING 5 LINES CONTAIN PROBLEM-DEPENDENT DATA

```

      DIMENSION BK(3,3),BM(3,3),U(3,3),UI(3,3),UTI(3,3),BMU(3,3),
      2  UMU(3,3),XF(3,3),EV(3,3)
      DATA BK/2.,-1.,0.,-1.,2.,-1.,0.,-1.,3./
      DATA BM/1.,0.,0.,0.,1.,0.,0.,0.,2./
      ND=3
  
```

C END OF PROBLEM-DEPENDENT DATA

UPPER TRIANGULAR MATRIX [U]:

0.141421E+01	-0.707107E+00	0.000000E+00
0.000000E+00	0.122474E+01	-0.816497E+00
0.000000E+00	0.000000E+00	0.152753E+01

INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

0.707107E+00	0.408248E+00	0.218218E+00
0.000000E+00	0.816497E+00	0.436436E+00
0.000000E+00	0.000000E+00	0.654654E+00

MATRIX [UMU] = [UTI][M][UI]:

0.500000E+00	0.288675E+00	0.154303E+00
0.288675E+00	0.833333E+00	0.445435E+00
0.154303E+00	0.445435E+00	0.109524E+01

EIGENVALUES:

0.151668E+01	0.595606E+00	0.316285E+00
--------------	--------------	--------------

EIGENVECTORS (COLUMNWISE):

0.610531E+00	-0.468617E+00	0.349193E+00
0.818509E+00	-0.150438E+00	-0.405654E+00
0.486814E+00	0.420340E+00	0.122055E+00

7.37

The main program which calls the subroutine MITER and output are given.

```

C =====
C
C PROGRAM 14
C MAIN PROGRAM FOR CALLING THE SUBROUTINE MITER
C
C =====
C FOLLOWING 8 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(3,3),X(3),XS(3),B(3,3),C(3,3),XX(3),XM(3,3),FREQ(3),
      2  EIG(3,3)
  
```

```

N=3
NVEC=3
DATA D/3.,2.,1.,2.,2.,1.,1.,1.,1./
DATA XM/1.,0.,0.,0.,1.,0.,0.,0.,1./
DATA XS/1.,1.,1./
EPS=0.00001
C END OF PROBLEM-DEPENDENT DATA
CALL MITER (D,X,XS,N,NVEC,B,C,XX,XM,EPS,FREQ,EIG)
WRITE (18,10)
10  FORMAT (//,34H SOLUTION OF EIGENVALUE PROBLEM BY,/,
2    24H MATRIX ITERATION METHOD)
WRITE (18,20) (FREQ(I),I=1,NVEC)
20  FORMAT (//,20H NATURAL FREQUENCIES,/,3(E15.8,1X))
WRITE (18,30)
30  FORMAT (//,26H MODE SHAPES (COLUMNWISE):,/)
DO 40 I=1,NVEC
40  WRITE (18,50) (EIG(I,J),J=1,N)
50  FORMAT (3(E15.8,1X),/)
STOP
END

```

```

-----
ITER= 0  0.10000000E+01 0.10000000E+01 0.10000000E+01
ITER= 1  0.10000000E+01 0.83333331E+00 0.50000000E+00
ITER= 2  0.10000000E+01 0.80645168E+00 0.45161295E+00
ITER= 3  0.10000000E+01 0.80254781E+00 0.44585988E+00
ITER= 4  0.10000000E+01 0.80201757E+00 0.44514498E+00
ITER= 5  0.10000000E+01 0.80194795E+00 0.44505492E+00
ITER= 6  0.10000000E+01 0.80193895E+00 0.44504350E+00
ITER= 7  0.10000000E+01 0.80193788E+00 0.44504207E+00
FREQ= 0.44504169E+00

```

```

ITER= 0  0.10000000E+01 0.10000000E+01 0.10000000E+01
ITER= 1  0.10000000E+01-0.36259139E+00-0.15934708E+01
ITER= 2  0.10000000E+01-0.46800232E+00-0.14036615E+01
ITER= 3  0.10000000E+01-0.51440042E+00-0.13200551E+01
ITER= 4  0.10000000E+01-0.53577375E+00-0.12815419E+01
ITER= 5  0.10000000E+01-0.54582363E+00-0.12634329E+01
ITER= 6  0.10000000E+01-0.55059475E+00-0.12548355E+01
ITER= 7  0.10000000E+01-0.55287021E+00-0.12507354E+01
ITER= 8  0.10000000E+01-0.55395770E+00-0.12487757E+01
ITER= 9  0.10000000E+01-0.55447805E+00-0.12478381E+01
ITER= 10 0.10000000E+01-0.55472714E+00-0.12473894E+01
ITER= 11 0.10000000E+01-0.55484635E+00-0.12471745E+01
ITER= 12 0.10000000E+01-0.55490345E+00-0.12470715E+01
ITER= 13 0.10000000E+01-0.55493081E+00-0.12470224E+01
ITER= 14 0.10000000E+01-0.55494392E+00-0.12469988E+01
ITER= 15 0.10000000E+01-0.55495018E+00-0.12469875E+01
ITER= 16 0.10000000E+01-0.55495316E+00-0.12469820E+01

```

FREQ= 0.12469784E+01

```

ITER= 0  0.10000000E+01 0.10000000E+01 0.10000000E+01
ITER= 1  0.10000000E+01-0.22478061E+01 0.18022429E+01
ITER= 2  0.10000000E+01-0.22469645E+01 0.18019053E+01
ITER= 3  0.10000000E+01-0.22469645E+01 0.18019053E+01

```


FREQ= 0.18019379E+01

SOLUTION OF EIGENVALUE PROBLEM BY
MATRIX ITERATION METHOD

NATURAL FREQUENCIES

0.44504169E+00 0.12469784E+01 0.18019379E+01

MODE SHAPES (COLUMNWISE):

0.10000000E+01 0.10000000E+01 0.10000000E+01

0.80193788E+00 -0.55495316E+00 -0.22469645E+01

0.44504207E+00 -0.12469820E+01 0.18019053E+01

7.38

The problem-dependent data for Program 13 and results are given.

C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA

DIMENSION D(3,3),E(3,3)

DATA N,ITMAX,EPS/3,200,1.0E-05/

DATA D/5.,-1.,1.,-1.,6.,-4.,1.,-4.,3./

C END OF PROBLEM-DEPENDENT DATA

EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX

0.500000E+01	-0.100000E+01	0.100000E+01
-0.100000E+01	0.600000E+01	-0.400000E+01
0.100000E+01	-0.400000E+01	0.300000E+01

EIGEN VALUES ARE

0.923048E+01 0.455549E+01 0.214034E+00

EIGEN VECTORS

FIRST	SECOND	THIRD
0.312915E+00	0.948172E+00	-0.552614E-01
-0.775620E+00	0.288684E+00	0.561315E+00
0.548176E+00	-0.132782E+00	0.825755E+00

7.39

The problem-dependent data for Program 13 and results are given.

C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA

DIMENSION D(3,3),E(3,3)

DATA N,ITMAX,EPS/3,200,1.0E-05/

DATA D/16.,-20.,-24.,-20.,89.,-50.,-24.,-50.,280./

C END OF PROBLEM-DEPENDENT DATA

EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX

$$\begin{array}{ccc} 0.160000E+02 & -0.200000E+02 & -0.240000E+02 \\ -0.200000E+02 & 0.890000E+02 & -0.500000E+02 \\ -0.240000E+02 & -0.500000E+02 & 0.280000E+03 \end{array}$$

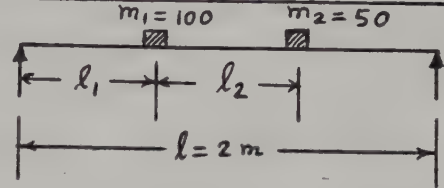
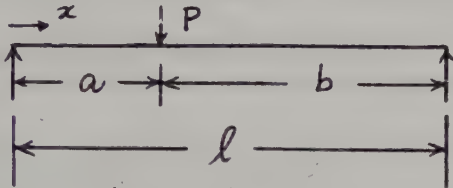
EIGEN VALUES ARE

$$0.293547E+03 \quad 0.855832E+02 \quad 0.586943E+01$$

EIGEN VECTORS

FIRST	SECOND	THIRD
0.673506E-01	-0.332765E+00	0.940601E+00
0.230709E+00	-0.922386E+00	-0.309802E+00
-0.970689E+00	0.196140E+00	0.138895E+00

7.40



Basic relationship:

$$w(x) = \begin{cases} \frac{Pbx}{6EI} (l^2 - b^2 - x^2); & 0 \leq x \leq a \\ -\frac{Pa(l-x)}{6EI} (a^2 + x^2 - 2lx); & a \leq x \leq l \end{cases} \quad (E_1)$$

Deflection of mass m_1 due to load $m_1 g$:

Using $x = l_1$, $b = 2 - l_1$ and $l = 2$ in (E_1) :

$$w_1' = \frac{(100 \times 9.81)(2 - l_1)l_1}{6EI(2)} \{4 - (2 - l_1)^2 - l_1^2\} = \frac{981 l_1^2 (2 - l_1)^2}{6EI} \quad (E_3)$$

Deflection of mass m_2 due to load $m_1 g$:

Using $x = l_1 + l_2$, $a = l_1$, $b = 2 - l_1$, $l = 2$ in (E_2) :

$$\begin{aligned} w_2' &= -\frac{(100 \times 9.81)l_1(2 - l_1 - l_2)}{6EI(2)} \{l_1^2 + (l_1 + l_2)^2 - 2(2)(l_1 + l_2)\} \\ &= -\frac{981 l_1 (2 - l_1 - l_2)}{12EI} (2l_1^2 + l_2^2 + 2l_1 l_2 - 4l_1 - 4l_2) \quad (E_4) \end{aligned}$$

Deflection of mass m_1 due to load $m_2 g$:

Using $x = l_1$, $l = 2$, $b = (2 - l_1 - l_2)$ in (E_1) :

$$\begin{aligned} w_1'' &= \frac{(50 \times 9.81)(2 - l_1 - l_2)l_1}{6EI(2)} \{4 - (2 - l_1 - l_2)^2 - l_1^2\} \\ &= \frac{490.5 l_1 (2 - l_1 - l_2) (-2l_1^2 - l_2^2 + 4l_1 + 4l_2 - 2l_1 l_2)}{12EI} \quad (E_5) \end{aligned}$$

Deflection of mass m_2 due to load $m_2 g$:

Using $x = l_1 + l_2$, $l = 2$ and $b = 2 - l_1 - l_2$ in (E₁):

$$\begin{aligned} w_2'' &= \frac{(50 \times 9.81)(2 - l_1 - l_2)(l_1 + l_2)}{6EI(2)} \{4 - (2 - l_1 - l_2)^2 - (l_1 + l_2)^2\} \\ &= \frac{490.5 (l_1 + l_2)(2 - l_1 - l_2)(-2l_1^2 - 2l_2^2 + 4l_1 + 4l_2 - 4l_1l_2)}{12EI} \quad (E_6) \end{aligned}$$

Total deflection of masses m_1 and m_2 are:

$$\begin{aligned} w_1 &= w_1' + w_1'' = \frac{981 l_1^2 (2 - l_1)^2}{6EI} + \\ &+ \frac{490.5 l_1 (2 - l_1 - l_2)(-2l_1^2 - l_2^2 + 4l_1 + 4l_2 - 2l_1l_2)}{12EI} \quad (E_7) \end{aligned}$$

$$\begin{aligned} w_2 &= w_2' + w_2'' = \frac{-981 l_1 (2 - l_1 - l_2)(2l_1^2 + l_2^2 + 2l_1l_2 - 4l_1 - 4l_2)}{12EI} \\ &+ \frac{490.5 (l_1 + l_2)(2 - l_1 - l_2)(-2l_1^2 - 2l_2^2 + 4l_1 + 4l_2 - 4l_1l_2)}{12EI} \quad (E_8) \end{aligned}$$

Fundamental natural frequency is given by

$$\omega = \left\{ \frac{g(m_1 w_1 + m_2 w_2)}{(m_1 w_1^2 + m_2 w_2^2)} \right\}^{\frac{1}{2}} = 3.1321 \left(\frac{2w_1 + w_2}{2w_1^2 + w_2^2} \right) \quad (E_9)$$

To maximize ω , we can maximize ω^2 .

Problem is:

Find l_1 and l_2

$$\text{to maximize } f = \left(\frac{2w_1 + w_2}{2w_1^2 + w_2^2} \right)$$

where w_1 and w_2 are given by (E₇) and (E₈).

Problem can be solved as follows:

Treat f as a function of l_1 and l_2 .

$$\text{Set } \frac{\partial f}{\partial l_1} = 0 \quad \text{and} \quad \frac{\partial f}{\partial l_2} = 0 \quad (E_{10})$$

Solve Eqs. (E₁₀) for l_1 and l_2 .

7.41 Stiffness of one girder (simply supported beam)

$$k_{g1} = \frac{48EI}{l^3} = \frac{48(30 \times 10^6)(\frac{1}{12}a^4)}{(30 \times 12)^3} = 2.572 a^4$$

where a = width and depth of cross-section (inch) of girder.

$$k_g = 2k_{g1} = 5.144 a^4 \text{ lb/in}$$

$$m_t = \text{mass of trolley} = \frac{40000}{386.4} = 103.5197 \text{ lb-sec}^2/\text{in}$$

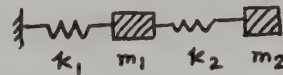
$$\text{stiffness of rope} = k_r = \frac{AE}{l} = \frac{\frac{\pi d^2}{4}(30 \times 10^6)}{(20 \times 12)} = 98175 d^2$$

where d = diameter of the rope (inch).

$$m_l = \text{mass of lifted load} = \frac{10000}{386.4} = 25.8799 \text{ lb-sec}^2/\text{in}$$

From section 5.3, the natural frequencies of a 2 d.o.f. system

(shown in adjacent figure) are given by



$$\omega_{1,2}^2 = \frac{1}{2} \left\{ \frac{(k_1 + k_2)m_2 + k_2 m_1}{m_1 m_2} \right\} \mp \left[\left\{ \frac{(k_1 + k_2)m_2 + k_2 m_1}{m_1 m_2} \right\}^2 - 4 \left\{ \frac{(k_1 + k_2)k_2 - k_2^2}{m_1 m_2} \right\} \right]^{\frac{1}{2}} \quad (E_1)$$

Here $k_1 = k_g$, $k_2 = k_r$, $m_1 = m_t$, $m_2 = m_l$

and hence ω_1^2 and ω_2^2 can be expressed as functions of a and d :

$$\omega_1^2 = \omega_1^2(a, d) \quad ; \quad \omega_2^2 = \omega_2^2(a, d) \quad (E_2)$$

Requirement is

$$\omega_1^2(a, d) > (157.08)^2 \quad ; \quad \omega_2^2(a, d) > (157.08)^2 \quad (E_3)$$

since $1500 \text{ rpm} = 157.08 \text{ rad/sec}$.

choose a and d such that the inequalities (E_3) are satisfied. A trial and error procedure can be used for this purpose.

Chapter 8

Continuous Systems

8.1 $c = \left(\frac{P}{\rho}\right)^{1/2} = \left(\frac{4000}{5}\right)^{1/2} = 28.2843 \text{ m/s}$

8.2 $\rho = \text{mass density} \times \text{area} = 7830 \left\{ \frac{\pi}{4} (2 \times 10^{-3})^2 \right\} = 24.5987 \times 10^{-3} \text{ kg/m}$

$P = 250 \text{ N}, \quad \ell = 2 \text{ m}$

(b) $c = \left(\frac{P}{\rho}\right)^{1/2} = \left(\frac{250}{24.5987 \times 10^{-3}}\right)^{1/2} = 100.8124 \text{ m/s}$

(a) $\omega_1 = \frac{c\pi}{\ell} = \frac{100.8124 \pi}{2} = 158.3561 \text{ rad/s} = 25.2031 \text{ Hz}$

8.3 $\omega_1 = 3000 (2\pi) \text{ rad/sec} = \frac{\pi c}{\ell} = \frac{\pi c}{2}$

$c = (6000 \pi \times 2 / \pi) = 12000 \text{ m/s}$

$\omega_3 = 3\pi c / \ell = 3\omega_1 = 9000 \text{ Hz}$

$c_{\text{original}} = (P/\rho)^{1/2} = 12000 \text{ m/s}$

$c_{\text{new}} = (1.2 P/\rho)^{1/2} = 1.0954 (P/\rho)^{1/2} = 1.0954 c_{\text{original}}$

$\therefore \omega_1 \text{ and } \omega_3 \text{ are increased by } 9.54\%$

8.4 $P = 30000 \text{ N}, \quad \rho = 2 \text{ kg/m}$

$c = \left(\frac{P}{\rho}\right)^{1/2} = (30000/2)^{1/2} = 122.4745 \text{ m/s}$

$\text{Time taken} = \frac{300}{122.4745} = 2.4495 \text{ s}$

8.5 At $x=0$: $P \frac{\partial^2 w}{\partial x^2} = m \frac{\partial^2 w}{\partial t^2} \quad (E_1)$

At $x=\ell$: $P \frac{\partial w}{\partial x} = -k w \quad (E_2)$

General solution is

$w(x, t) = W(x) \cdot T(t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right) (C \cos \omega t + D \sin \omega t) \quad (E_3)$

Equations (E_1) and (E_3) give:

$\left\{ P \left(-A \frac{\omega}{c} \sin \frac{\omega x}{c} + B \frac{\omega}{c} \cos \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t) \right.$
 $\left. = -m \omega^2 \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t) \right\}_{x=0}$

i.e., $A (m \omega^2) + B \left(\frac{P \omega}{c} \right) = 0 \quad (E_4)$

Eqs. (E₂) and (E₃) yield

$$P \left(-\frac{\omega}{c} A \sin \frac{\omega l}{c} + B \frac{\omega}{c} \cos \frac{\omega l}{c} \right) = -k \left(A \cos \frac{\omega l}{c} + B \sin \frac{\omega l}{c} \right)$$

i.e.,

$$A \left(-\frac{P\omega}{c} \sin \frac{\omega l}{c} + k \cos \frac{\omega l}{c} \right) + B \left(\frac{P\omega}{c} \cos \frac{\omega l}{c} + k \sin \frac{\omega l}{c} \right) = 0 \quad (E_5)$$

Eqs. (E₄) and (E₅) give the frequency equation:

$$\begin{vmatrix} (m\omega^2) & (P\omega/c) \\ \left(-\frac{P\omega}{c} \sin \frac{\omega l}{c} + k \cos \frac{\omega l}{c} \right) & \left(\frac{P\omega}{c} \cos \frac{\omega l}{c} + k \sin \frac{\omega l}{c} \right) \end{vmatrix} = 0$$

which, upon simplification, becomes

$$\tan \alpha = \left\{ \frac{Pk - \left(\frac{Pm c^2}{l^2} \right) \alpha^2}{\left(\frac{c^2 m k}{l} \right) \alpha + \left(\frac{P^2 c}{l} \right) \alpha} \right\} \quad (E_6)$$

8.6

$$l = 2 \text{ m}, \quad d = 0.5 \text{ mm}, \quad \rho = 7800 \text{ kg/m}^3, \quad \omega_n = \frac{n c \pi}{l}, \quad c = \sqrt{\frac{P}{\rho}}$$

$$(a) \quad \omega_1 = 1(2\pi) \text{ rad/sec} = \frac{\pi c}{l} = \frac{\pi}{2} \sqrt{\frac{P}{7800}}$$

$$\text{i.e.,} \quad (2\pi)^2 = \frac{\pi^2}{4} \left(\frac{P}{7800} \right)$$

$$\text{i.e.,} \quad P = 124,800 \text{ N}$$

$$(b) \quad \omega_1 = 5(2\pi) \text{ rad/sec} = \frac{\pi c}{l} = \frac{\pi}{2} \sqrt{\frac{P}{7800}}$$

$$\text{i.e.,} \quad 100\pi^2 = \frac{\pi^2}{4(7800)} P$$

$$\text{i.e.,} \quad P = 3,120,000 \text{ N}$$

8.7

Let $x=0$ be fixed and $x=l$ be connected to the pin which can move in a frictionless slot.

$$W(x) = A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}$$

$$w(0,t) = 0 \Rightarrow W(0) = 0 \Rightarrow A = 0$$

$$\frac{\partial w}{\partial x}(l,t) = 0 \Rightarrow \frac{dW}{dx}(l) = 0 \Rightarrow B \cos \frac{\omega l}{c} = 0 \Rightarrow \cos \frac{\omega l}{c} = 0$$

$$\therefore \frac{\omega l}{c} = (2n+1) \frac{\pi}{2}; \quad n = 0, 1, 2, \dots$$

$$\text{or} \quad \omega_n = \frac{(2n+1) \pi c}{2l}; \quad n = 0, 1, 2, \dots$$

8.8

$$w(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left[C_n \cos \frac{n c \pi t}{l} + D_n \sin \frac{n c \pi t}{l} \right]$$

where

$$C_n = \frac{2}{l} \int_0^l w_0(x) \sin \frac{n\pi x}{l} dx, \quad D_n = \frac{2}{\pi c n} \int_0^l \dot{w}_0(x) \sin \frac{n\pi x}{l} dx$$

Since $w_0(x) = w(x, 0) = 0$, $C_n = 0$

$$D_n = \frac{2}{\pi c n} \left[\int_0^{l/2} \frac{2ax}{l} \sin \frac{n\pi x}{l} dx + \int_{l/2}^l 2a \left(1 - \frac{x}{l}\right) \sin \frac{n\pi x}{l} dx \right]$$

$$= \frac{8al}{\pi^3 c n^3} \sin \frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} \frac{8al}{\pi^3 n^3 c} & \text{if } n \text{ is odd} \end{cases}$$

$$w(x, t) = \frac{8al}{\pi^3 c} \sum_{n=1,3,5,\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3} \sin \frac{n\pi x}{l} \sin \frac{n\pi c t}{l}$$

8.9

Eq. (8.18) is $c^2 \frac{d^2 W}{dx^2} = a W(x)$

Multiply this equation by $W(x)$ and integrate from 0 to l :

$$c^2 \int_0^l W(x) \frac{d^2 W(x)}{dx^2} dx = a \int_0^l [W(x)]^2 dx$$

This shows that the sign of a will be same as that of the integral on the left side. Integration by parts gives

$$I = \int_0^l W(x) \frac{d^2 W(x)}{dx^2} dx = W(l) \frac{dW}{dx}(l) - W(0) \frac{dW}{dx}(0) - \int_0^l \left[\frac{dW}{dx}(x) \right]^2 dx$$

Since the first two terms on the right side of this equation are zero for common boundary conditions, I and hence a will be negative.

Common boundary conditions (examples):

Fixed at ends: $W(0) = W(l) = 0$

Free at ends: $\frac{dW}{dx}(0) = \frac{dW}{dx}(l) = 0$

One end fixed and other end free: $W(0) = \frac{dW}{dx}(l) = 0$

One end fixed and other end connected to a spring:

$$W(0) = 0; \frac{dW}{dx}(l) = -k W(l)$$

8.10

$l = 2000 \text{ m}$, $\rho = 8890 \text{ kg/m}^3$, $0 \leq \omega_1 \text{ to } \omega_4 \leq 20 \text{ Hz}$

$$\omega_n = \frac{n c \pi}{l} = \frac{n \pi}{l} \sqrt{\frac{P}{\rho}}$$

$$\text{For } \omega_4 \leq 20 (2\pi) \text{ rad/sec, } \frac{4\pi}{l} \sqrt{\frac{P}{\rho}} \leq 40\pi$$

$$\text{i.e., } \sqrt{\frac{P}{8890}} \leq \frac{40\pi (2000)}{4\pi}$$

$$\text{i.e., } P \leq 35560 \times 10^8 \text{ N}$$

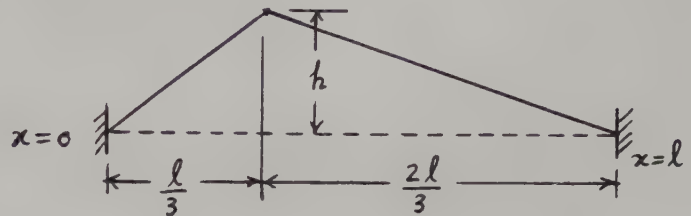
Let the permissible (yield) stress of the material be $300 \text{ MPa} = 3 \times 10^8 \text{ N/m}^2$.

Assuming the diameter of cable as 0.1 m, area of cross-section is $\frac{\pi}{4} (0.1)^2 = 0.007854 \text{ m}^2$, and the permissible tension (with factor of safety of one) is

$$(0.007854)(3 \times 10^8) = 2,356,200 \text{ N}$$

$$\therefore \text{Initial tension} = 2.3562 \times 10^6 \text{ N}$$

8.11



Solution is given by
Eq. (8.30):

$$w(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l} \quad (E_1)$$

where

$$C_n = \frac{2}{l} \int_0^l w_0(x) \sin \frac{n\pi x}{l} dx \quad (E_2)$$

The initial deflection $w_0(x)$ is given by

$$w_0(x) = \begin{cases} 3hx/l & \text{for } 0 \leq x \leq l/3 \\ 3h(l-x)/2l & \text{for } l/3 \leq x \leq l \end{cases} \quad (E_3)$$

Substitution of Eq. (E₃) into (E₂) gives

$$\begin{aligned} C_n &= \frac{2}{l} \left\{ \int_0^{l/3} \frac{3hx}{l} \sin \frac{n\pi x}{l} dx + \int_{l/3}^l \frac{3h(l-x)}{2l} \sin \frac{n\pi x}{l} dx \right\} \\ &= \frac{6h}{l^2} \left\{ \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} - \frac{x l}{n\pi} \cos \frac{n\pi x}{l} \right\}_0^{l/3} - \frac{3h}{l} \left(\frac{l}{n\pi} \right) \left(\cos \frac{n\pi x}{l} \right)_{l/3}^l \\ &\quad - \frac{3h}{l^2} \left\{ \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} - \frac{x l}{n\pi} \cos \frac{n\pi x}{l} \right\}_{l/3}^l \\ &= \frac{9h}{n^2\pi^2} \sin \frac{n\pi}{3} \\ &= \frac{9h\sqrt{3}}{2\pi^2} \text{ for } n=1, \quad \frac{9h\sqrt{3}}{8\pi^2} \text{ for } n=2, \quad 0 \text{ for } n=3, \\ &\quad - \frac{9h\sqrt{3}}{32\pi^2} \text{ for } n=4, \quad - \frac{9h\sqrt{3}}{50\pi^2} \text{ for } n=5, \quad 0 \text{ for } n=6, \dots \end{aligned}$$

$$\begin{aligned} w(x,t) &= \frac{9\sqrt{3}}{2\pi^2} h \sin \frac{\pi x}{l} \cos \frac{c\pi t}{l} + \frac{9\sqrt{3}}{8\pi^2} h \sin \frac{2\pi x}{l} \cos \frac{2c\pi t}{l} \\ &\quad - \frac{9\sqrt{3}}{32\pi^2} h \sin \frac{4\pi x}{l} \cos \frac{4c\pi t}{l} - \dots \end{aligned}$$

At $t=0$:

$$w(x,0) = \left\{ \frac{9\sqrt{3}}{2\pi^2} h \sin \frac{\pi x}{l} + \frac{9\sqrt{3}}{8\pi^2} h \sin \frac{2\pi x}{l} - \frac{9\sqrt{3}}{32\pi^2} h \sin \frac{4\pi x}{l} - \frac{9\sqrt{3}}{50\pi^2} h \sin \frac{5\pi x}{l} \right\}$$

At $t = \frac{l}{4c}$:

$$w(x, \frac{l}{4c}) = \frac{9\sqrt{3}}{2\pi^2} \frac{h}{\sqrt{2}} \sin \frac{\pi x}{l} - \frac{9\sqrt{3}}{32\pi^2} h \sin \frac{4\pi x}{l} + \frac{9\sqrt{3}}{50\pi^2} \frac{h}{\sqrt{2}} \sin \frac{5\pi x}{l}$$

At $t = \frac{l}{3c}$:

$$w(x, \frac{l}{3c}) = \frac{9\sqrt{3}}{4\pi^2} h \left\{ \sin \frac{\pi x}{l} - \frac{1}{4} \sin \frac{2\pi x}{l} + \frac{1}{16} \sin \frac{4\pi x}{l} - \frac{1}{25} \sin \frac{5\pi x}{l} \right\}$$

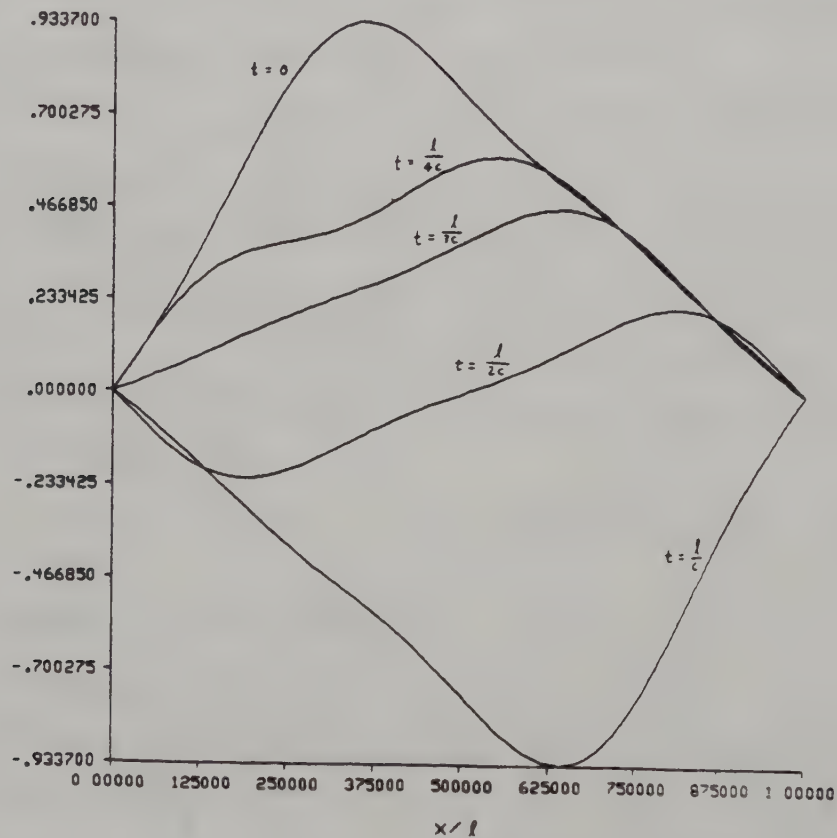
At $t = \frac{l}{2c}$:

$$w(x, \frac{l}{2c}) = -\frac{9\sqrt{3}}{8\pi^2} h \left\{ \sin \frac{2\pi x}{l} + \frac{1}{4} \sin \frac{4\pi x}{l} \right\}$$

At $t = \frac{l}{c}$:

$$w(x, \frac{l}{c}) = \frac{9\sqrt{3}}{2\pi^2} h \left\{ -\sin \frac{\pi x}{l} + \frac{1}{4} \sin \frac{2\pi x}{l} - \frac{1}{16} \sin \frac{4\pi x}{l} + \frac{1}{25} \sin \frac{5\pi x}{l} \right\}$$

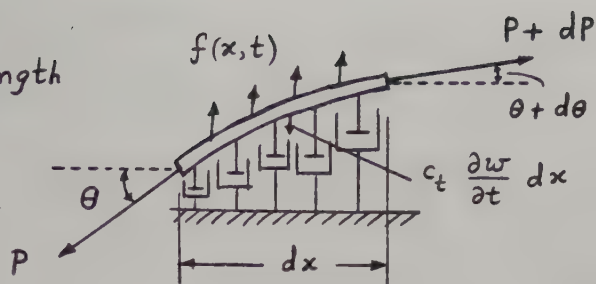
These deflection shapes are shown below:



8.12

c_t = viscous damping coefficient
= force/unit velocity/unit length

$$\begin{aligned} (P + dP) \sin(\theta + d\theta) + f(x, t) dx \\ - c_t \frac{\partial w}{\partial t} dx - P \sin \theta \\ = \rho dx \frac{\partial^2 w}{\partial t^2} \quad \dots (E_1) \end{aligned}$$



with $dP = \frac{\partial P}{\partial x} dx$ and $\sin(\theta + d\theta) \approx \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx$

Eg. (E₁) can be rewritten as

$$\frac{\partial}{\partial x} \left(P \frac{\partial w}{\partial x} \right) + f(x, t) = \rho(x) \frac{\partial^2 w(x, t)}{\partial t^2} + c_t \frac{\partial w(x, t)}{\partial t}$$

8.13

Free vibration solution is given by

$$w(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left\{ C_n \cos \frac{nc\pi t}{l} + D_n \sin \frac{nc\pi t}{l} \right\}$$

with $C_n = \frac{2}{l} \int_0^l w_0(x) \sin \frac{n\pi x}{l} dx$

and $D_n = \frac{2}{nc\pi} \int_0^l \dot{w}_0(x) \sin \frac{n\pi x}{l} dx$

Since $\dot{w}_0(x) = \frac{\partial w}{\partial t}(x, t=0) = 0$, $D_n = 0$.

For $w_0(x) = w_0 \sin \frac{\pi x}{l}$, $C_n = \frac{2w_0}{l} \int_0^l \sin \frac{\pi x}{l} \sin \frac{n\pi x}{l} dx$

Using the relations

$$\int \sin m_1 x \sin n_1 x dx = \frac{\sin(m_1 - n_1)x}{2(m_1 - n_1)} - \frac{\sin(m_1 + n_1)x}{2(m_1 + n_1)}; \quad m_1 \neq n_1$$

$$\int \sin^2 m_1 x dx = \frac{x}{2} - \frac{1}{4m_1} \sin 2m_1 x,$$

we get for $n=1$,

$$C_1 = \frac{2w_0}{l} \int_0^l \sin^2 \frac{\pi x}{l} dx = w_0$$

and for $n=2, 3, \dots$,

$$C_n = \frac{2w_0}{l} \left\{ \frac{\sin \frac{\pi}{l}(n-1)x}{2 \frac{\pi}{l}(n-1)} - \frac{\sin \frac{\pi}{l}(n+1)x}{2 \frac{\pi}{l}(n+1)} \right\}_0^l = 0$$

$$\therefore w(x, t) = w_0 \sin \frac{\pi x}{l} \cos \frac{c\pi t}{l}$$

8.14

$$u(x, t) = \left(\tilde{A} \cos \frac{\omega x}{c} + \tilde{B} \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$$

$$\frac{\partial u}{\partial x} = \frac{\omega}{c} \left(-\tilde{A} \sin \frac{\omega x}{c} + \tilde{B} \cos \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$$

$$\frac{\partial u}{\partial x}(0, t) = 0 \Rightarrow \tilde{B} = 0$$

$$\frac{\partial u}{\partial x}(l, t) = 0 \Rightarrow -\frac{\omega}{c} \tilde{A} \sin \frac{\omega l}{c} (C \cos \omega t + D \sin \omega t) = 0$$

$$\Rightarrow \sin \frac{\omega l}{c} = 0 ; \quad \frac{\omega l}{c} = n\pi$$

$$\omega_n = \frac{n\pi c}{l} = \frac{n\pi}{l} \sqrt{\frac{E}{\rho}}$$

$$u(x, t) = \sum_{n=1}^{\infty} \cos \frac{n\pi x}{l} \left[C_n \cos \frac{n\pi c t}{l} + D_n \sin \frac{n\pi c t}{l} \right]$$

$$\text{where } C_n = \frac{2}{l} \int_0^l u_0(x) \cos \frac{n\pi x}{l} dx$$

$$\text{and } D_n = \frac{2}{n\pi c} \int_0^l \dot{u}_0(x) \cos \frac{n\pi x}{l} dx$$

8.15

$$(a) \quad u(x, t) = \left(\tilde{A} \cos \frac{\omega x}{c} + \tilde{B} \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$$

$$\text{At } x=0: \quad M_1 \frac{\partial^2 u}{\partial t^2} = AE \frac{\partial u}{\partial x}$$

$$-\omega^2 M_1 \tilde{A} = AE \frac{\omega}{c} \tilde{B} \Rightarrow \tilde{B} = -\left(\frac{\omega M_1 c}{AE} \right) \tilde{A}$$

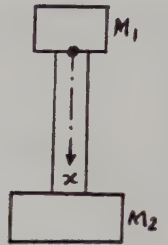
$$\text{At } x=l: \quad M_2 \frac{\partial^2 u}{\partial t^2} = -AE \frac{\partial u}{\partial x}$$

$$-\omega^2 M_2 \left(\tilde{A} \cos \frac{\omega l}{c} + \tilde{B} \sin \frac{\omega l}{c} \right) = -AE \frac{\omega}{c} \left(-\tilde{A} \sin \frac{\omega l}{c} + \tilde{B} \cos \frac{\omega l}{c} \right)$$

$$\text{i.e. } \tilde{A} \left(-\omega^2 M_2 \cos \frac{\omega l}{c} - AE \frac{\omega}{c} \sin \frac{\omega l}{c} \right) = \tilde{B} \left(-\frac{AE\omega}{c} \cos \frac{\omega l}{c} + \omega^2 M_2 \sin \frac{\omega l}{c} \right)$$

$$\text{i.e. } \omega^2 M_2 \cos \frac{\omega l}{c} + \frac{AE\omega}{c} \sin \frac{\omega l}{c} + \frac{\omega M_1 c}{AE} \left(-\omega^2 M_2 \sin \frac{\omega l}{c} + \frac{AE\omega}{c} \cos \frac{\omega l}{c} \right) = 0$$

This is the frequency equation.



$$(b) \quad u(x, t) = \left(\tilde{A} \cos \frac{\omega x}{c} + \tilde{B} \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t) \quad \dots (E_1)$$

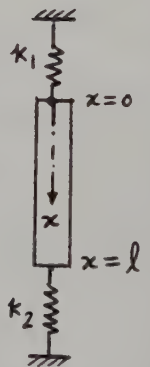
$$\text{At } x=0, \quad k_1 u = AE \frac{\partial u}{\partial x} \Rightarrow \tilde{B} = \frac{k_1 c}{AE \omega} \tilde{A} \quad \dots (E_2)$$

$$\text{At } x=l, \quad k_2 u = -AE \frac{\partial u}{\partial x}$$

$$\Rightarrow k_2 \left(\tilde{A} \cos \frac{\omega l}{c} + \tilde{B} \sin \frac{\omega l}{c} \right) = -AE \frac{\omega}{c} \left\{ -\tilde{A} \sin \frac{\omega l}{c} + \tilde{B} \cos \frac{\omega l}{c} \right\} \quad \dots (E_3)$$

Substituting (E₂) into (E₃), we get

$$\tilde{A} \left[\left(k_2 - \frac{k_1 c}{AE \omega} \right) \cos \frac{\omega l}{c} + \left(\frac{k_1 k_2 c}{AE \omega} - \frac{AE \omega}{c} \right) \sin \frac{\omega l}{c} \right] = 0$$



Hence the frequency equation is given by

$$\left(k_2 - \frac{k_1 c}{AE\omega}\right) \cos \frac{\omega l}{c} + \left(\frac{k_1 k_2 c}{AE\omega} - \frac{AE\omega}{c}\right) \sin \frac{\omega l}{c} = 0$$

$$\text{i.e., } \tan \frac{\omega l}{c} = \left(\frac{k_1 c^2 - k_2 AE\omega c}{k_1 k_2 c^2 - A^2 E^2 \omega^2} \right)$$

$$(C) \quad u(x, t) = \left(\tilde{A} \cos \frac{\omega x}{c} + \tilde{B} \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t) \quad \dots (E_1)$$

$$\text{At } x=0, \quad k u = AE \frac{\partial u}{\partial x} \Rightarrow \tilde{B} = \frac{k c}{AE\omega} \tilde{A} \quad \dots (E_2)$$

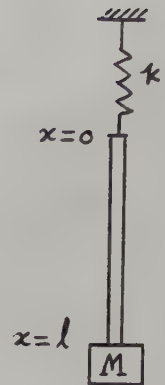
$$\begin{aligned} \text{At } x=l, \quad AE \frac{\partial u}{\partial x} &= -M \frac{\partial^2 u}{\partial t^2} \\ \Rightarrow AE \left(-\tilde{A} \frac{\omega}{c} \sin \frac{\omega l}{c} + \tilde{B} \frac{\omega}{c} \cos \frac{\omega l}{c} \right) &= M \left(\tilde{A} \cos \frac{\omega l}{c} + \tilde{B} \sin \frac{\omega l}{c} \right) \omega^2 \quad \dots (E_3) \end{aligned}$$

Substituting (E₂) into (E₃), we get

$$\frac{AE\omega}{c} \tilde{A} \left(-\sin \frac{\omega l}{c} + \frac{k c}{AE\omega} \cos \frac{\omega l}{c} \right) = M \omega^2 \tilde{A} \left(\cos \frac{\omega l}{c} + \frac{k c}{AE\omega} \sin \frac{\omega l}{c} \right)$$

This gives the frequency equation

$$\tan \frac{\omega l}{c} = \left\{ \frac{AE\omega c (k - M \omega^2)}{A^2 E^2 \omega^2 - M \omega^2 k c^2} \right\}$$



8.16

For a clamped-free bar in longitudinal vibration, fundamental frequency = $\frac{\pi c}{2l}$

If the bar is fixed at $x=0$ and attached to a mass M at $x=l$, its fundamental frequency = $\frac{\alpha_1 c}{l}$.

$$\text{Here } \frac{\alpha_1 c}{l} = \frac{1}{2} \left(\frac{\pi c}{2l} \right) \Rightarrow \alpha_1 = \frac{\pi}{4} = 0.7854$$

As an approximation, use linear relationship for the values in Table 8.1. When $\beta = 0.1$, $\alpha_1 = 0.3113$

$$\text{When } \beta = 1.0, \quad \alpha_1 = 0.8602$$

$$\text{This gives } \alpha_1(\beta) = a + b\beta \equiv 0.6099\beta + 0.2503$$

$$\text{For } \alpha_1 = 0.7854, \text{ we get } \beta = \frac{0.7854 - 0.2503}{0.6099} = 0.8774 = \frac{m}{M}$$

$$\therefore \text{Mass to be attached} = M \approx \frac{m}{0.8774} = 1.1397 m.$$

8.17

$$\text{Equation of motion for free vibration } c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad (E_1)$$

Assume separation of variables in Eq. (E₁): $u(x, t) = U(x) \cdot T(t)$

$$\text{so that } c^2 \frac{d^2 U}{dx^2} T = U \frac{d^2 T}{dt^2} \quad \text{or} \quad c^2 \frac{1}{U} \frac{d^2 U}{dx^2} = \frac{1}{T} \frac{d^2 T}{dt^2} = \omega^2 \quad \dots (E_2)$$

$$\text{i.e., } c^2 \frac{d^2 U(x)}{dx^2} + \omega^2 U(x) = 0 \quad (E_3)$$

$$\text{and } \frac{d^2 T(t)}{dt^2} + \omega^2 T(t) = 0 \quad (E_4)$$

To show that the constant a on the right hand side of Eq. (E₂) is a negative quantity, multiply Eq. (E₃) by $U(x)$ and integrate w.r.t. x from 0 to l :

$$c^2 \int_0^l U(x) \frac{d^2 U}{dx^2} dx = a \underbrace{\int_0^l \{U(x)\}^2 dx}_{\text{positive always}} \quad (E_5)$$

Eq. (E₅) shows that a will have the same sign as the integral on the left side:

$$I = \int_0^l U(x) \frac{d^2 U(x)}{dx^2} dx \quad (E_6)$$

Integrate (E₆) by parts to get

$$I = U(l) \cdot \frac{dU(l)}{dx} - U(0) \cdot \frac{dU(0)}{dx} - \int_0^l \left\{ \frac{dU(x)}{dx} \right\}^2 dx \quad (E_7)$$

Since $U(0) = 0$, the second term on the r.h.s. of (E₇) will be zero. Also, $AE \frac{dU(l)}{dx} = -k U(l)$ (E₈)

at $x=l$. Multiplication of (E₈) by $U(l)$ gives

$$U(l) \cdot \frac{dU}{dx}(l) = -k \{U(l)\}^2 < 0$$

This shows that $I < 0$ and hence $a < 0$.

In Eqs. (E₃) and (E₄), there exists an eigen (normal) function for each frequency (constant) ω . Let $U_m(x)$ and $U_n(x)$ denote the normal functions corresponding to the frequencies ω_m and ω_n , respectively. Eq. (E₃) gives

$$c^2 \frac{d^2 U_m}{dx^2} + \omega_m^2 U_m = 0 \quad \dots (E_9); \quad c^2 \frac{d^2 U_n}{dx^2} + \omega_n^2 U_n = 0 \quad \dots (E_{10})$$

Multiply (E₉) by U_n , (E₁₀) by U_m and subtract the resulting equations one from the other to get

$$c^2 U_n \frac{d^2 U_m}{dx^2} - c^2 U_m \frac{d^2 U_n}{dx^2} + (\omega_m^2 - \omega_n^2) U_m U_n = 0 \quad (E_{11})$$

Integrate (E₁₁) with respect to x from 0 to l to get, after integration by parts, the desired orthogonality relation:

$$\int_0^l U_m(x) U_n(x) dx = 0, \quad \omega_m \neq \omega_n \quad (E_{12})$$

8.18

Set up two coordinates x_1 and x_2 as shown.

$$u_1(x_1, t) = \left(\tilde{A}_1 \cos \frac{\omega x_1}{c_1} + \tilde{B}_1 \sin \frac{\omega x_1}{c_1} \right) (C \cos \omega t + D \sin \omega t)$$

$$u_2(x_2, t) = \left(\tilde{A}_2 \cos \frac{\omega x_2}{c_2} + \tilde{B}_2 \sin \frac{\omega x_2}{c_2} \right) (C \cos \omega t + D \sin \omega t)$$

$$u_1(0, t) = 0 \Rightarrow \tilde{A}_1 = 0$$

$$u_1(l_1, t) = u_2(0, t) \Rightarrow \tilde{B}_1 \sin \frac{\omega l_1}{c_1} = \tilde{A}_2$$

$$A_1 E_1 \frac{\partial u_1}{\partial x_1}(l_1, t) = A_2 E_2 \frac{\partial u_2}{\partial x_2}(0, t) = \text{tensile force same in both areas}$$

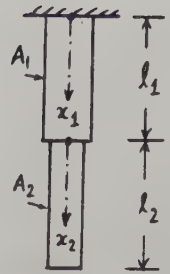
$$\text{i.e. } A_1 E_1 \tilde{B}_1 \frac{\omega}{c_1} \cos \frac{\omega l_1}{c_1} = A_2 E_2 \frac{\omega}{c_2} \tilde{B}_2 \Rightarrow \tilde{B}_2 = \frac{A_1 E_1 c_2}{A_2 E_2 c_1} \cos \frac{\omega l_1}{c_1} \cdot \tilde{B}_1$$

$$u_2(x_2, t) = \tilde{B}_1 \left(\sin \frac{\omega l_1}{c_1} \cos \frac{\omega x_2}{c_2} + \frac{A_1 E_1 c_2}{A_2 E_2 c_1} \cos \frac{\omega l_1}{c_1} \sin \frac{\omega x_2}{c_2} \right) (C \cos \omega t + D \sin \omega t)$$

$$\frac{\partial u_2}{\partial x_2}(l_2, t) = 0 \Rightarrow \tilde{B}_1 \frac{\omega}{c_2} \left\{ -\sin \frac{\omega l_1}{c_1} \sin \frac{\omega l_2}{c_2} + \frac{A_1 E_1 c_2}{A_2 E_2 c_1} \cos \frac{\omega l_1}{c_1} \cos \frac{\omega l_2}{c_2} \right\} = 0$$

$\therefore$ Frequency equation is

$$\tan \frac{\omega l_1}{c_1} \cdot \tan \frac{\omega l_2}{c_2} = \frac{A_1 E_1 c_2}{A_2 E_2 c_1}$$



8.19

(a) Axial vibration:

$$\beta = \frac{m_0}{m} = \frac{\text{mass of rod}}{\text{end mass}}$$

Using $\rho = 76.5 \text{ kN/m}^3$ for steel, we find

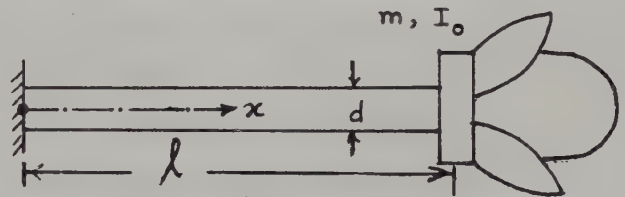
$$m_0 = \frac{\pi d^2 \ell \rho}{4} = \frac{\pi}{4} \left(\frac{5}{100} \right)^2 (1) \left(\frac{76.5 (10^3)}{9.81} \right) = 15.3117 \text{ kg}$$

$$\frac{m_0}{m} = \frac{15.3117}{100} = 0.1531$$

From Table 8.1, the value of α_1 for $\beta = 0.1531$ (using linear interpolation between values of $\beta = 0.1$ and $\beta = 1.0$) is:

$$\alpha_1 = 0.6099 (0.1531) + 0.2503 = 0.3437$$

$$\omega_1 = \frac{\alpha_1 c}{\ell} = \frac{\alpha_1}{\ell} \sqrt{\frac{E}{\rho}} = \frac{0.3437}{1} \sqrt{\frac{207 (10^9) (9.81)}{76500}} = 1770.7958 \text{ rad/sec}$$



(b) Torsional vibration:

In this case, we use the result of Example 8.6.

$$\beta = \frac{\tilde{J}_{\text{rod}}}{I_0} = \frac{J \rho \ell}{I_0}$$

where J = polar moment of inertia of the shaft:

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32} (0.05)^4 = 2.4544 (10^{-8}) \text{ m}^4$$

$$\rho = 76500/9.81 \text{ kg/m}^3 ; \ell = 1 \text{ m}$$

$$\tilde{J}_{\text{rod}} = (2.4544 (10^{-8})) \frac{76500}{9.81} (1) = 1.9140 (10^{-4}) \text{ kg-m}^2$$

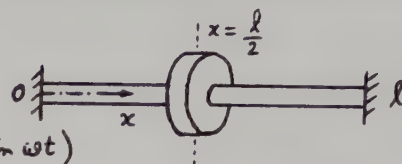
$$\beta = \frac{1.9140 (10^{-4})}{10} = 1.9140 (10^{-5})$$

Since β is close to zero, we have from Example 8.6,

$$\omega_1 \approx \frac{c}{\ell} \sqrt{\beta} = \sqrt{\frac{G\beta}{\rho}} = \sqrt{\frac{(80 (10^9)) (9.81)}{76500}} = 14.0126 \text{ rad/sec}$$

8.20

Consider half the shaft and half the disc.



$$\theta(x, t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$$

$$\theta(0, t) = 0 \Rightarrow A = 0$$

$$GJ \frac{\partial \theta}{\partial x} \left(\frac{l}{2}, t \right) = - \frac{J_0}{2} \frac{\partial^2 \theta}{\partial t^2} \left(\frac{l}{2}, t \right)$$

$$\Rightarrow GJ \frac{\omega}{c} \cdot B \cos \frac{\omega l}{2c} = \frac{J_0 \omega^2}{2} \cdot B \sin \frac{\omega l}{2c}$$

$$\Rightarrow \tan \frac{\omega l}{2c} = \frac{2GJ}{J_0 \omega c} = \frac{2c}{\omega l} \cdot \frac{GJl}{J_0 c^2}$$

Frequency equation: $\alpha \tan \alpha = \beta$ where $\alpha = \frac{\omega l}{2c}$ and $\beta = \frac{GJl}{J_0 c^2} = \left(\frac{J_0 l}{J_0} \right)$

If roots of this equation are given by α_n ,

$$\omega_n = \frac{2\alpha_n c}{l} \text{ and}$$

$$\theta(x, t) = \sum_{n=1}^{\infty} \sin \frac{\omega_n x}{c} \left(C_n \cos \omega_n t + D_n \sin \omega_n t \right) \quad \text{---- (E}_1\text{)}$$

$$\theta(x, 0) = \sum_{n=1}^{\infty} \sin \frac{\omega_n x}{c} (C_n) = 0 \Rightarrow C_n = 0 \quad \text{---- (E}_2\text{)}$$

$$\dot{\theta}(x, 0) = \sum_{n=1}^{\infty} \sin \frac{\omega_n x}{c} (\omega_n D_n) = \frac{2 \dot{\theta}_0 x}{l} \quad \text{---- (E}_3\text{)}$$

as $\dot{\theta} \Big|_{\frac{l}{2}, t=0} = \dot{\theta}_0$ and hence $\dot{\theta}(x, 0) = \frac{2 \dot{\theta}_0 x}{l}$

Multiply Eq. (E₃) by $\sin \frac{\omega_n x}{c}$ and integrate from 0 to $\frac{\pi c}{2\omega_n}$:

$$\omega_n D_n \int_0^{\left(\frac{\pi c}{2\omega_n}\right)} \sin^2 \frac{\omega_n x}{c} dx = \frac{2 \dot{\theta}_0}{l} \int_0^{\left(\frac{\pi c}{2\omega_n}\right)} x \sin \frac{\omega_n x}{c} dx \quad \text{(E}_4\text{)}$$

i.e., $D_n = \frac{8 c \dot{\theta}_0}{\pi l \omega_n^2} \quad \text{(E}_5\text{)}$

$$\therefore \theta(x, t) = \sum_{n=1}^{\infty} \left(\frac{8c \dot{\theta}_0}{\pi l \omega_n^2} \right) \sin \frac{\omega_n x}{c} \sin \omega_n t \quad (E_6)$$

8.21

$$\theta(x, t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$$

$$\theta(0, t) = 0 \Rightarrow A = 0$$

$$\theta(l, t) = 0 \Rightarrow \sin \frac{\omega l}{c} = 0 \quad ; \quad \frac{\omega l}{c} = n\pi$$

$$\therefore \omega_n = \frac{n\pi c}{l} = \frac{n\pi}{l} \sqrt{\frac{G}{J}} \quad ; \quad n = 1, 2, 3, \dots$$

8.22

Boundary conditions:

$$\text{At } x=0, \quad GJ \frac{\partial \theta}{\partial x}(0, t) = k_{t1} \theta(0, t) + c_{t1} \frac{\partial \theta}{\partial t}(0, t) + J_1 \frac{\partial^2 \theta}{\partial t^2}(0, t)$$

$$\text{At } x=l, \quad GJ \frac{\partial \theta}{\partial x}(l, t) = -k_{t2} \theta(l, t) - c_{t2} \frac{\partial \theta}{\partial t}(l, t) - J_2 \frac{\partial^2 \theta}{\partial t^2}(l, t)$$

8.23

$$\theta(x, t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t)$$

$$\theta(0, t) = 0 \Rightarrow A = 0$$

$$\frac{\partial \theta}{\partial x}(l, t) = 0 \Rightarrow \cos \frac{\omega l}{c} = 0 \quad ; \quad \frac{\omega l}{c} = \left(n - \frac{1}{2}\right) \pi$$

$$\therefore \omega_n = \left(n - \frac{1}{2}\right) \pi \sqrt{\frac{G}{Jl^2}} \quad ; \quad n = 1, 2, \dots$$

8.24

$$\theta(x, t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \right) (C \cos \omega t + D \sin \omega t) \quad \text{---- (E}_1\text{)}$$

$$\text{at } x=0: \quad J_1 \frac{\partial^2 \theta}{\partial t^2} = GJ \frac{\partial \theta}{\partial x} \quad \text{---- (E}_2\text{)}$$

$$\text{at } x=l: \quad -J_2 \frac{\partial^2 \theta}{\partial t^2} = GJ \frac{\partial \theta}{\partial x} \quad \text{---- (E}_3\text{)}$$

$$\text{Eqs. (E}_1\text{) and (E}_2\text{) give} \quad (J_1 \omega^2) A + \left(\frac{GJ\omega}{c}\right) B = 0 \quad \text{---- (E}_4\text{)}$$

$$\text{(E}_1\text{) and (E}_3\text{) give} \quad \left(J_2 \omega^2 \cos \frac{\omega l}{c} + \frac{GJ\omega}{c} \sin \frac{\omega l}{c}\right) A + \left(J_2 \omega^2 \sin \frac{\omega l}{c} - \frac{GJ\omega}{c} \cos \frac{\omega l}{c}\right) B = 0 \quad \text{---- (E}_5\text{)}$$

Setting the determinant of the coefficients of A and B in (E₄) and (E₅) equal to zero, we get the frequency equation

$$\begin{vmatrix} J_1 \omega^2 & \frac{GJ\omega}{c} \\ \left(J_2 \omega^2 \cos \frac{\omega l}{c} + \frac{GJ\omega}{c} \sin \frac{\omega l}{c}\right) & \left(J_2 \omega^2 \sin \frac{\omega l}{c} - \frac{GJ\omega}{c} \cos \frac{\omega l}{c}\right) \end{vmatrix} = 0$$

i.e.

$$J_1 J_2 \omega^2 \sin \frac{\omega l}{c} - (J_1 + J_2) \frac{GJ\omega}{c} \cos \frac{\omega l}{c} - \frac{G^2 J^2}{c^2} \sin \frac{\omega l}{c} = 0$$

8.25 Equation of motion: $c^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$ ----- (E₁)

Let steady state vibration be $\theta(x, t) = X(x) \cdot \cos \omega t$ ----- (E₂)

(E₁) and (E₂) give $\frac{d^2 X}{dx^2} + \frac{\omega^2}{c^2} X = 0$

Solution is $X(x) = A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}$

$\theta(x=0, t) = 0 \Rightarrow X(x=0) = 0 \Rightarrow A = 0$

At free end, $GJ \frac{\partial \theta}{\partial x} (x=l, t) = M_{t0} \cos \omega t$

$B GJ \frac{\omega}{c} \cos \frac{\omega l}{c} \cos \omega t = M_{t0} \cos \omega t$

$B = \frac{M_{t0} c}{GJ \omega} \sec \frac{\omega l}{c}$

$\therefore \theta(x, t) = \frac{M_{t0} c}{GJ \omega} \sec \frac{\omega l}{c} \sin \frac{\omega x}{c} \cdot \cos \omega t$

8.26 From solution of problem 8.21, $\omega_n = \frac{n\pi}{l} \sqrt{\frac{G}{J}}$
 $\therefore \omega_1 = \frac{\pi}{l} \sqrt{\frac{G}{J}} = \frac{\pi}{2} \sqrt{\frac{0.8 \times 10^{11}}{7800}} = 5030.5861 \text{ rad/sec}$

8.27 Let the angular displacement of the shaft be measured from the position occupied at the instant the shaft is stopped. Then the initial conditions are

$\theta(x, 0) = 0, \quad \frac{\partial \theta}{\partial t} (x, 0) = \omega$ ----- (E₁)

Angular displacement of shaft is

$\theta(x, t) = (A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}) (C \cos \omega t + D \sin \omega t)$ ---- (E₂)

As the shaft is supported at $x=0$ and free at $x=l$,

$\theta(0, t) = 0, \quad \frac{\partial \theta}{\partial x} (l, t) = 0$ ----- (E₃)

Eqs. (E₂) and (E₃) give

$A = 0$

$B \frac{\omega}{c} \cos \frac{\omega l}{c} = 0 \quad \text{or} \quad \frac{\omega_n l}{c} = \frac{n\pi}{2}; n=1, 3, \dots$

$\theta_n(x, t) = \sin \frac{n\pi x}{2l} (C_n \cos \omega_n t + D_n \sin \omega_n t)$ ---- (E₄)

$\theta(x, t) = \sum_{n=1, 3, \dots}^{\infty} \sin \frac{n\pi x}{2l} (C_n \cos \omega_n t + D_n \sin \omega_n t)$ ---- (E₅)

At $t=0$, $\theta(x, 0) = \sum_{n=1, 3, \dots}^{\infty} \sin \frac{n\pi x}{2l} C_n$ ---- (E₆)

$\frac{\partial \theta}{\partial t} (x, 0) = \sum_{n=1, 3, \dots}^{\infty} \sin \frac{n\pi x}{2l} \omega_n D_n = \omega$ ---- (E₇)

Since $\theta(x, 0) = 0$, $C_n = 0$. By multiplying both sides of (E₇) by $\sin \frac{n\pi x}{2l}$, integrating from 0 to l , and noting that

$$\int_0^l \sin^2 \frac{n\pi x}{2l} dx = \frac{l}{2} \quad \text{and} \quad \int_0^l \sin \frac{n\pi x}{2l} dx = \frac{2l}{n\pi},$$

we get $\frac{\omega}{n\pi} = \frac{\omega_n l}{2} D_n = \frac{n\pi c}{4} D_n \Rightarrow D_n = \frac{8l\omega}{n^2\pi^2 c}; n=1,3,\dots$

$$\theta(x,t) = \sum_{n=1,3,\dots}^{\infty} \sin \frac{n\pi x}{2l} \left(\frac{8l\omega}{n^2\pi^2 c} \right) \sin \omega_n t$$

8.28

$$I = \frac{1}{12} (0.1) (0.3)^3 = 2.25 \times 10^{-4} \text{ m}^4$$

$$A = 0.03 \text{ m}^2, \quad l = 2 \text{ m}, \quad E = 20.5 \times 10^{10} \text{ N/m}^2, \quad \rho = 7.83 \times 10^3 \text{ kg/m}^3$$

Fig. 8.15 gives the values of $\beta_n l$:

$$\omega_n = (\beta_n l)^2 \sqrt{\frac{EI}{\rho A l^4}}$$

$$\text{Here } \sqrt{\frac{EI}{\rho A l^4}} = \left\{ \frac{(20.5 \times 10^{10})(2.25 \times 10^{-4})}{7830 \times 0.03 \times 16} \right\}^{\frac{1}{2}} = 110.7814$$

(a) For pinned-pinned beam:

$$\beta_1 l = \pi, \quad \omega_1 = \pi^2 (110.7814) = 1093.3737 \text{ rad/sec}$$

$$\beta_2 l = 2\pi, \quad \omega_2 = 4\omega_1 = 4373.4948 \text{ rad/sec}$$

$$\beta_3 l = 3\pi, \quad \omega_3 = 9\omega_1 = 9840.3634 \text{ rad/sec}$$

(b) For fixed-fixed beam:

$$\beta_1 l = 4.730841, \quad \omega_1 = 2479.3826 \text{ rad/sec}$$

$$\beta_2 l = 7.853205, \quad \omega_2 = 6832.2023 \text{ rad/sec}$$

$$\beta_3 l = 10.995608, \quad \omega_3 = 13393.8474 \text{ rad/sec}$$

(c) For fixed-free beam:

$$\beta_1 l = 1.875104, \quad \omega_1 = 389.5091 \text{ rad/sec}$$

$$\beta_2 l = 4.694091, \quad \omega_2 = 2441.0117 \text{ rad/sec}$$

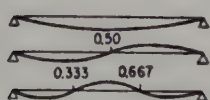
$$\beta_3 l = 7.854757, \quad \omega_3 = 6834.9030 \text{ rad/sec}$$

(d) For free-free beam:

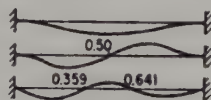
$$\beta_1 l = 0, \quad \omega_1 = 0; \quad \beta_2 l = 4.730841, \quad \omega_2 = 2479.3826 \text{ rad/sec}$$

$$\beta_3 l = 7.853205, \quad \omega_3 = 6832.2023 \text{ rad/sec.}$$

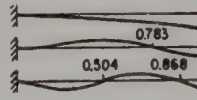
Mode shapes: The mode shapes are given in Fig. 8.15 (equations only). They result in the following.



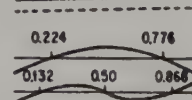
Pinned-pinned



Fixed-fixed



Fixed-free



Free-free

8.29

Boundary conditions are:

$$\text{At } x=0, \text{ fixed end} \Rightarrow W(0)=0 \dots (E_1), \quad \frac{dW}{dx}(0)=0 \dots (E_2)$$

$$\text{At } x=l, \text{ free end} \Rightarrow \frac{d^2W}{dx^2}(l)=0 \dots (E_3), \quad \frac{d^3W}{dx^3}(l)=0 \dots (E_4)$$

The deflection (normal) function is given by

$$W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x \quad (E_5)$$

from which

$$\frac{dW}{dx}(x) = \beta [-C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x] \quad (E_6)$$

$$\text{Eqs. (E}_1\text{) and (E}_5\text{) give } C_1 + C_3 = 0 \quad (E_7)$$

$$\text{Eqs. (E}_2\text{) and (E}_6\text{) yield } \beta (C_2 + C_4) = 0 \quad (E_8)$$

Use of (E₇) and (E₈) in (E₅) leads to

$$W(x) = C_1 (\cos \beta x - \cosh \beta x) + C_2 (\sin \beta x - \sinh \beta x) \quad (E_9)$$

Use of Eqs. (E₃), (E₄) and (E₉) yields

$$C_1 (\cos \beta l + \cosh \beta l) + C_2 (\sin \beta l + \sinh \beta l) = 0 \quad (E_{10})$$

$$C_1 (\sin \beta l - \sinh \beta l) - C_2 (\cos \beta l + \cosh \beta l) = 0 \quad (E_{11})$$

The frequency equation can be obtained by setting the coefficient matrix in (E₁₀) and (E₁₁) to zero as

$$\begin{vmatrix} \cos \beta l + \cosh \beta l & \sin \beta l + \sinh \beta l \\ \sin \beta l - \sinh \beta l & -\cos \beta l - \cosh \beta l \end{vmatrix} = 0 \quad (E_{12})$$

Upon simplification, Eq. (E₁₂) yields the frequency equation

$$\cos \beta l \cdot \cosh \beta l = -1 \quad (E_{13})$$

First four roots of (E₁₃) are given by

$$\beta_1 l = 1.875104, \quad \beta_2 l = 4.694091, \quad \beta_3 l = 7.854757, \quad \beta_4 l = 10.995541.$$

8.30

Boundary conditions are:

$$\text{At } x=0; \quad EI \frac{d^3 W}{dx^3} = 0 \quad \dots (E_1), \quad EI \frac{d^2 W}{dx^2} = -k_t \frac{dW}{dx} \quad \dots (E_2)$$

$$\text{At } x=l; \quad EI \frac{d^3 W}{dx^3} = k W \quad \dots (E_3), \quad EI \frac{d^2 W}{dx^2} = 0 \quad \dots (E_4)$$

Eq. (8.105) gives

$$\int_0^l W_i W_j dx = - \frac{c^2}{\omega_i^2 - \omega_j^2} \left[W_i W_j''' - W_j W_i''' + W_j' W_i'' - W_i' W_j'' \right]_0^l \quad \dots (E_5)$$

At $x=l$, Eq. (E4) gives

$$W_i'' = 0 \quad \dots (E_6), \quad W_j'' = 0 \quad \dots (E_7)$$

Eq. (E3) gives

$$EI W_i''' = k W_i \quad \dots (E_8), \quad EI W_j''' = k W_j \quad \dots (E_9)$$

Multiplying (E8) by W_j , (E9) by W_i , and subtracting one from the other, we get

$$EI (W_j''' W_i - W_i''' W_j) = k (W_j W_i - W_i W_j) = 0 \quad \dots (E_{10})$$

This shows that the r.h.s. of (E5) is zero at $x=l$.At $x=0$, Eq. (E1) gives

$$W_i''' = 0 \quad \dots (E_{11}), \quad W_j''' = 0 \quad \dots (E_{12})$$

Eq. (E2) gives

$$EI W_i'' = -k_t W_i' \quad \dots (E_{13}), \quad EI W_j'' = -k_t W_j' \quad \dots (E_{14})$$

Multiplying (E13) by W_j' , (E14) by W_i' , and subtracting one from the other, we get

$$EI (W_i'' W_j' - W_j'' W_i') = -k_t (W_i' W_j' - W_j' W_i') = 0 \quad \dots (E_{15})$$

This shows that the r.h.s. of Eq. (E5) is zero at $x=0$.

8.31

$$w(0,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = w(l,t) = \frac{\partial^2 w}{\partial x^2}(l,t) = 0; \quad t \geq 0$$

$$W(x) = C_1 (\cos \beta x + \cosh \beta x) + C_2 (\cos \beta x - \cosh \beta x)$$

$$+ C_3 (\sin \beta x + \sinh \beta x) + C_4 (\sin \beta x - \sinh \beta x)$$

$$\frac{d^2 W}{dx^2}(x) = C_1 \beta^2 (-\cos \beta x + \cosh \beta x) + C_2 \beta^2 (-\cos \beta x - \cosh \beta x)$$

$$+ C_3 \beta^2 (-\sin \beta x + \sinh \beta x) + C_4 \beta^2 (-\sin \beta x - \sinh \beta x)$$

$$W(0) = 0 \Rightarrow C_1 = 0$$

$$\frac{d^2 W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$$

$$W(l) = 0 \Rightarrow C_3 (\sin \beta l + \sinh \beta l) + C_4 (\sin \beta l - \sinh \beta l) = 0 \quad \dots (E_1)$$

$$\frac{d^2 W}{dx^2}(l) = 0 \Rightarrow C_3 (-\sin \beta l + \sinh \beta l) - C_4 (\sin \beta l + \sinh \beta l) = 0 \quad \dots (E_2)$$

Setting the determinant of the coefficient matrix of C_3 and C_4 in (E_1) and (E_2) to zero, we get the frequency equation

$$-(\sin \beta l + \sinh \beta l)^2 + (\sin \beta l - \sinh \beta l)^2 = 0$$

$$\text{or } \sin \beta l \sinh \beta l = 0$$

Since $\sinh \beta l \neq 0$, the frequency equation becomes

$$\sin \beta l = 0$$

$$\therefore \beta_n l = n\pi ; \quad \omega_n = \frac{n^2 \pi^2}{l^2} \sqrt{\frac{EI}{\rho A}} ; n = 1, 2, \dots$$

8.32

$$w(0, t) = \frac{\partial^2 w}{\partial x^2}(0, t) = \frac{\partial^2 w}{\partial x^2}(l, t) = \frac{\partial^3 w}{\partial x^3}(l, t) = 0 ; t \geq 0$$

$$W(x) = C_1(\cos \beta x + \cosh \beta x) + C_2(\cos \beta x - \cosh \beta x) \\ + C_3(\sin \beta x + \sinh \beta x) + C_4(\sin \beta x - \sinh \beta x)$$

$$\frac{d^2 W}{dx^2}(x) = C_1 \beta^2 (-\cos \beta x + \cosh \beta x) + C_2 \beta^2 (-\cos \beta x - \cosh \beta x) \\ + C_3 \beta^2 (-\sin \beta x + \sinh \beta x) + C_4 \beta^2 (-\sin \beta x - \sinh \beta x)$$

$$\frac{d^3 W}{dx^3}(x) = C_1 \beta^3 (\sin \beta x + \sinh \beta x) + C_2 \beta^3 (\sin \beta x - \sinh \beta x) \\ + C_3 \beta^3 (-\cos \beta x + \cosh \beta x) + C_4 \beta^3 (-\cos \beta x - \cosh \beta x)$$

$$W(0) = 0 \Rightarrow C_1 = 0$$

$$\frac{d^2 W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$$

$$\frac{d^2 W}{dx^2}(l) = 0 \Rightarrow C_3(-\sin \beta l + \sinh \beta l) - C_4(\sin \beta l + \sinh \beta l) = 0 \quad \text{---- (E}_1\text{)}$$

$$\frac{d^3 W}{dx^3}(l) = 0 \Rightarrow C_3(-\cos \beta l + \cosh \beta l) - C_4(\cos \beta l + \cosh \beta l) = 0 \quad \text{---- (E}_2\text{)}$$

Setting the determinant of coefficients of C_3 and C_4 in Eqs. (E_1) and (E_2) to zero, the frequency equation can be obtained:

$$-(-\sin \beta l + \sinh \beta l)(\cos \beta l + \cosh \beta l) + (\sin \beta l + \sinh \beta l)(-\cos \beta l + \cosh \beta l) = 0$$

$$\text{or } \tan \beta l - \tanh \beta l = 0$$

8.33

For a simply supported beam, the first three natural frequencies are given by

$$\omega_1 = (\beta_1 l)^2 (EI/\rho A l^4)^{\frac{1}{2}} = \pi^2 (EI/\rho A l^4)^{\frac{1}{2}}$$

$$\omega_2 = (\beta_2 l)^2 (EI/\rho A l^4)^{\frac{1}{2}} = 4\pi^2 (EI/\rho A l^4)^{\frac{1}{2}}$$

$$\omega_3 = (\beta_3 l)^2 (EI/\rho A l^4)^{\frac{1}{2}} = 9\pi^2 (EI/\rho A l^4)^{\frac{1}{2}}$$

For steel, $E = 2.07 \times 10^{11} \text{ N/m}^2$, $\rho = 7880 \text{ kg/m}^3$, $l = 1.0 \text{ m}$.

Setting $\omega_1 \geq 1500 \text{ Hz} = 9424.8 \text{ rad/sec} = \pi^2 \left\{ \frac{(2.07 \times 10^{11}) I}{(7880) A (1)} \right\}^{\frac{1}{2}}$
we get

$$(9424.8)^2 \geq \pi^4 \left(\frac{2.07 \times 10^{11}}{7880} \right) \left(\frac{I}{A} \right)$$

or $\frac{I}{A} \leq 0.03471 \quad (E_1)$

Setting $\omega_3 \leq 5000 \text{ Hz} = 31416.0 \text{ rad/sec} = 9\pi^2 \left\{ \frac{(2.07 \times 10^{11}) I}{7880 A (1)} \right\}^{\frac{1}{2}}$
we get

$$\frac{I}{A} \geq 0.0047617$$

Let the beam have a rectangular section

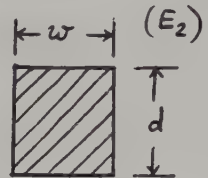
$$A = wd, \quad I = \frac{1}{12} wd^3, \quad \frac{I}{A} = \frac{d^2}{12}$$

Let $\frac{I}{A} = 0.005$ to satisfy inequalities (E_1) and (E_2) .

Then $d^2 = 0.06$ or $d = 0.2449 \text{ m}$

Taking $w = 0.1 \text{ m}$ (w can have any value), we get

$$A = wd = 0.02449 \text{ m}^2 \quad \text{and} \quad I = 1.2240 \times 10^{-4} \text{ m}^4.$$



8.34

From the solution of problem 8.31, we find for $\sin \beta l = 0$,

$$C_3 = C_4$$

Hence $w_n(x) = C_n \sin \beta_n x = C_n \sin \frac{n\pi x}{l}$

General free vibration solution is

$$w(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

When beam vibrates in the first mode,

$$w(x, t) = \sin \frac{\pi x}{l} (A_1 \cos \omega_1 t + B_1 \sin \omega_1 t)$$

Bending moment in the beam is

$$\begin{aligned} M(x, t) &= EI \frac{\partial^2 w}{\partial x^2}(x, t) = -\frac{\pi^2 EI}{l^2} \sin \frac{\pi x}{l} (A_1 \cos \omega_1 t + B_1 \sin \omega_1 t) \\ &= -\frac{\pi^2}{l^2} EI w(x, t) \end{aligned}$$

If the amplitude of vibration is Y , the maximum bending moment is

$$|M_{\max}| = \frac{\pi^2}{l^2} EI Y$$

For the given data,

$$|M_{\max}| = \frac{\pi^2}{(1)^2} (20.5 \times 10^{10}) \left(\frac{10^3}{10^{12}} \right) \left(\frac{10}{1000} \right) = 20.2328 \text{ N-m}$$

8.35

$$W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x$$

$$\frac{d^2 W}{dx^2}(x) = -C_1 \beta^2 \cos \beta x - C_2 \beta^2 \sin \beta x + C_3 \beta^2 \cosh \beta x + C_4 \beta^2 \sinh \beta x$$

$$\frac{d^3 W}{dx^3}(x) = C_1 \beta^3 \sin \beta x - C_2 \beta^3 \cos \beta x + C_3 \beta^3 \sinh \beta x + C_4 \beta^3 \cosh \beta x$$

$$EI \frac{d^2 W}{dx^2}(0) = 0 \Rightarrow C_1 = C_3 \quad \text{--- (E}_1\text{)}$$

$$EI \frac{d^3 W}{dx^3}(0) = -\tilde{k}_1 W(0) \Rightarrow EI \beta^3 (-C_2 + C_4) + \tilde{k}_1 (C_1 + C_3) = 0$$

$$C_2 (-EI \beta^3) + C_3 (+2 \tilde{k}_1) + C_4 (EI \beta^3) = 0 \quad \text{--- (E}_2\text{)}$$

$$EI \frac{d^2 W}{dx^2}(l) = 0 \Rightarrow C_2 (-\sin \beta l) + C_3 (\cosh \beta l - \cos \beta l) + C_4 (\sinh \beta l) = 0$$

$$EI \frac{d^3 W}{dx^3}(l) = +\tilde{k}_2 W(l) \Rightarrow \quad \text{--- (E}_3\text{)}$$

$$C_2 (-EI \beta^3 \cos \beta l - \tilde{k}_2 \sin \beta l) + C_3 (EI \beta^3 \sin \beta l + EI \beta^3 \sinh \beta l - \tilde{k}_2 \cos \beta l - \tilde{k}_2 \cosh \beta l) + C_4 (EI \beta^3 \cosh \beta l - \tilde{k}_2 \sinh \beta l) = 0$$

Setting the determinant of the coefficients of C_2 , C_3 and C_4 in (E₂) to (E₄) to zero, we get the frequency equation:

$-EI \beta^3$	$+2 \tilde{k}_1$	$EI \beta^3$	$= 0$
$-\sin \beta l$	$(\cosh \beta l - \cos \beta l)$	$\sinh \beta l$	
$(-EI \beta^3 \cos \beta l - \tilde{k}_2 \sin \beta l)$	$(EI \beta^3 \{\sin \beta l + \sinh \beta l\} - \tilde{k}_2 \{\cos \beta l + \cosh \beta l\})$	$(EI \beta^3 \cosh \beta l - \tilde{k}_2 \sinh \beta l)$	

8.36

Because of symmetry, consider only half of beam to the left of M. Boundary conditions are:

$$W(0, t) = 0, \quad EI \frac{\partial^2 W}{\partial x^2}(0, t) = 0$$

$$\text{For symmetry, } \frac{\partial W}{\partial x} = 0 \text{ at } x = \frac{l}{2}.$$

$$EI \frac{\partial^3 W}{\partial x^3} = +\frac{M}{2} \frac{\partial^2 W}{\partial t^2} \text{ at } x = \frac{l}{2}$$

$$W(x) = C_1 (\cos \beta x + \cosh \beta x) + C_2 (\cos \beta x - \cosh \beta x) + C_3 (\sin \beta x + \sinh \beta x) + C_4 (\sin \beta x - \sinh \beta x)$$

$$W(0) = 0 \Rightarrow C_1 = 0$$

$$\frac{d^2 W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$$

$$W(x) = C_3 (\sin \beta x + \sinh \beta x) + C_4 (\sin \beta x - \sinh \beta x)$$

$$\frac{dW}{dx}(x) = \beta C_3 (\cos \beta x + \cosh \beta x) + \beta C_4 (\cos \beta x - \cosh \beta x)$$

$$\frac{d^3 W}{dx^3}(x) = \beta^3 C_3 (-\cos \beta x + \cosh \beta x) - \beta^3 C_4 (\cos \beta x + \cosh \beta x)$$

$$\frac{dW}{dx}\left(\frac{l}{2}\right) = 0 \Rightarrow C_3 \left(\cos \frac{\beta l}{2} + \cosh \frac{\beta l}{2}\right) + C_4 \left(\cos \frac{\beta l}{2} - \cosh \frac{\beta l}{2}\right) = 0 \quad \text{--- (E}_1\text{)}$$

$$EI \frac{d^3 W}{dx^3}\left(\frac{l}{2}\right) = -\frac{M}{2} \omega^2 W\left(\frac{l}{2}\right) \Rightarrow$$

$$EI \beta^3 \left[C_3 \left(-\cos \frac{\beta l}{2} + \cosh \frac{\beta l}{2}\right) - C_4 \left(\cos \frac{\beta l}{2} + \cosh \frac{\beta l}{2}\right) \right] = -\frac{M}{2} \omega^2 \left[C_3 \left(\sin \frac{\beta l}{2} + \sinh \frac{\beta l}{2}\right) + C_4 \left(\sin \frac{\beta l}{2} - \sinh \frac{\beta l}{2}\right) \right] \quad \text{--- (E}_2\text{)}$$

i.e.

$$R_1 \cos \frac{\beta l}{2} + R_2 \cosh \frac{\beta l}{2} = 0$$

$$R_1 \left(-\cos \frac{\beta l}{2} + \lambda \sin \frac{\beta l}{2}\right) + R_2 \left(\cosh \frac{\beta l}{2} + \lambda \sinh \frac{\beta l}{2}\right) = 0$$

where

$$R_1 = C_3 + C_4, \quad R_2 = C_3 - C_4, \quad \lambda = \frac{M \omega^2}{2EI \beta^3}$$

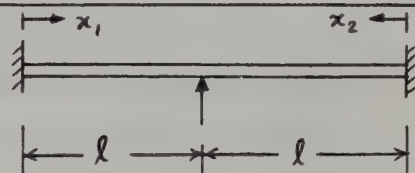
By setting the coefficient matrix of R_1 and R_2 equal to zero, we obtain the frequency equation:

$$\cos \frac{\beta l}{2} (\cosh \frac{\beta l}{2} + \lambda \sinh \frac{\beta l}{2}) + \cosh \frac{\beta l}{2} (\cos \frac{\beta l}{2} + \lambda \sin \frac{\beta l}{2}) = 0$$

$$\text{or} \quad \lambda (-\tanh \frac{\beta l}{2} + \tan \frac{\beta l}{2}) = 2$$

8.37

For convenience, set up two coordinates x_1 and x_2 as shown.



$$W_1(x_1) = C_1 \cos \beta x_1 + C_2 \sin \beta x_1 + C_3 \cosh \beta x_1 + C_4 \sinh \beta x_1$$

$$\frac{dW_1}{dx_1}(x_1) = -C_1 \beta \sin \beta x_1 + C_2 \beta \cos \beta x_1 + C_3 \beta \sinh \beta x_1 + C_4 \beta \cosh \beta x_1$$

$$W_1(0) = 0 \Rightarrow C_1 + C_3 = 0; \quad C_1 = -C_3$$

$$\frac{dW_1}{dx_1}(0) = 0 \Rightarrow C_2 + C_4 = 0; \quad C_2 = -C_4$$

$$W_1(x_1) = C_3 (\cosh \beta x_1 - \cos \beta x_1) + C_4 (\sinh \beta x_1 - \sin \beta x_1)$$

$$W_1(l) = 0 \Rightarrow C_3 = -C_4 \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right)$$

$$W_1(x_1) = C_4 \left[(\sinh \beta x_1 - \sin \beta x_1) - \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right) (\cosh \beta x_1 - \cos \beta x_1) \right] \quad \text{--- (E}_1\text{)}$$

$$W_2(x_2) = C'_1 \cos \beta x_2 + C'_2 \sin \beta x_2 + C'_3 \cosh \beta x_2 + C'_4 \sinh \beta x_2$$

$$\frac{dW_2}{dx_2}(x_2) = -c'_1 \beta \sin \beta x_2 + c'_2 \beta \cos \beta x_2 + c'_3 \beta \sinh \beta x_2 + c'_4 \beta \cosh \beta x_2$$

$$W_2(0) = 0 \Rightarrow c'_1 = -c'_3$$

$$\frac{dW_2}{dx_2}(0) = 0 \Rightarrow c'_2 = -c'_4$$

$$W_2(l) = 0 \Rightarrow c'_3 = -c'_4 \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right)$$

$$W_2(x_2) = c'_4 \left[(\sinh \beta x_2 - \sin \beta x_2) - \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right) (\cosh \beta x_2 - \cos \beta x_2) \right] \quad \text{--- (E}_2\text{)}$$

$$\frac{dW_1}{dx_1}(x_1=l) = - \frac{dW_2}{dx_2}(x_2=l) \quad \text{gives}$$

$$c_4 \left[(\cosh \beta l - \cos \beta l) - \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right) (\sinh \beta l + \sin \beta l) \right] + c'_4 \left[(\cosh \beta l - \cos \beta l) - \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right) (\sinh \beta l + \sin \beta l) \right] = 0 \quad \text{--- (E}_3\text{)}$$

$$\frac{d^2 W_1}{dx_1^2}(x_1=l) = - \frac{d^2 W_2}{dx_2^2}(x_2=l) \quad \text{gives}$$

$$c_4 \left[(\sinh \beta l + \sin \beta l) - \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right) (\cosh \beta l + \cos \beta l) \right] + c'_4 \left[(\sinh \beta l + \sin \beta l) - \left(\frac{\sin \beta l - \sinh \beta l}{\cos \beta l - \cosh \beta l} \right) (\cosh \beta l + \cos \beta l) \right] = 0 \quad \text{--- (E}_4\text{)}$$

Since the determinant of the coefficients of c_4 and c'_4 in (E₃) and (E₄) is identically zero, we set the coefficients of c_4 and c'_4 in (E₃) and (E₄) to zero. This leads to the frequency equations

$$\cos \beta l \cosh \beta l = 1$$

$$\text{and } \tan \beta l = \tanh \beta l$$

8.38

General free vibration solution of a simply supported beam is

$$w(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

$$\text{At } t=0, \quad \frac{\partial w}{\partial t}(x, 0) = 0 \quad \text{and} \quad EI \frac{\partial^4 w}{\partial x^4}(x, 0) = f_0$$

$$\text{Thus } \sum_{n=1}^{\infty} B_n \omega_n \sin \frac{n\pi x}{l} = 0 \quad \text{or} \quad B_n = 0$$

$$\text{and } EI \sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{l} \right)^4 \sin \frac{n\pi x}{l} = f_0 \quad \text{--- (E}_1\text{)}$$

Multiplying Eq. (E₁) by $\sin \frac{n\pi x}{l}$ and integrating from 0 to l, we obtain

$$A_n = \frac{2l^3}{EI n^4 \pi^4} \int_0^l f_0 \sin \frac{n\pi x}{l} dx = \frac{2l^4 f_0}{EI n^5 \pi^5} (1 - \cos n\pi) \\ = \frac{4f_0 l^4}{EI n^5 \pi^5} ; n = 1, 3, 5, \dots$$

$\therefore$ Solution is $w(x, t) = \frac{4f_0 l^4}{\pi^5 EI} \sum_{n=1,3,\dots}^{\infty} \frac{1}{n^5} \sin \frac{n\pi x}{l} \cos \omega_n t$

8.39

Let $W(x) = (1 - \frac{x}{l})^2$; $\frac{dW}{dx} = -\frac{2}{l}(1 - \frac{x}{l})$; $\frac{d^2W}{dx^2} = \frac{2}{l^2}$

$$N = \int_0^l EI \left(\frac{d^2W}{dx^2} \right)^2 dx = \frac{EI_0}{l} \int_0^l x \left(\frac{4}{l^2} \right) dx = \frac{4EI_0}{l^3}$$

$$D = \int_0^l \rho A \{W(x)\}^2 dx = \frac{\rho A_0}{l} \int_0^l x \left(1 - \frac{x}{l} \right)^4 dx = \frac{\rho A_0 l}{30}$$

$$\omega^2 = \frac{N}{D} \simeq \frac{4EI_0 (30)}{l^3 (\rho A_0 l)} = \frac{120 EI_0}{\rho A_0 l^4}$$

$$\omega \simeq \sqrt{120} \left(\frac{EI_0}{\rho A_0 l^4} \right)^{1/2}$$

8.40

(a) Equation of motion for a uniform beam is

$$EI \frac{\partial^4 w}{\partial x^4} = f(x, t) - \rho A \frac{\partial^2 w}{\partial t^2} \quad \text{---- (E}_1\text{)}$$

Let $w(x, t) = \sum_{n=1}^{\infty} W_n(x) T_n(t)$ where $W_n(x) = n^{\text{th}}$ normal mode satisfying the boundary conditions and the equation

$$EI \frac{d^4 W}{dx^4} = \omega_n^2 \rho A W_n ; n = 1, 2, \dots \quad \text{---- (E}_2\text{)}$$

Eq. (E₁) becomes for the assumed solution,

$$EI \sum_{n=1}^{\infty} \frac{d^4 W}{dx^4} T_n(t) = f(x, t) - \rho A \sum_{n=1}^{\infty} W_n(x) \frac{d^2 T_n}{dt^2} \quad \text{---- (E}_3\text{)}$$

which, in view of (E₂), becomes

$$\sum_{n=1}^{\infty} \omega_n^2 W_n(x) T_n(t) = \frac{1}{\rho A} f(x, t) - \sum_{n=1}^{\infty} W_n(x) \frac{d^2 T_n}{dt^2} \quad \text{---- (E}_4\text{)}$$

Multiplying Eq. (E₄) by $W_m(x)$ and integrating from 0 to l, we get

$$\frac{d^2 T_n}{dt^2} + \omega_n^2 T_n = \frac{2}{\rho A l} F_n(t) \quad \text{---- (E}_5\text{)}$$

where $F_n(t) = \int_0^l f(x, t) W_n(x) dx \quad \text{---- (E}_6\text{)}$

Note that the orthogonality of normal modes, namely,

$$\int_0^l W_m(x) W_n(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ l/2 & \text{if } m = n \end{cases}$$

is used in deriving (E5). The solution of Eq. (E5) can be written as

$$T_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{2}{\rho A l \omega_n} \int_0^l F_n(\tau) \sin \omega_n(t-\tau) d\tau$$

where A_n and B_n are constants to be determined from initial conditions. Thus the total solution can be expressed as ----- (E7)

$$w(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{2}{\rho A l \omega_n} \int_0^t F_n(\tau) \sin \omega_n(t-\tau) d\tau \right] W_n(x) \quad \text{----- (E8)}$$

(b) For a simply supported beam,

$$W_n(x) = \sin \frac{n\pi x}{l}$$

For the given loading, (E6) becomes $F_n(t) = F_0 \sin \frac{n\pi a}{l} \sin \omega t$

Hence (E7) becomes

$$T_n(t) = \frac{2 F_0 l^3}{EI \pi^4} \sin \frac{n\pi a}{l} \left[\frac{1}{n^4 - \left(\frac{\omega}{\omega_1}\right)^2} \sin \omega t - \frac{\left(\frac{\omega}{\omega_1}\right)}{n^2 \{n^4 - \left(\frac{\omega}{\omega_1}\right)^2\}} \sin n^2 \omega_1 t \right]$$

where $\omega_1 = \frac{\pi^2}{l^2} \sqrt{\frac{EI}{\rho A}}$ = fundamental frequency of the beam

Thus the forced response of the beam is given by

$$w(x,t) = \frac{2 F_0 l^3}{\pi^4 EI} \left[\sum_{n=1}^{\infty} \frac{\sin \left(\frac{n\pi a}{l}\right)}{n^4 - \left(\frac{\omega}{\omega_1}\right)^2} \sin \frac{n\pi x}{l} \right] \sin \omega t \\ - \frac{2 \left(\frac{\omega}{\omega_1}\right) F_0 l^3}{\pi^4 EI} \left[\sum_{n=1}^{\infty} \frac{\sin \left(\frac{n\pi a}{l}\right)}{n^2 \{n^4 - \left(\frac{\omega}{\omega_1}\right)^2\}} \sin \frac{n\pi x}{l} \sin n^2 \omega_1 t \right]$$

8.41

Derivation of Eq. (E5):

If shear deformation is zero, $M = EI \frac{\partial^2 w}{\partial x^2}$

$$\text{Eq. (8.99)} \Rightarrow \frac{\partial V}{\partial x} = -\rho A \frac{\partial^2 w}{\partial t^2} + f$$

$$\text{Eq. (8.100)} \Rightarrow \frac{\partial M}{\partial x} - V = \rho I \frac{\partial^2 \phi}{\partial t^2}, \quad \frac{\partial^2 M}{\partial x^2} - \frac{\partial V}{\partial x} = \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2}$$

$$\text{or } \frac{\partial^2 M}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} - f = \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2}$$

$$\text{or } EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} - f = 0 \quad \text{----- (E.1)}$$

Derivation of Eq. (E6):

Eq. (E.1) becomes, for $f=0$,

$$\frac{EI}{\rho A} \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - \frac{I}{A} \frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \quad \text{---- (E.2)}$$

If $w(x, t) = C \sin \frac{n\pi x}{l} \cos \omega_n t$, Eq. (E.2) becomes

$$\alpha^2 \left(\frac{n\pi}{l} \right)^4 - \omega_n^2 - r^2 \left(\frac{n\pi}{l} \right)^2 \omega_n^2 = 0$$

$$\text{i.e.} \quad \omega_n^2 = \frac{\alpha^2 \left(\frac{n\pi}{l} \right)^4}{1 + r^2 \left(\frac{n\pi}{l} \right)^2} = \frac{\alpha^2 n^4 \pi^4}{l^4 \left(1 + \frac{n^2 \pi^2 r^2}{l^2} \right)} \quad \left\{ \begin{array}{l} \alpha^2 = \frac{EI}{\rho A} \\ r^2 = \frac{I}{A} \end{array} \right.$$

8.42

Derivation of (E.7):

If rotary inertia is neglected,

$$M = EI \frac{\partial \phi}{\partial x} \quad (E.1)$$

$$V = \kappa AG \left(\phi - \frac{\partial w}{\partial x} \right) \quad (E.2)$$

Eq. (8.99) $\Rightarrow$

$$-\frac{\partial V}{\partial x} + f = \rho A \frac{\partial^2 w}{\partial t^2} \quad (E.3)$$

Eq. (8.100) $\Rightarrow$

$$\frac{\partial M}{\partial x} - V = 0 \quad (E.4)$$

Using (E.1) and (E.2), (E.3) and (E.4) can be written as

$$-\kappa AG \left(\frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2} \right) + f = \rho A \frac{\partial^2 w}{\partial t^2}$$

$$\text{or} \quad \frac{\partial \phi}{\partial x} = \frac{\partial^2 w}{\partial x^2} + \frac{f}{\kappa AG} - \frac{\rho A}{\kappa AG} \frac{\partial^2 w}{\partial t^2} \quad (E.5)$$

$$\text{and} \quad EI \frac{\partial^2 \phi}{\partial x^2} - \kappa AG \left(\phi - \frac{\partial w}{\partial x} \right) = 0$$

$$\text{or} \quad EI \frac{\partial^2}{\partial x^2} \left(\frac{\partial \phi}{\partial x} \right) - \kappa AG \frac{\partial \phi}{\partial x} + \kappa AG \frac{\partial^2 w}{\partial x^2} = 0 \quad (E.6)$$

Substitution of (E.5) into (E.6) gives

$$EI \frac{\partial^4 w}{\partial x^4} + \frac{EI}{\kappa AG} \frac{\partial^2 f}{\partial x^2} - \frac{\rho A EI}{\kappa AG} \frac{\partial^4 w}{\partial x^2 \partial t^2} - f + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \quad (E.7)$$

$$\text{If } f=0, \text{ (E.7) becomes} \quad EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \frac{\rho EI}{\kappa G} \frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \quad (E.8)$$

Derivation of (E.8):

With $w(x, t) = C \sin \frac{n\pi x}{l} \cos \omega_n t$, (E.8) gives

$$\omega_n^2 = \frac{\alpha^2 n^4 \pi^4}{l^4 \left(1 + \frac{E}{\kappa G} \cdot \frac{r^2 n^2 \pi^2}{l^2} \right)}$$

8.43

Multiply Eq. (8.83) by $w(x)$ and integrate from 0 to l :

$$\int_0^l w(x) \frac{d^4 w}{dx^4} dx = \frac{\omega^2}{c^2} \int_0^l [w(x)]^2 dx$$

This shows that the sign of ω^2 will be same as that of the

left hand side term. Integrating the left hand side expression by parts, we get

$$\left. \frac{d^3 W}{dx^3}(x) \cdot W(x) \right|_0^l - \left. \frac{d^2 W}{dx^2}(x) \cdot \frac{dW}{dx}(x) \right|_0^l + \int_0^l \left[\frac{d^2 W}{dx^2}(x) \right]^2 dx \quad (E_1)$$

For common boundary conditions, the first two terms in (E_1) will be zero, and (E_1) will be positive. Hence ω^2 is positive.

Common boundary conditions:

Simply supported end: $W(x) = \frac{d^2 W}{dx^2} = 0$

Fixed end: $W(x) = \frac{dW}{dx} = 0$

Free end: $\frac{d^2 W}{dx^2}(x) = \frac{d^3 W}{dx^3} = 0$

8.44

Equation of motion:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = F_0 \sin \omega t \quad (1)$$

For steady state response, we assume the particular solution as

$$w(x, t) = W(x) \sin \omega t \quad (2)$$

Substitution of Eq. (2) into (1) gives

$$\frac{d^4 W}{dx^4} - \frac{\omega^2}{c^2} W = \frac{F_0}{\rho A c^2} \quad (3)$$

$$\text{where } c^2 = \frac{EI}{\rho A} \quad (4)$$

The complete solution of Eq. (3) can be written as

$$W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x - \frac{F_0}{\rho A c^2} \quad (5)$$

$$\text{where } \beta^2 = \frac{\omega^2}{c^2} \quad (6)$$

The boundary conditions for a simply supported beam are

$$W(x=0) = 0 \quad (7)$$

$$\frac{d^2 W}{dx^2}(x=0) = 0 \quad (8)$$

$$W(x=l) = 0 \quad (9)$$

$$\frac{d^2 W}{dx^2}(x=l) = 0 \quad (10)$$

Equations (7) and (5) give:

$$C_1 + C_3 - \frac{F_0}{\rho A c^2} = 0 \quad (11)$$

Equations (8) and (5) yield:

$$-C_1 \beta^2 + C_3 \beta^2 = 0 \quad \text{or} \quad C_1 = C_3 \quad (12)$$

Eqs. (11) and (12) provide:

$$C_1 = C_3 = \frac{F_0}{2 \rho A c^2} \quad (13)$$

Eqs. (9) and (5) give:

$$C_2 \sin \beta \ell + C_4 \sinh \beta \ell = -\frac{F_0}{\rho A c^2} \left(\frac{\cos \beta \ell + \cosh \beta \ell}{2} - 1 \right) \quad (14)$$

Eqs. (10) and (5) yield:

$$-C_2 \sin \beta \ell + C_4 \sinh \beta \ell = \frac{F_0}{2 \rho A c^2} (\cos \beta \ell - \cosh \beta \ell) \quad (15)$$

Solution of Eqs. (14) and (15) gives:

$$C_4 = \frac{F_0}{2 \rho A c^2 \sinh \beta \ell} (1 - \cosh \beta \ell) = -\frac{F_0}{2 \rho A c^2} \tanh \frac{\beta \ell}{2} \quad (16)$$

$$C_2 = \frac{F_0}{2 \rho A c^2 \sin \beta \ell} (1 - \cos \beta \ell) = \frac{F_0}{2 \rho A c^2} \tan \frac{\beta \ell}{2} \quad (17)$$

Thus the complete solution becomes

$$w(x, t) = \frac{F_0}{2 \rho A c^2} \left[\left(\cos \beta x + \cosh \beta x \right) + \tan \frac{\beta \ell}{2} \sin \beta x - \tanh \frac{\beta \ell}{2} \sinh \beta x - 2 \right] \sin \omega t$$

8.45

$$\omega = \frac{1000}{60} (2 \pi) = 104.72 \text{ rad/sec.}$$

$$F(t) = m e \omega^2 \sin \omega t = (0.5) (104.72^2) \sin 104.72 t = F_0 \sin \omega t \quad (1)$$

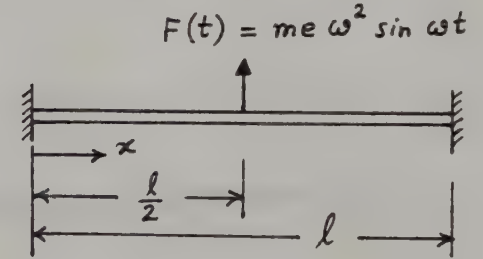
where $F_0 = 5483.1392 \text{ N}$; $\omega = 104.72 \text{ rad/sec}$

Steady state response of the beam can be expressed as:

$$w(x, t) = \sum_{n=1}^{\infty} W_n(x) q_n(t) \quad (2)$$

where the normal modes, $W_n(x)$, are given by (see Fig. 8.15):

$$W_n(x) = \sinh \beta_n x - \sin \beta_n x + \alpha_n (\cosh \beta_n x - \cos \beta_n x) \quad (3)$$



$$\text{and } \alpha_n = \frac{\sinh \beta_n \ell - \sin \beta_n \ell}{\cos \beta_n \ell - \cosh \beta_n \ell} \quad (4)$$

The generalized force, $Q_n(t)$, can be expressed as:

$$Q_n(t) = \int_0^\ell f(x,t) W_n(x) dx = F_0 W_n\left(\frac{\ell}{2}\right) \sin \omega t \quad (5)$$

The steady state values of the generalized coordinates, $q_n(t)$, are given by Eq. (8.117):

$$\begin{aligned} q_n(t) &= \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) d\tau \\ &= \frac{F_0 W_n\left(\frac{\ell}{2}\right)}{\rho A \ell \omega_n} \int_0^t \sin \omega \tau \sin \omega_n (t - \tau) d\tau \end{aligned} \quad (6)$$

$$\text{where } b = \int_0^\ell W_n^2(x) dx = \ell \quad (7)$$

The integral of Eq. (6) can be evaluated as:

$$\begin{aligned} &\int_0^t \sin \omega \tau \left\{ \sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau \right\} d\tau \\ &= \sin \omega_n t \int_0^t \frac{1}{2} \sin 2 \omega_n \tau d\tau - \cos \omega_n t \int_0^t \frac{1}{2} (1 - \cos 2 \omega_n \tau) d\tau \\ &= \frac{\sin \omega_n t}{2 \omega_n} - \frac{t \cos \omega_n t}{2} \end{aligned} \quad (8)$$

Thus $q_n(t)$ can be expressed as

$$\begin{aligned} q_n(t) &= \frac{F_0 W_n\left(\frac{\ell}{2}\right)}{\rho A \ell \omega_n} \left\{ \frac{\sin \omega_n t}{2 \omega_n} - \frac{t \cos \omega_n t}{2} \right\} \\ &= \frac{18.2771 W_n\left(\frac{\ell}{2}\right)}{\omega_n} \left\{ \sin \omega_n t - t \cos \omega_n t \right\} \end{aligned} \quad (9)$$

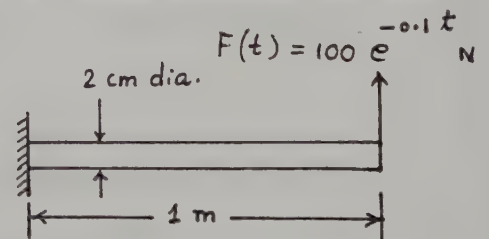
The steady state response of Eq. (2) can be expressed as

$$w(x,t) = 18.2771 \sum_{n=1}^{\infty} \frac{W_n\left(\frac{\ell}{2}\right)}{\omega_n} W_n(x) \left(\sin \omega_n t - t \cos \omega_n t \right) \text{ m} \quad (10)$$

8.46

Steady state displacement of the beam can be expressed as

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x) q_n(t) \quad (1)$$



where the normal modes, $W_n(x)$, are given by (see Fig. 8.15)

$$W_n(x) = C_n \left[\sin \beta_n x - \sinh \beta_n x - \alpha_n (\cos \beta_n x - \cosh \beta_n x) \right] \quad (2)$$

$$\alpha_n = \left(\frac{\sin \beta_n \ell + \sinh \beta_n \ell}{\cos \beta_n \ell + \cosh \beta_n \ell} \right) \quad (3)$$

and the generalized coordinates, $q_n(t)$, are given by

$$\frac{d^2 q_n(t)}{dt^2} + \omega_n^2 q_n(t) = \frac{1}{\rho A b} Q_n(t) \quad (5)$$

$$\text{where } Q_n(t) = \int_0^\ell f(x, t) W_n(x) dx = F(t) W_n(\ell) \quad (6)$$

The steady state solution of Eq. (5) is given by

$$q_n(t) = \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) d\tau \quad (7)$$

$$\text{where } b = \int_0^\ell W_n^2(x) dx = \ell \quad (8)$$

$$\begin{aligned} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) d\tau &= 100 W_n(\ell) \int_0^t e^{-0.1 \tau} \sin \omega_n (t - \tau) d\tau \\ &= -100 W_n(\ell) e^{-0.1 t} \int_{t-\tau=t}^{t-\tau=0} e^{0.1(t-\tau)} \sin \omega_n (t - \tau) (-d\tau) \end{aligned} \quad (9)$$

Using the formula

$$\int e^{ax} \sin bx dx = \frac{1}{a^2 + b^2} e^{ax} \left\{ a \sin bx - b \cos bx \right\} \quad (10)$$

Eq. (9) can be evaluated to obtain

$$q_n(t) = \frac{100 W_n(\ell)}{\rho A \ell \omega_n (\omega_n^2 + 0.01)} \left\{ \omega_n e^{-0.1 t} + 0.1 \sin \omega_n t - \omega_n \cos \omega_n t \right\} \quad (11)$$

Using $\rho = 7500 \text{ kg/m}^3$, $\ell = 1 \text{ m}$, $A = \frac{\pi}{4} (0.02^2) = 3.1416 (10^{-4}) \text{ m}^2$, Eq. (11) can be written as

$$q_n(t) = \frac{42.4412 W_n(\ell)}{\omega_n (\omega_n^2 + 0.01)} \left\{ \omega_n e^{-0.1 t} + 0.1 \sin \omega_n t - \omega_n \cos \omega_n t \right\} \quad (12)$$

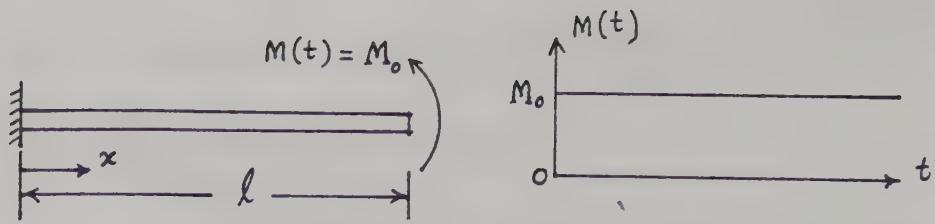
The total steady state solution can be found from Eq. (1).

8.47

The solution is assumed as

$$w(x, t) = \sum_{n=1}^{\infty} W_n(x) q_n(t) \quad (1)$$

where the normal modes of a cantilever beam are given by (Fig. 8.15):



$$W_n(x) = \sin \beta_n x - \sinh \beta_n x - \alpha_n (\cos \beta_n x - \cosh \beta_n x) \quad (2)$$

$$\alpha_n = \left(\frac{\sin \beta_n \ell + \sinh \beta_n \ell}{\cos \beta_n \ell + \cosh \beta_n \ell} \right) \quad (3)$$

where $\beta_n \ell$ are given by the frequency equation:

$$\cos \beta_n \ell \cosh \beta_n \ell = -1 \quad (4)$$

The generalized force $Q_n(t)$ given by Eq. (8.115) becomes

$$Q_n(t) = M_0 \left. \frac{dW_n}{dx} \right|_{x=\ell} \quad (5)$$

where

$$\left. \frac{dW_n}{dx} \right|_{x=\ell} = \beta_n (\cos \beta_n \ell - \cosh \beta_n \ell) + \alpha_n \beta_n (\sin \beta_n \ell + \sinh \beta_n \ell) \quad (6)$$

The steady state response of the beam is given by Eq. (1) with

$$\begin{aligned} q_n(t) &= \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) d\tau \\ &= \frac{1}{\rho A b \omega_n} M_0 \left. \frac{dW_n}{dx} \right|_{x=\ell} \int_0^t \sin \omega_n (t - \tau) d\tau \end{aligned} \quad (7)$$

$$\text{where } b = \int_0^\ell W_n^2(x) dx = \ell \quad (8)$$

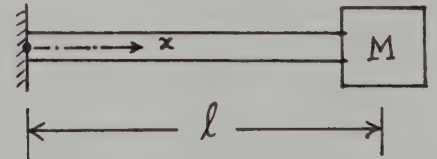
Noting that

$$\int_0^t \sin \omega_n (t - \tau) d\tau = \frac{1}{\omega_n} (1 - \cos \omega_n t) \quad (9)$$

Eq. (7) can be written as

$$q_n(t) = \frac{M_0}{\rho A \ell \omega_n^2} \left. \frac{dW_n}{dx} \right|_{x=\ell} (1 - \cos \omega_n t) \quad (10)$$

8.48



$$w(x, t) = W(x) \sin \omega t \quad (1)$$

$$\text{where } W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x \quad (2)$$

Boundary conditions:

$$w(0, t) = 0 \quad (3)$$

$$\frac{\partial w}{\partial x}(0, t) = 0 \quad (4)$$

$$\frac{\partial^2 w}{\partial x^2}(x = \ell, t) = 0 \quad (5)$$

$$EI \frac{\partial^3 w}{\partial x^3}(x = \ell, t) = m \frac{\partial^2 w}{\partial t^2}(x = \ell, t) \quad (6)$$

Eq. (2) gives:

$$\frac{dW}{dx} = -\beta C_1 \sin \beta x + \beta C_2 \cos \beta x + \beta C_3 \sinh \beta x + \beta C_4 \cosh \beta x \quad (7)$$

$$\frac{d^2 W}{dx^2} = \beta^2 \left\{ -C_1 \cos \beta x - C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x \right\} \quad (8)$$

$$\frac{d^3 W}{dx^3} = \beta^3 \left\{ C_1 \sin \beta x - C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \right\} \quad (9)$$

Eqs. (2) and (3) give:

$$C_1 + C_3 = 0 \quad (10)$$

Eqs. (2) and (4) lead to:

$$C_2 + C_4 = 0 \quad (11)$$

Eqs. (2) and (5) yield:

$$-C_1 \cos \beta \ell - C_2 \sin \beta \ell + C_3 \cosh \beta \ell + C_4 \sinh \beta \ell = 0 \quad (12)$$

Eqs. (2) and (6) result in:

$$C_1 (\sin \beta \ell + k \cos \beta \ell) + C_2 (-\cos \beta \ell + k \sin \beta \ell) + C_3 (\sinh \beta \ell + k \cosh \beta \ell) + C_4 (\cosh \beta \ell + k \sinh \beta \ell) = 0 \quad (13)$$

where

$$k = \frac{m \omega^2}{EI \beta^3} \quad (14)$$

Eqs. (10) to (13) can be expressed in matrix form as

$$[A] \vec{C} = \vec{0} \quad (15)$$

where

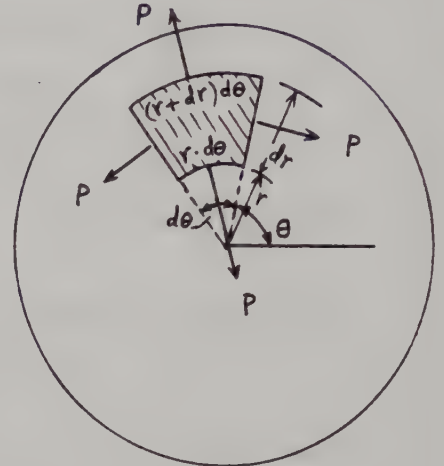
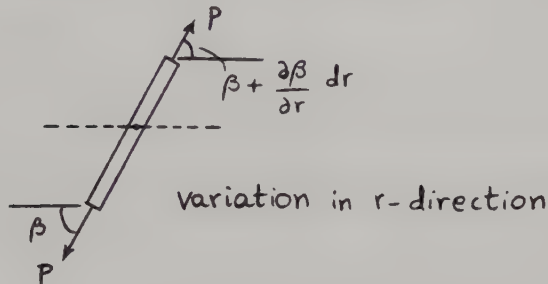
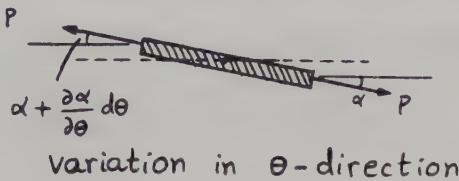
$$[A] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos \beta \ell & -\sin \beta \ell & \cosh \beta \ell & \sinh \beta \ell \\ (\sin \beta \ell + k \cos \beta \ell) & (-\cos \beta \ell + k \sin \beta \ell) & (\sinh \beta \ell + k \cosh \beta \ell) & (\cosh \beta \ell + k \sinh \beta \ell) \end{bmatrix} \quad (16)$$

$$\vec{C} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} ; \quad \vec{0} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

By setting the determinant of the coefficient matrix, $[A]$, to zero, we obtain the frequency equation:

$$|[A]| = 0 \quad (17)$$

8.49



Forces in radial direction are $P r d\theta$ and $P(r+dr) d\theta$

vertical component of radial forces

$$= P(r+dr) d\theta \left(\beta + \frac{\partial \beta}{\partial r} dr \right) - P r d\theta \beta = P r \left(\frac{\partial \beta}{\partial r} + \frac{1}{r} \beta \right) dr d\theta$$

$$= P r \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) dr d\theta \quad \text{since } \beta = \frac{\partial w}{\partial r}$$

Forces in tangential direction are $P dr$ and $P dr$.

vertical component of tangential forces

$$= P dr \left(\alpha + \frac{\partial \alpha}{\partial \theta} d\theta \right) - P dr \alpha = P \frac{\partial \alpha}{\partial \theta} dr d\theta$$

$$= P \cdot \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} r dr d\theta \quad \text{since } \alpha = \frac{1}{r} \frac{\partial w}{\partial \theta} \text{ and } \frac{\partial \alpha}{\partial \theta} = \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2}$$

Equating the total vertical force to mass times acceleration, we get

$$P r dr d\theta \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right] = \rho \frac{\partial^2 w}{\partial t^2} r dr d\theta$$

$$\text{i.e.} \quad \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = \frac{\rho}{P} \frac{\partial^2 w}{\partial t^2}$$

8.50

For harmonic motion, equation of motion becomes

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2} = -\frac{\rho}{P} \omega^2 W$$

Let $W(r, \theta) = X(r) Y(\theta)$

$$\frac{r^2}{X} \left(\frac{d^2 X}{dr^2} + \frac{1}{r} \frac{dX}{dr} + \frac{\rho \omega^2}{P} X \right) = -\frac{1}{Y} \frac{d^2 Y}{d\theta^2} = \alpha^2 \text{ (say)}$$

$$\therefore \frac{d^2 Y}{d\theta^2} + \alpha^2 Y = 0 \quad (E_1)$$

$$\frac{d^2 X}{dr^2} + \frac{1}{r} \frac{dX}{dr} + \left(\frac{\rho \omega^2}{P} - \frac{\alpha^2}{r^2} \right) X = 0 \quad (E_2)$$

Solution of (E₂) is $X(r) = C_1 J_m(\gamma \cdot r) + C_2 I_m(\gamma \cdot r)$; $m = 0, 1, 2, \dots$

where J_m and I_m are Bessel functions of the first and second kinds, respectively, and $\gamma = \frac{\rho \omega^2}{P}$

Since $I_m(\gamma \cdot r) \rightarrow \infty$ when $r \rightarrow 0$, $C_2 = 0$ to keep $X(r)$ and hence $w(r, \theta, t)$ finite.

$$\therefore X(r) = C_1 J_m(\gamma \cdot r)$$

$$\text{At } r = R, X(r) = 0 \text{ i.e., } J_m(\gamma \cdot R) = 0 \quad (E_3)$$

Eg. (E₃) has several roots: $\gamma_1 R, \gamma_2 R, \dots, \gamma_n R, \dots$

$$\therefore \omega_{mn}^2 = \frac{\gamma_n^2 P}{\rho}$$

8.51

$$(a) \text{ Equation of motion: } \rho \frac{\partial^2 w}{\partial t^2} = P \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + f \quad (E_1)$$

Let the forced response of the rectangular membrane be of the form

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} T_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (E_2)$$

where $T_{mn}(t)$ is to be determined. Eqs. (E₁) and (E₂) give

$$\begin{aligned} \pi^2 P \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) T_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ + \rho \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{d^2 T_{mn}(t)}{dt^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} - f = 0 \end{aligned} \quad (E_3)$$

Multiply (E₃) by $\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$ and integrate with respect to x from 0 to a and with respect to y from 0 to b to get

$$\frac{d^2 T_{mn}(t)}{dt^2} + \frac{P \pi^2}{\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) T_{mn}(t) = \frac{4}{ab\rho} F_{mn}(t); \quad m, n = 1, 2, \dots \quad (E_4)$$

where

$$F_{mn}(t) = \int_0^a \int_0^b f(x, y, t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \quad (E_5)$$

Solution of (E_4) can be expressed as

$$T_{mn}(t) = A_{mn} \cos \pi \sqrt{\frac{P}{\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} t + B_{mn} \sin \pi \sqrt{\frac{P}{\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} t \\ + \frac{4}{\pi \rho a b \sqrt{\frac{P}{\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}} \int_0^t F_{mn}(\tau) \cdot \sin \left\{ \pi \sqrt{\frac{P}{\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} (t - \tau) \right\} d\tau \quad \dots (E_6)$$

where A_{mn} and B_{mn} are determined from the known initial conditions of the membrane. Thus the general solution is given by Eq. (E_2) with $T_{mn}(t)$ shown in the last equation, (E_6) .

(b) For $f(x, y, t) = f_0$, (E_5) becomes

$$F_{mn}(t) = \int_0^a \int_0^b f_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ = \frac{f_0 a b}{\pi^2 m n} (1 - \cos m\pi) (1 - \cos n\pi) \\ = \begin{cases} 0 & \text{for } m \text{ or } n \text{ even} \\ \frac{4 f_0 a b}{\pi^2 m n} & \text{for } m \text{ and } n \text{ odd} \end{cases}$$

Since the membrane is at rest initially, $A_{mn} = B_{mn} = 0$ and Eq. (E_6) gives

$$T_{mn}(t) = \frac{4}{\pi \rho a b \Delta} * \frac{4 f_0 a b}{\pi^2 m n} \int_0^t \sin \pi \Delta (t - \tau) \cdot d\tau \\ = \frac{16 f_0}{\pi^3 m n \rho \Delta} * \frac{(1 - \cos \pi \Delta t)}{\pi \Delta} \\ = \frac{16 f_0}{\pi^4 m n \rho \Delta^2} (1 - \cos \pi \Delta t) \quad \text{for } m \text{ and } n \text{ odd}$$

where $\Delta = \left\{ \frac{P}{\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \right\}^{1/2}$

Final solution is

$$w(x, y, t) = \frac{16 f_0}{\pi^4 \rho} \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} \frac{1}{m n \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cdot (1 - \cos \pi \Delta t)$$

8.52

From example 8.11,

$$w(x, y, t) = X(x) Y(y) T(t) = (C_1 \cos \alpha x + C_2 \sin \alpha x) (C_3 \cos \beta y + C_4 \sin \beta y) * \\ (A \cos \omega t + B \sin \omega t)$$

Boundary conditions:

$$w(0, y, t) = w(a, y, t) = 0; \quad 0 \leq y \leq b \text{ and } t \geq 0$$

$$w(x, 0, t) = w(x, b, t) = 0; \quad 0 \leq x \leq a \text{ and } t \geq 0$$

i.e. $X(0) = X(a) = 0, \quad Y(0) = Y(b) = 0$

i.e. $C_1 = 0, C_3 = 0 \Rightarrow \sin \alpha a = 0, \sin \beta b = 0$

$$\alpha_m = \frac{m\pi}{a} \quad (m = 1, 2, \dots), \quad \beta_n = \frac{n\pi}{b} \quad (n = 1, 2, \dots)$$

$$\therefore \omega_{mn}^2 = c^2 \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right); \quad m, n = 1, 2, \dots$$

and the corresponding displacement solution is

$$w_{mn}(x, y, t) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t)$$

General solution is

$$\begin{aligned} w(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}(x, y, t) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t) \end{aligned}$$

8.53

General solution

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t) \quad (E_1)$$

If $w(x, y, 0) = w_0(x, y)$

and $\frac{\partial w}{\partial t}(x, y, 0) = \dot{w}_0(x, y) \quad \left. \begin{array}{l} 0 \leq x \leq a, \\ 0 \leq y \leq b \end{array} \right\} \quad (E_2)$

Eg. (E₁) gives

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = w_0(x, y) \quad (E_3)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \dot{w}_0(x, y) \quad (E_4)$$

These are double Fourier sine series expansions so that

$$\begin{aligned} A_{mn} &= \frac{4}{ab} \int_{x=0}^a \int_{y=0}^b w_0(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ &= \frac{4}{ab} \int_{x=0}^a w_0 \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} dx \int_{y=0}^b \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b} dy \quad (E_5) \end{aligned}$$

$$\begin{aligned} B_{mn} &= \frac{4}{ab \omega_{mn}} \int_{x=0}^a \int_{y=0}^b \dot{w}_0(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ &= 0 \text{ for given data} \quad (E_6) \end{aligned}$$

Using the relation $\int \sin \alpha t \sin \beta t dt = \frac{1}{2} \left\{ \frac{\sin(\alpha - \beta)t}{\alpha - \beta} - \frac{\sin(\alpha + \beta)t}{\alpha + \beta} \right\}$; $\alpha \neq \beta$

E_5 can be simplified as

$$A_{mn} = \frac{4\omega_0}{ab} \left\{ \frac{1}{2} \left[\frac{\sin \frac{(m-1)\pi x}{a}}{\frac{(m-1)\pi}{a}} - \frac{\sin \frac{(m+1)\pi x}{a}}{\frac{(m+1)\pi}{a}} \right]_0^a \right\} \left\{ \frac{1}{2} \left[\frac{\sin \frac{(n-1)\pi y}{b}}{\frac{(n-1)\pi}{b}} - \frac{\sin \frac{(n+1)\pi y}{b}}{\frac{(n+1)\pi}{b}} \right]_0^b \right\}$$

$$= 0 \text{ for } m > 1 \text{ and/or } n > 1 \quad (E_7)$$

For $m=1$ and $n=1$, E_5 can be simplified as

$$A_{11} = \frac{4\omega_0}{ab} \int_0^a \sin^2 \frac{\pi x}{a} dx \int_0^b \sin^2 \frac{\pi y}{b} dy = \omega_0 \quad (E_8)$$

$$\therefore w(x, y, t) = \omega_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \cos \omega_{11} t$$

$$\text{with } \omega_{11} = \left\{ c^2 \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \right\}^{1/2}$$

(from the solution of problem 8.52)

8.54

General solution:

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t)$$

$$\left. \begin{aligned} \text{If } w(x, y, 0) &= w_0(x, y) \\ \text{and } \frac{\partial w}{\partial t}(x, y, 0) &= \dot{w}_0(x, y) \end{aligned} \right\}; \quad 0 \leq x \leq a, \quad 0 \leq y \leq b$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = w_0(x, y)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \dot{w}_0(x, y)$$

These are double Fourier sine series expansions of $w_0(x, y)$ and $\dot{w}_0(x, y)$.

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b w_0(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy$$

$$= 0 \text{ for given data}$$

$$B_{mn} = \frac{4}{ab \omega_{mn}} \int_0^a \int_0^b \dot{w}_0(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy$$

$$= \frac{4 \dot{w}_0}{ab \omega_{mn}} \int_0^a \sin \frac{m\pi x}{a} \sin \frac{\pi x}{a} dx \int_0^b \sin \frac{n\pi y}{b} \sin \frac{2\pi y}{b} dy \quad \dots (E_1)$$

using the relation

$$\int \sin \alpha t \cdot \sin \beta t \cdot dt = \frac{1}{2} \left\{ \frac{\sin(\alpha - \beta)t}{\alpha - \beta} - \frac{\sin(\alpha + \beta)t}{\alpha + \beta} \right\} \text{ for } \alpha \neq \beta,$$

Eq. (E₁) becomes

$$B_{mn} = \frac{4 \dot{w}_0}{ab \omega_{mn}} \left\{ \frac{1}{2} \left(\frac{\sin \frac{(m-1)\pi x}{a}}{m-1} - \frac{\sin \frac{(m+1)\pi x}{a}}{m+1} \right) \right|_0^a \right\} x$$

$$\left\{ \frac{1}{2} \left(\frac{\sin \frac{(n-2)\pi y}{b}}{n-2} - \frac{\sin \frac{(n+2)\pi y}{b}}{n+2} \right) \right|_0^b \right\} y$$

= 0 for $m \neq 1$ and $n \neq 2$.

$$B_{12} = \frac{4 \dot{w}_0}{ab \omega_{mn}} \left(\int_0^a \sin^2 \frac{\pi x}{a} dx \right) \left(\int_0^b \sin^2 \frac{2\pi y}{b} dy \right)$$

$$= \frac{4 \dot{w}_0}{ab \omega_{mn}} \left\{ \left(\frac{x}{2} - \frac{\sin \frac{\pi x}{a} \cos \frac{\pi x}{a}}{(2\pi/a)} \right) \right|_0^a \right\} \left\{ \left(\frac{y}{2} - \frac{\sin \frac{2\pi y}{b} \cos \frac{2\pi y}{b}}{(4\pi/b)} \right) \right|_0^b \right\}$$

$$= \frac{4 \dot{w}_0}{ab \omega_{12}} \left(\frac{a}{2} \right) \left(\frac{b}{2} \right) = \frac{\dot{w}_0}{\omega_{12}}$$

$$\therefore w(x, y, t) = \frac{\dot{w}_0}{\omega_{12}} \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} \sin \omega_{12} t$$

8.55

Fundamental natural frequency of transverse vibration is given by:

$$\omega_{11}^2 = c^2 \pi^2 \left(\frac{1}{a_1^2} + \frac{1}{b_1^2} \right) \quad \text{for rectangular membrane of sides } \dots (E_1)$$

a_1 and b_1

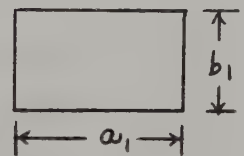
$$\text{with } c^2 = \frac{P}{\rho} \quad \dots (E_2)$$

$$(a) \text{ For } a_1 = b_1 = a, \quad \omega_{11}^2 = c^2 \pi^2 \left(\frac{2}{a^2} \right) \quad \dots (E_3)$$

$$(c) \text{ For } a_1 = 2b_1, \quad \text{area} = a_1 b_1 = a^2 = 2b_1^2$$

$$\therefore b_1 = a/\sqrt{2}, \quad a_1 = \sqrt{2} a$$

$$\omega_{11}^2 = c^2 \pi^2 \left(\frac{1}{a_1^2} + \frac{1}{b_1^2} \right) = c^2 \pi^2 \left(\frac{5}{2a^2} \right) \quad \dots (E_4)$$

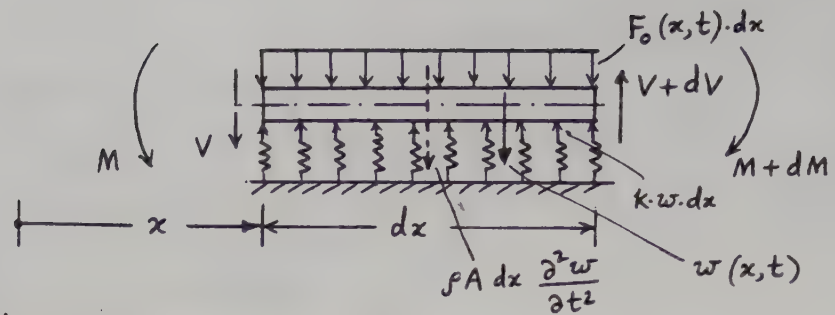


(b) From solution of problem 8.50, the fundamental natural frequency of a circular membrane of radius R is given by

$$\omega_{01}^2 = \frac{\gamma_1^2 P}{\rho} = c^2 \gamma_1^2 \quad \text{where } \gamma_1 R = 2.404 = \text{first zero of Bessel function of the first kind.}$$

Since $\pi R^2 = a^2$, $R = a/\sqrt{\pi}$ and

$$\omega_{01}^2 = c^2 \left(\frac{2.404}{R} \right)^2 = c^2 \left(\frac{4.2610}{a} \right)^2 \quad \dots (E_5)$$



(a) If load due to the car moves with a constant velocity v_0 in the x -direction,

$$F(x,t) = F(x - v_0 t) \quad (E_1)$$

Equilibrium equations are

$$(V + dV) - V - F_0(x,t) dx + k w dx = -\rho A dx \frac{\partial^2 w}{\partial t^2}$$

$$\text{or } \frac{\partial V}{\partial x} - F_0(x,t) + k w = -\rho A \frac{\partial^2 w}{\partial t^2} \quad (E_2)$$

$$\text{and } (M + dM) - M - (V + dV) dx + F(x,t) dx \cdot \frac{dx}{2} = 0$$

$$\text{or } \frac{\partial M}{\partial x} - V = 0 \quad (E_3)$$

Using (E₃), (E₂) can be rewritten as

$$\frac{\partial^2 M}{\partial x^2} - F_0(x,t) + k w = -\rho A \frac{\partial^2 w}{\partial t^2} \quad (E_4)$$

$$\text{But } M = EI \frac{\partial^2 w}{\partial x^2} \quad (E_5)$$

Eqs. (E₄) and (E₅) lead to

$$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) - F_0(x,t) + k w = -\rho A \frac{\partial^2 w}{\partial t^2} \quad (E_6)$$

For a uniform beam, (E₆) simplifies to, in view of (E₁),

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} + k w = F_0(x - v_0 t) \quad (E_7)$$

(b) Solution:

$$\text{Defining } y = x - v_0 t \quad (E_8)$$

Eq. (E₇) can be expressed as

$$EI \frac{d^4 w}{dy^4} + \rho A v_0^2 \frac{d^2 w}{dy^2} + k w = F_0(y) \quad (E_9)$$

Let F_0 = concentrated load. Then the governing equation at all points of the beam, except at $y=0$, is

$$EI \frac{d^4 w}{dy^4} + \rho A v_0^2 \frac{d^2 w}{dy^2} + k w = 0 \quad (E_{10})$$

Solution of (E₁₀) can be expressed as

$$w(y) = e^{\alpha y} \quad (E_{11})$$

Substitution of (E_{11}) into (E_{10}) gives

$$EI \alpha^4 + \rho A v_0^2 + k = 0 \quad (E_{12})$$

whose roots can be expressed as

$$\alpha_{1,2} = \pm (a + ib) \quad , \quad \alpha_{3,4} = \pm (a - ib) \quad (E_{13})$$

with

$$\left. \begin{aligned} a &= \sqrt{1-c} \, d, \quad b = \sqrt{1+c} \, d, \quad c = \frac{v_0^2}{\sqrt{\frac{4EI k}{\rho^2 A^2}}} \\ \text{and } d &= \left(\frac{k}{4EI} \right)^{1/4} \end{aligned} \right\} \quad (E_{14})$$

Thus the solution of (E_{10}) becomes

$$w(y) = A_1 e^{\alpha_1 y} + A_2 e^{\alpha_2 y} + A_3 e^{\alpha_3 y} + A_4 e^{\alpha_4 y} \quad (E_{15})$$

where A_i ($i=1, 2, 3, 4$) are constants which can be determined from the following conditions:

$$\left. \begin{aligned} w &= 0 \text{ at } y = \infty \\ \frac{d^2 w}{dy^2} &= 0 \text{ at } y = \infty \end{aligned} \right\} \begin{array}{l} \text{deflection \& bending moment are} \\ \text{zero at } y = \infty \end{array}$$

$$\left. \frac{dw}{dy} = 0 \text{ at } y = 0 \right\} \text{slope is zero under the load}$$

$$\left. \begin{aligned} EI \frac{d^3 w}{dy^3} \Big|_{y=0^+} - EI \frac{d^3 w}{dy^3} \Big|_{y=0^-} &= P \\ \text{i.e., } EI \frac{d^2 w}{dy^2} &= \frac{P}{2} \end{aligned} \right\} \begin{array}{l} \text{shear force has} \\ \text{discontinuity} \\ \text{under the load} \end{array}$$

8.57

$$W(x) = \frac{c_0 x^2}{24 EI} (l-x)^2 = \frac{c_0}{24 EI} (x^2 l^2 + x^4 - 2l x^3)$$

$$\frac{dW}{dx} = \frac{c_0}{24 EI} (2x l^2 + 4x^3 - 6l x^2)$$

$$\frac{d^2 W}{dx^2} = \frac{c_0}{24 EI} (2l^2 + 12x^2 - 12lx)$$

$$\begin{aligned} N &= EI \int_0^l \left(\frac{d^2 W}{dx^2} \right)^2 dx = 4EI \left(\frac{c_0}{24 EI} \right)^2 \int_0^l (l^2 + 6x^2 - 6lx)^2 dx \\ &= \frac{c_0^2 l^5}{720 EI} \end{aligned} \quad (E_1)$$

$$\begin{aligned} D &= \rho A \int_0^l (W(x))^2 dx = \rho A \left(\frac{c_0}{24 EI} \right)^2 \int_0^l (x^2 l^2 + x^4 - 2l x^3)^2 dx \\ &= \frac{\rho A l^9 c_0^2}{362880 E^2 I^2} \end{aligned} \quad (E_2)$$

$$\omega^2 = \frac{N}{D} = \left(\frac{c_0^2 l^5}{720 EI} \right) \left(\frac{362880 E^2 I^2}{\rho A l^3 c_0^2} \right) = 504 \frac{EI}{\rho A l^4}$$

$$\therefore \omega = 22.4499 \sqrt{\frac{EI}{\rho A l^4}}$$

This can be compared with the exact solution (Fig. 8.15):

$$\omega_{\text{exact}} = (4.730041)^2 \sqrt{\frac{EI}{\rho A l^4}} = 22.3733 \sqrt{\frac{EI}{\rho A l^4}}$$

8.58

$$W(x) = c_0 \left(1 - \cos \frac{2\pi x}{l} \right)$$

$$\frac{dW}{dx} = c_0 \frac{2\pi}{l} \sin \frac{2\pi x}{l} ; \quad \frac{d^2 W}{dx^2} = c_0 \left(\frac{2\pi}{l} \right)^2 \cos \frac{2\pi x}{l}$$

$$N = EI \int_0^l \left(\frac{d^2 W}{dx^2} \right)^2 dx = EI c_0^2 \left(\frac{2\pi}{l} \right)^4 \int_0^l \cos^2 \frac{2\pi x}{l} dx$$

$$= 8EI c_0^2 \pi^4 / l^3$$

$$D = \rho A \int_0^l (W(x))^2 dx = \rho A c_0^2 \int_0^l \left(1 + \cos^2 \frac{2\pi x}{l} - 2 \cos \frac{2\pi x}{l} \right) dx$$

$$= 3c_0^2 \rho A l / 2$$

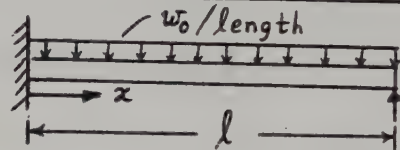
$$\omega^2 = \frac{N}{D} = \frac{16\pi^4}{3} \cdot \frac{EI}{\rho A l^4}$$

$$\therefore \omega = 22.7930 \sqrt{\frac{EI}{\rho A l^4}} \quad \text{Compare this with}$$

$$\omega_{\text{exact}} = (4.730041)^2 \sqrt{\frac{EI}{\rho A l^4}} = 22.3733 \sqrt{\frac{EI}{\rho A l^4}}$$

8.59

From strength of materials, the static deflection curve is given by



$$W(x) = \frac{w_0 x^2}{48EI} (l-x)(2x-3l) = \frac{c_0}{48EI} (5lx^3 - 2x^4 - 3l^2 x^2)$$

$$\frac{dW}{dx} = \frac{c_0}{48EI} (15lx^2 - 8x^3 - 6l^2 x)$$

$$\frac{d^2 W}{dx^2} = \frac{c_0}{8EI} (5lx - 4x^2 - l^2)$$

$$N = EI \int_0^l \left(\frac{d^2 W}{dx^2} \right)^2 dx = \frac{c_0^2 l^5}{320 EI}$$

$$D = \rho A \int_0^l (W(x))^2 dx = 1.3090 \times 10^{-5} \frac{c_0^2 \rho A l^9}{E^2 I^2}$$

$$\omega^2 = \frac{N}{D} = 238.7319 \frac{EI}{\rho A l^4}$$

$$\therefore \omega = 15.4510 \sqrt{\frac{EI}{\rho A l^4}}$$

This can be compared with the exact value

$$\omega_{\text{exact}} = (3.926602)^2 \sqrt{\frac{EI}{\rho A l^4}} = 15.4182 \sqrt{\frac{EI}{\rho A l^4}}$$

8.60

$$W(x) = \begin{cases} \frac{x^2}{48EI} (-4x + 3l) & ; 0 \leq x \leq \frac{l}{2} \\ \frac{(l-x)^2}{48EI} (4x - l) & ; \frac{l}{2} \leq x \leq l \end{cases}$$



$$\frac{d^2 W}{dx^2} = \begin{cases} \frac{1}{8EI} (-4x + l) & ; 0 \leq x \leq \frac{l}{2} \\ \frac{1}{8EI} (4x - l) & ; \frac{l}{2} \leq x \leq l \end{cases}$$

$$\begin{aligned} T_{\text{max of beam}} &= \frac{\omega^2}{2} \int_0^l \rho A [W(x)]^2 dx \\ &= \frac{\omega^2 \rho A}{2} \left(\frac{1}{48EI} \right)^2 \left[\int_0^{l/2} x^4 (16x^2 + 9l^2 - 24xl) dx + \int_{l/2}^l (l-x)^4 (4x-l)^2 dx \right] \\ &= \frac{\rho A \omega^2}{4608 E^2 I^2} \left\{ \left[\frac{16x^7}{7} + \frac{9l^2 x^5}{5} - \frac{24lx^6}{6} \right]_0^{l/2} \right. \\ &\quad \left. + \left[\frac{16x^7}{7} - \frac{72lx^6}{6} + \frac{129l^2 x^5}{5} - \frac{116l^3 x^4}{4} + \frac{54l^4 x^3}{3} - \frac{12l^5 x^2}{2} + l^6 x \right]_{l/2}^l \right\} \\ &= 5 \times 10^{-6} \left(\frac{\rho A \omega^2 l^7}{E^2 I^2} \right) \end{aligned}$$

$$T_{\text{max of mass}} = \frac{1}{2} m \dot{w}_{\text{max}} \Big|_{x=\frac{l}{2}} = \frac{1}{2} m \omega^2 W^2(x=\frac{l}{2})$$

$$= \frac{1}{2} m \omega^2 \left(\frac{l^3}{192EI} \right)^2 = 13.6 \times 10^{-6} \left(\frac{m \omega^2 l^6}{E^2 I^2} \right)$$

$$T_{\text{max total}} = 5 \times 10^{-6} \frac{\rho A \omega^2 l^7}{E^2 I^2} + 13.6 \times 10^{-6} \frac{m \omega^2 l^6}{E^2 I^2}$$

$$\begin{aligned}
 V_{\max} &= \frac{1}{2} \int_0^l EI \left(\frac{d^2 W}{dx^2} \right)^2 dx \\
 &= \frac{1}{128 EI} \left[\int_0^{\frac{l}{2}} (16x^2 + l^2 - 8lx) dx + \int_{\frac{l}{2}}^l (16x^2 + 9l^2 - 24lx) dx \right] \\
 &= \frac{1}{128 EI} \left[\left(\frac{16}{3} x^3 + l^2 x - \frac{8l}{2} x^2 \right) \Big|_0^{\frac{l}{2}} + \left(\frac{16}{3} x^3 + 9l^2 x - \frac{24l}{2} x^2 \right) \Big|_{\frac{l}{2}}^l \right] \\
 &= \frac{l^3}{384 EI}
 \end{aligned}$$

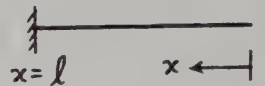
$$T_{\max} \text{ total} = V_{\max} \text{ gives}$$

$$\omega^2 = \frac{l^3}{384 EI} \cdot \frac{E^2 I^2}{(5 \times 10^{-6} \rho A l^7 + 13.6 \times 10^{-6} \text{ m } l^6)} = \frac{2590.7 EI}{l^3 (5 \rho A l + 13.6 \text{ m})}$$

$$\therefore \omega = 50.8987 \sqrt{\frac{EI}{l^3 (5 M_b + 13.6 \text{ m})}} \quad \text{where } M_b = \rho A l = \text{mass of beam}$$

8.61

$$\text{Let } W(x) = c_0 \left(1 - \frac{x}{l} \right)^2 = c_0 \left(1 + \frac{x^2}{l^2} - \frac{2x}{l} \right)$$



$$\frac{d^2 W}{dx^2} = 2c_0 \left(\frac{1}{l^2} \right)$$

$$T_{\max} = \frac{\omega^2}{2} \int_0^l \rho A(x) [W(x)]^2 dx = \frac{\rho \omega^2 A_0 c_0^2}{2l} \int_0^l x \left(1 + \frac{x^2}{l^2} - \frac{2x}{l} \right)^2 dx$$

$$= \frac{\rho \omega^2 A_0 c_0^2}{2l} \left[\frac{x^2}{2} + \frac{x^6}{6l^4} + \frac{4x^4}{4l^2} + \frac{2x^4}{4l^2} - \frac{4x^5}{5l^3} - \frac{4x^3}{3l} \right]_0^l$$

$$= \frac{\rho \omega^2 A_0 c_0^2 l}{60}$$

$$V_{\max} = \frac{1}{2} \int_0^l EI(x) \left(\frac{d^2 W}{dx^2} \right)^2 dx = \frac{E}{2} \int_0^l \left(\frac{I_0 x}{l} \right) \left(\frac{4c_0^2}{l^4} \right) dx = \frac{EI_0 c_0^2}{l^3}$$

$$T_{\max} = V_{\max} \text{ gives}$$

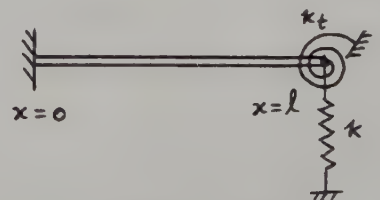
$$\omega^2 = \frac{EI_0 c_0^2}{l^3} \cdot \frac{60}{\rho A_0 c_0^2 l} = \frac{60 EI_0}{\rho A_0 l^4}$$

$$\omega = 7.7460 \sqrt{\frac{EI_0}{\rho A_0 l^4}}$$

8.62

$$\text{Let } W(x) = \frac{c_0}{24EI} (x^4 - 4lx^3 + 6l^2x^2)$$

$$\frac{d^2 W}{dx^2} = \frac{c_0}{2EI} (x^2 - 2lx + l^2)$$



$$V_{\max} = \frac{EI}{2} \int_0^l \left(\frac{d^2 W}{dx^2} \right)^2 dx + \frac{1}{2} k [W(x=l)]^2 + \frac{1}{2} k_t \left[\frac{dW}{dx}(x=l) \right]^2$$

Since $\frac{\kappa}{2} [W(x=l)]^2 = \frac{\kappa}{2} c_0^2 \frac{l^8}{64 E^2 I^2}$,

$$\frac{\kappa_t}{2} \left[\frac{dW}{dx}(x=l) \right]^2 = \frac{\kappa_t}{2} c_0^2 \frac{l^6}{36 E^2 I^2},$$

and $\frac{1}{2} EI \int_0^l \left(\frac{d^2 W}{dx^2} \right)^2 dx = \frac{c_0^2 l^5}{40 EI}$ (from solution of problem 8.55)

$$V_{\max} = \frac{c_0^2 l^5}{40 EI} + \frac{\kappa c_0^2 l^8}{128 E^2 I^2} + \frac{\kappa_t c_0^2 l^6}{72 E^2 I^2}$$

$$T_{\max} = \frac{\omega^2 \rho A}{2} \int_0^l [W(x)]^2 dx = \omega^2 \rho A \left(\frac{13 c_0^2 l^9}{6480 E^2 I^2} \right) \quad \text{from problem 8.55}$$

$V_{\max} = T_{\max}$ gives

$$\omega^2 = \left(12.4615 \frac{EI}{\rho A l^4} + 3.8942 \frac{\kappa}{\rho A l} + 6.9231 \frac{\kappa_t}{\rho A l^4} \right)^{1/2}$$

8.63

$$W(x) = c_1 \left(1 - \cos \frac{2\pi x}{l} \right), \quad \frac{d^2 W}{dx^2} = c_1 \left(\frac{2\pi}{l} \right)^2 \cos \frac{2\pi x}{l}$$

$$\begin{aligned} T_{\max} &= \frac{\omega^2}{2} \int_0^l \rho A [W(x)]^2 dx = \frac{\omega^2 \rho A c_1^2}{2} \int_0^l \left(1 + \cos^2 \frac{2\pi x}{l} - 2 \cos \frac{2\pi x}{l} \right) dx \\ &= \frac{\rho A \omega^2 c_1^2}{2} \left[x + \left(\frac{x}{2} + \frac{l}{8\pi} \sin \frac{4\pi x}{l} \right) + \left(\frac{l}{\pi} \sin \frac{2\pi x}{l} \right) \right]_0^l \\ &= \frac{3}{4} \rho A \omega^2 c_1^2 l \end{aligned}$$

$$\begin{aligned} V_{\max} &= \frac{EI}{2} \int_0^l \left(\frac{d^2 W}{dx^2} \right)^2 dx = \frac{EI}{2} \int_0^l c_1^2 \left(\frac{2\pi}{l} \right)^4 \cos^2 \frac{2\pi x}{l} dx \\ &= \frac{8EI c_1^2 \pi^4}{l^4} \left(\frac{x}{2} + \frac{l}{8\pi} \sin \frac{4\pi x}{l} \right)_0^l = \frac{4EI c_1^2 \pi^4}{l^3} \end{aligned}$$

$T_{\max} = V_{\max}$ gives

$$\omega^2 = \frac{4EI c_1^2 \pi^4}{l^3} \cdot \frac{4}{3 \rho A c_1^2 l} = \frac{519.52 EI}{\rho A l^4}$$

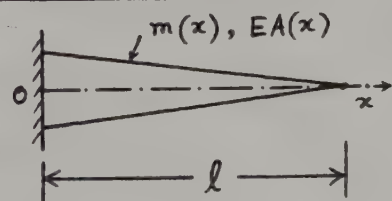
$$\omega = 22.7930 \sqrt{\frac{EI}{\rho A l^4}}$$

8.64

$$U(x) = c_1 \sin \frac{\pi x}{2l}$$

$$\frac{dU}{dx} = c_1 \frac{\pi}{2l} \cos \frac{\pi x}{2l}$$

$$V_{\max} = \frac{1}{2} \int_0^l EA(x) \left(\frac{dU}{dx} \right)^2 dx$$



$$\begin{aligned}
&= \frac{1}{2} \int_0^l 2EA_0 \left(1 - \frac{x}{l}\right) c_1^2 \left(\frac{\pi}{2l}\right)^2 \cos^2 \frac{\pi x}{2l} \cdot dx \\
&= EA_0 c_1^2 \left(\frac{\pi}{2l}\right)^2 \left[\int_0^l \cos^2 \frac{\pi x}{2l} dx - \int_0^l \frac{x}{l} \cos^2 \frac{\pi x}{2l} dx \right] \\
&= EA_0 c_1^2 \left(\frac{\pi}{2l}\right)^2 \left[\left(\frac{x}{2} + \frac{l}{2\pi} \sin \frac{\pi x}{l}\right)_0^l - \frac{1}{l} \left(\frac{x^2}{4} + \frac{x \sin \frac{\pi x}{l}}{\left(\frac{2\pi}{l}\right)} + \frac{\cos \frac{\pi x}{l}}{\left(\frac{8\pi^2}{4l^2}\right)} \right)_0^l \right] \\
&= \frac{EA_0 c_1^2 \pi^2}{16l} \left(1 + \frac{4}{\pi^2}\right) \\
T_{\max} &= \frac{\omega^2}{2} \int_0^l m(x) U^2 dx = \frac{\omega^2}{2} \int_0^l 2m_0 \left(1 - \frac{x}{l}\right) c_1^2 \sin^2 \frac{\pi x}{2l} \cdot dx \\
&= \omega^2 m_0 c_1^2 \left[\int_0^l \sin^2 \frac{\pi x}{2l} dx - \int_0^l \frac{x}{l} \sin^2 \frac{\pi x}{2l} dx \right] \\
&= \omega^2 m_0 c_1^2 \left[\left(\frac{x}{2} - \frac{l}{2\pi} \sin \frac{\pi x}{l}\right)_0^l - \frac{1}{l} \left(\frac{x^2}{4} - \frac{x \sin \frac{\pi x}{l}}{\left(\frac{2\pi}{l}\right)} - \frac{\cos \frac{\pi x}{l}}{\left(\frac{8\pi^2}{4l^2}\right)} \right)_0^l \right] \\
&= \omega^2 m_0 c_1^2 \frac{l}{4} \left(1 - \frac{4}{\pi^2}\right) \\
V_{\max} &= T_{\max} \text{ gives} \\
\omega^2 &= \frac{EA_0 \pi^2}{16l} \frac{\left(1 + \frac{4}{\pi^2}\right) 4}{m_0 l \left(1 - \frac{4}{\pi^2}\right)} = 5.8303 \frac{EA_0}{m_0 l^2} \\
\therefore \omega_1 &= 2.4146 \sqrt{\frac{EA_0}{m_0 l^2}}
\end{aligned}$$

8.65

We take the deflection curve satisfying the boundary conditions

$$w(0, y) = w(a, y) = w(x, 0) = w(x, b) = 0 \text{ as}$$

$$w(x, y) = c_1 xy(x-a)(y-b)$$

$$\frac{\partial w}{\partial x} = c_1 y(x-a)(y-b) + c_1 xy(y-b)$$

$$\frac{\partial w}{\partial y} = c_1 x(x-a)(y-b) + c_1 xy(x-a)$$

$$\begin{aligned}
\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 &= c_1^2 y^2(y-b)^2(x-a)^2 + c_1^2 x^2 y^2(y-b)^2 \\
&\quad + c_1^2 x^2(x-a)^2(y-b)^2 + c_1^2 x^2 y^2(x-a)^2
\end{aligned}$$

$$\begin{aligned}
&= c_1^2 [2x^2 y^4 + x^2 y^2(2a^2 + 2b^2) + x^2 y^3(-4b) + y^4(a^2) + y^2(a^2 b^2) \\
&\quad + y^3(-2ba^2) + xy^4(-2a) + xy^2(-2ab^2) + xy^3(4ab) \\
&\quad + 2x^4 y^2 + x^4(b^2) + x^4 y(-2b) + x^2(a^2 b^2) + x^2 y(-2ba^2) \\
&\quad + x^3 y^2(-2a) + x^3(-2ab^2) + x^3 y(4ab) + x^3 y^2(-2a)]
\end{aligned}$$

$$\int_0^a \int_0^b \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dx dy = c_1^2 \left(\frac{a^3 b^5}{45} + \frac{a^5 b^3}{45} \right)$$

$$w^2(x, y) = c_1^2 \left[x^4 y^4 + a^2 x^2 y^4 + b^2 x^4 y^2 + a^2 b^2 x^2 y^2 - 2a x^3 y^4 - 2b x^4 y^3 + 2ab x^3 y^3 + 2ab x^3 y^3 - 2a^2 b x^2 y^3 - 2ab^2 x^3 y^2 \right]$$

$$\int_0^a \int_0^b w^2 dx dy = c_1^2 a^5 b^5 / 900$$

$$V_{\max} = \frac{P}{2} \frac{c_1^2}{45} (a^3 b^3) (a^2 + b^2)$$

$$T_{\max} = \frac{\omega^2 \rho}{2} \cdot \frac{c_1^2 a^5 b^5}{900}$$

$$V_{\max} = T_{\max} \text{ gives } \frac{\omega^2 \rho}{P} = 20 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$$

$$\therefore \omega_1 = 4.4721 \sqrt{\frac{P}{\rho} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)}$$

$$\text{Exact value of } \omega_1 = \pi \sqrt{\frac{P}{\rho} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)}$$

8.66

For a shaft under torsion, the shear stress (τ) induced at a radius r from the center of the cross section is given by

$$\tau = \frac{M_t(x) r}{J} \quad (1)$$

where $M_t(x)$ = torque at section x . The potential energy of the shaft is given by the strain energy:

$$V = \frac{1}{2} \int \frac{1}{G} \tau^2 dA dx \quad (2)$$

Using Eq. (8.61):

$$M_t(x) = GJ \frac{\partial \theta}{\partial x} \quad (3)$$

Eq. (2) can be expressed as

$$V = \frac{1}{2} \int_0^l GJ \left(\frac{\partial \theta}{\partial x} \right)^2 dx \quad (4)$$

The kinetic energy of the shaft can be written as

$$T = \frac{1}{2} \int_0^l \rho J \left(\frac{\partial \theta}{\partial t} \right)^2 dx \quad (5)$$

By assuming a harmonic variation of $\theta(x, t)$ as

$$\theta(x, t) = \Theta(x) \cos \omega t \quad (6)$$

and equating $V_{\max}$ to $T_{\max}$, we obtain

$$\omega^2 = \frac{\int_0^{\ell} GJ \left(\frac{d\Theta}{dx} \right)^2 dx}{\int_0^{\ell} \rho J \Theta^2 dx} \quad (7)$$

For the shaft shown in Fig. 8.36,

$$G = 80 (10^9) \text{ N/m}^2 ; \quad \rho g = 76.5 \text{ kN/m}^3$$

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32} (0.05^4) = 61.3594 (10^{-8}) \text{ m}^4$$

Let $\Theta(x)$ be assumed to vary linearly on either side of the disc for simplicity:

$$\begin{aligned} \Theta(x) &= \frac{\theta_0 x}{0.8} ; \quad 0 \leq x \leq 0.8 \\ &= \frac{\theta_0 (1-x)}{0.2} ; \quad 0.8 \leq x \leq 1 \end{aligned} \quad (8)$$

where θ_0 is the angular displacement of the disc. Equation (8) satisfies the boundary conditions and gives:

$$\begin{aligned} \frac{d\Theta}{dx} &= \frac{\theta_0}{0.8} ; \quad 0 \leq x \leq 0.8 \\ &= -\frac{\theta_0}{0.2} ; \quad 0.8 \leq x \leq 1.0 \end{aligned} \quad (9)$$

$$\int_0^{\ell} GJ \left(\frac{d\Theta}{dx} \right)^2 dx = 306\,797 \theta_0^2 ; \quad \int_0^{\ell} \rho J \Theta^2 dx = 159.5384 (10^{-5}) \theta_0^2$$

Thus Eq. (7) gives the approximate natural frequency as:

$$\omega^2 \approx \frac{306797 \theta_0^2}{159.5384 (10^{-5}) \theta_0^2} = 1923.0292 (10^5)$$

$$\omega \approx 13,867.3328 \text{ rad/sec}$$

For comparison, the torsional natural frequency of a fixed-fixed shaft with no intermediate disc, is given by (see Fig. 8.12):

$$\omega_1 = \frac{\pi c}{\ell} = \frac{\pi}{\ell} \sqrt{\frac{G}{\rho}} = \frac{\pi}{1.0} \sqrt{\frac{80 (10^9) (9.81)}{76500}} = 10,062.3557 \text{ rad/sec}$$

8.67

$$(a) W(x) = c_1 x(l-x), \quad \text{Let } P = \text{tension}$$

$$V_{\max} = \frac{1}{2} \int_0^l P \left(\frac{dW}{dx} \right)^2 dx = \frac{1}{2} c_1^2 P \int_0^l (l-2x)^2 dx$$

$$= \frac{1}{2} P c_1^2 \left[l^2 x + \frac{4}{3} x^3 - 2 l x^2 \right]_0^l = \frac{1}{6} P c_1^2 l^3$$

$$T_{\max} = \frac{\omega^2}{2} \int_0^l \rho [W]^2 dx = \frac{1}{2} \rho c_1^2 \omega^2 \int_0^l (x^2 l^2 + x^4 - 2 l x^3) dx$$

$$= \frac{1}{2} \rho c_1^2 \omega^2 \left[\frac{1}{3} x^3 l^2 + \frac{1}{5} x^5 - \frac{1}{2} l x^4 \right]_0^l = \frac{1}{60} \omega^2 \rho c_1^2 l^5$$

$$V_{\max} = T_{\max} \text{ gives}$$

$$\omega = \left(\frac{10 P}{\rho l^2} \right)^{1/2} = 3.1623 \sqrt{\frac{P}{\rho l^2}}$$

$$\text{Exact value is } \omega_1 = \pi \sqrt{\frac{P}{\rho l^2}}$$

$$(b) W(x) = c_1 x(l-x) + c_2 x^2(l-x)^2$$

$$\frac{dW}{dx} = c_1(l-2x) + c_2(2l^2x + 4x^3 - 6lx^2)$$

$$V_{\max} = \frac{1}{2} \int_0^l P \left(\frac{dW}{dx} \right)^2 dx = \frac{P}{2} \int_0^l [c_1^2(l^2 - 4lx + 4x^2) + c_2^2(4l^4x^2 + 16x^6 + 36l^2x^4 + 16l^2x^4 - 48lx^5 - 24l^3x^3) + 2c_1c_2(2l^3x + 4lx^3 - 6l^2x^2 - 4l^2x^2 - 8x^4 + 12lx^3)] dx$$

$$= \frac{P}{2} \left(\frac{1}{3} c_1^2 l^3 + \frac{2}{105} c_2^2 l^7 + \frac{2}{15} c_1 c_2 l^5 \right)$$

$$T_{\max} = \frac{\omega^2}{2} \int_0^l \rho [W(x)]^2 dx = \frac{\rho \omega^2}{2} \int_0^l [c_1^2(l^2x^2 + x^4 - 2lx^3) + c_2^2(l^4x^4 + x^8 + 6l^2x^6 - 4lx^7 - 4l^3x^5) + c_1c_2(2l^3x^3 + 6lx^5 - 6l^2x^4 - 2x^6)] dx$$

$$= \frac{\rho \omega^2}{2} \left[\frac{1}{30} c_1^2 l^5 + \frac{1}{630} c_2^2 l^9 + \frac{1}{70} c_1 c_2 l^7 \right]$$

$$X = P \left(\frac{1}{3} c_1^2 l^3 + \frac{2}{105} c_2^2 l^7 + \frac{2}{15} c_1 c_2 l^5 \right)$$

$$Y = \rho \left(\frac{1}{30} c_1^2 l^5 + \frac{1}{630} c_2^2 l^9 + \frac{1}{70} c_1 c_2 l^7 \right)$$

$$\frac{\partial X}{\partial c_1} = P \left(\frac{2}{3} c_1 l^3 + \frac{2}{15} c_2 l^5 \right), \quad \frac{\partial X}{\partial c_2} = P \left(\frac{4}{105} c_2 l^7 + \frac{2}{15} c_1 l^5 \right)$$

$$\frac{\partial Y}{\partial c_1} = \rho \left(\frac{1}{15} c_1 l^5 + \frac{1}{70} c_2 l^7 \right), \quad \frac{\partial Y}{\partial c_2} = \rho \left(\frac{1}{315} c_2 l^9 + \frac{1}{70} c_1 l^7 \right)$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0, \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \quad \text{give}$$

$$P \left(c_1 \frac{2}{3} l^3 + c_2 \frac{2}{15} l^5 \right) - \omega^2 P \left(c_1 \frac{l^5}{15} + c_2 \frac{l^7}{70} \right) = 0$$

$$P \left(c_2 \frac{4l^7}{105} + c_1 \frac{2l^5}{15} \right) - \omega^2 P \left(c_2 \frac{l^9}{315} + c_1 \frac{l^7}{70} \right) = 0$$

$$\text{i.e., } c_1 \left(\frac{2}{3} Pl^3 - \frac{P\omega^2 l^5}{15} \right) + c_2 \left(\frac{2}{15} Pl^5 - \frac{P\omega^2 l^7}{70} \right) = 0$$

$$c_1 \left(\frac{2}{15} Pl^5 - \frac{P\omega^2 l^7}{70} \right) + c_2 \left(\frac{4}{105} Pl^7 - \frac{P\omega^2 l^9}{315} \right) = 0$$

Setting the determinant of the coefficients of c_1 and c_2 to zero gives

$$\left(\frac{2}{3} Pl^3 - \frac{1}{15} P\omega^2 l^5 \right) \left(\frac{4}{105} Pl^7 - \frac{1}{315} P\omega^2 l^9 \right) - \left(\frac{2}{15} Pl^5 - \frac{1}{70} P\omega^2 l^7 \right) \left(\frac{2}{15} Pl^5 - \frac{1}{70} P\omega^2 l^7 \right) = 0$$

$$\text{i.e., } \omega^4 - 112 \left(\frac{P}{Pl^2} \right) \omega^2 + 1008 \left(\frac{P}{Pl^2} \right)^2 = 0$$

$$\omega^2 = 9.8697 \frac{P}{Pl^2}, 102.1302 \frac{P}{Pl^2}$$

$$\therefore \omega_1 = 3.1416 \sqrt{\frac{P}{Pl^2}}, \quad \omega_2 = 10.1059 \sqrt{\frac{P}{Pl^2}}$$

First mode

Third mode

Exact solution is :

$$\omega_1 = 3.1416 \sqrt{\frac{P}{Pl^2}}, \quad \omega_2 = 6.2832 \sqrt{\frac{P}{Pl^2}}, \quad \omega_3 = 9.4248 \sqrt{\frac{P}{Pl^2}}$$

8.68

$$(a) \quad V_{\max} = \frac{1}{2} \int_0^l EA \left(\frac{dU}{dx} \right)^2 dx$$

$$= \frac{1}{2} \int_0^l EA \left(\frac{c_1}{l} \right)^2 dx$$

$$= \frac{EA c_1^2}{2l} \text{ for uniform beam}$$

$$T_{\max} = \frac{P\omega^2}{2} \int_0^l A U^2 dx = \frac{P\omega^2}{2} \int_0^l A \frac{c_1^2}{l^2} x^2 dx$$

$$= \frac{\omega^2 P A c_1^2 l}{6} \text{ for uniform beam}$$

$$T_{\max} = V_{\max} \text{ gives}$$

$$\omega^2 = \frac{3E}{Pl^2}$$

$$\therefore \omega_1 = 1.73205 \sqrt{\frac{E}{Pl^2}}$$

$$U(x) = \frac{c_1}{l} x$$

$$\frac{dU}{dx} = \frac{c_1}{l}$$

$$(b) V_{\max} = \frac{EA}{2} \int_0^l \left(\frac{dU}{dx} \right)^2 dx$$

$$= \frac{EA}{2} \left[\frac{c_1^2}{l} + \frac{4}{3} \frac{c_2^2}{l} + \frac{2c_1 c_2}{l} \right]$$

$$T_{\max} = \frac{\rho \omega^2}{2} \int_0^l A \cdot U^2 dx$$

$$= \frac{\rho A \omega^2}{2} \left[\frac{1}{3} c_1^2 l + \frac{1}{5} c_2^2 l + \frac{1}{2} c_1 c_2 l \right]$$

$$U(x) = \frac{c_1 x}{l} + \frac{c_2 x^2}{l^2}$$

$$\frac{dU}{dx} = \frac{c_1}{l} + \frac{2c_2 x}{l^2}$$

Eqs. (E.1) and (E.2) of Example 8.12 give

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \quad \text{and} \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \quad (E.1)$$

$$\text{Where } X = \frac{EA}{2l} \left[c_1^2 + \frac{4}{3} c_2^2 + 2c_1 c_2 \right] \quad (E.2)$$

$$\text{and } Y = \frac{\rho A l}{2} \left[\frac{1}{3} c_1^2 + \frac{1}{5} c_2^2 + \frac{1}{2} c_1 c_2 \right] \quad (E.3)$$

Eqs. (E.1) to (E.3) give

$$c_1 \left(\frac{EA}{l} - \frac{1}{3} \omega^2 \rho A l \right) + c_2 \left(\frac{EA}{l} - \frac{1}{4} \omega^2 \rho A l \right) = 0$$

$$c_2 \left(\frac{4}{3} \frac{EA}{l} - \frac{1}{5} \omega^2 \rho A l \right) + c_1 \left(\frac{EA}{l} - \frac{1}{4} \omega^2 \rho A l \right) = 0$$

Frequency equation is

$$\begin{vmatrix} (1 - \frac{1}{3} \lambda) & (1 - \frac{1}{4} \lambda) \\ (1 - \frac{1}{4} \lambda) & (\frac{4}{3} - \frac{1}{5} \lambda) \end{vmatrix} = \frac{\lambda^2}{240} - \frac{13\lambda}{90} + \frac{1}{3} = 0$$

$$\text{Where } \lambda = \frac{\omega^2 \rho l^2}{E}$$

$$\lambda = 2.48596, 32.1807$$

$$\therefore \omega_1 = 1.57669 \sqrt{\frac{E}{\rho l^2}}, \quad \omega_2 = 5.67280 \sqrt{\frac{E}{\rho l^2}}$$

8.69

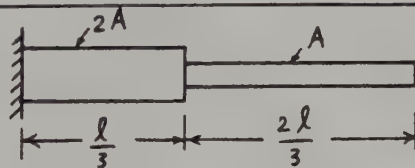
$$U(x) = c_1 \sin \frac{\pi x}{2l} + c_2 \sin \frac{3\pi x}{2l}$$

$$\frac{dU}{dx} = \frac{\pi}{2l} \left(c_1 \cos \frac{\pi x}{2l} + 3c_2 \cos \frac{3\pi x}{2l} \right)$$

$$V_{\max} = \frac{EA}{2} \left(\frac{\pi}{2l} \right)^2 \left[\int_0^{l/3} 2 \left(c_1 \cos \frac{\pi x}{2l} + 3c_2 \cos \frac{3\pi x}{2l} \right)^2 dx \right.$$

$$\left. + \int_{l/3}^l \left(c_1 \cos \frac{\pi x}{2l} + 3c_2 \cos \frac{3\pi x}{2l} \right)^2 dx \right]$$

$$\text{Using the relations } \int \cos^2 \frac{\pi x}{2l} dx = \frac{x}{2} + \frac{l}{2\pi} \sin \frac{\pi x}{l},$$



$$\int \cos^2 \frac{3\pi x}{2l} dx = \frac{x}{2} + \frac{l}{6\pi} \sin \frac{3\pi x}{l},$$

$$\int \cos \frac{\pi x}{2l} \cdot \cos \frac{3\pi x}{2l} dx = \frac{\sin \frac{\pi x}{l}}{(2\pi/l)} + \frac{\sin \frac{2\pi x}{l}}{(4\pi/l)},$$

$$V_{\max} = \frac{EA\pi^2}{8l^2} \left[C_1^2 \left(\frac{2}{3}l + \frac{l}{2\pi} \sin \frac{\pi}{3} \right) + C_2^2 (6l) \right. \\ \left. + C_1 C_2 \left(\frac{3l}{\pi} \sin \frac{\pi}{3} + \frac{3l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$T_{\max} = \frac{\rho\omega^2}{2} \left[\int_0^{\frac{l}{3}} 2A \left\{ C_1^2 \sin^2 \frac{\pi x}{2l} + C_2^2 \sin^2 \frac{3\pi x}{2l} \right. \right. \\ \left. \left. + 2C_1 C_2 \sin \frac{\pi x}{2l} \cdot \sin \frac{3\pi x}{2l} \right\} dx + \int_{\frac{l}{3}}^l A \left\{ C_1^2 \sin^2 \frac{\pi x}{2l} \right. \right. \\ \left. \left. + C_2^2 \sin^2 \frac{3\pi x}{2l} + 2C_1 C_2 \sin \frac{\pi x}{2l} \sin \frac{3\pi x}{2l} \right\} dx \right]$$

$$\text{But } \int \sin^2 \frac{\pi x}{2l} dx = \frac{x}{2} - \frac{l}{2\pi} \sin \frac{\pi x}{l},$$

$$\int \sin^2 \frac{3\pi x}{2l} dx = \frac{x}{2} - \frac{l}{6\pi} \sin \frac{3\pi x}{l},$$

$$\int \sin \frac{\pi x}{2l} \cdot \sin \frac{3\pi x}{2l} dx = \frac{\sin \frac{\pi x}{l}}{(2\pi/l)} - \frac{\sin \frac{2\pi x}{l}}{(4\pi/l)}$$

$$T_{\max} = \frac{\rho A \omega^2}{2} \left[C_1^2 \left(\frac{2l}{3} - \frac{l}{2\pi} \sin \frac{\pi}{3} \right) + C_2^2 \left(\frac{2l}{3} \right) + C_1 C_2 \left(\frac{l}{\pi} \sin \frac{\pi}{3} - \frac{l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$X = \frac{EA\pi^2}{4l^2} \left[C_1^2 \left(\frac{2l}{3} + \frac{l}{2\pi} \sin \frac{\pi}{3} \right) + C_2^2 (6l) + C_1 C_2 \left(\frac{3l}{\pi} \sin \frac{\pi}{3} + \frac{3l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$Y = \rho A \left[C_1^2 \left(\frac{2l}{3} - \frac{l}{2\pi} \sin \frac{\pi}{3} \right) + C_2^2 \left(\frac{2l}{3} \right) + C_1 C_2 \left(\frac{l}{\pi} \sin \frac{\pi}{3} - \frac{l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$\frac{\partial X}{\partial C_1} = \frac{EA\pi^2}{4l^2} \left[2C_1 \left(\frac{2l}{3} + \frac{l}{2\pi} \sin \frac{\pi}{3} \right) + C_2 \left(\frac{3l}{\pi} \sin \frac{\pi}{3} + \frac{3l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$= \frac{EA\pi^2}{4l^2} [2C_1(0.8045l) + C_2(1.2405l)]$$

$$\frac{\partial X}{\partial C_2} = \frac{EA\pi^2}{4l^2} \left[C_2(12l) + C_1 \left(\frac{3l}{\pi} \sin \frac{\pi}{3} + \frac{3l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$= \frac{EA\pi^2}{4l^2} [C_2(12l) + C_1(1.2405l)]$$

$$\frac{\partial Y}{\partial C_1} = \rho A \left[2C_1 \left(\frac{2l}{3} - \frac{l}{2\pi} \sin \frac{\pi}{3} \right) + C_2 \left(\frac{l}{\pi} \sin \frac{\pi}{3} - \frac{l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$= \rho A [2C_1(0.5288l) + C_2(0.1378l)]$$

$$\frac{\partial Y}{\partial C_2} = \rho A \left[C_2 \left(\frac{4l}{3} \right) + C_1 \left(\frac{l}{\pi} \sin \frac{\pi}{3} - \frac{l}{2\pi} \sin \frac{2\pi}{3} \right) \right]$$

$$= \rho A [C_2(1.3333l) + C_1(0.1378l)]$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \quad \text{and} \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \quad \text{give}$$

$$c_1 \left(\frac{EA\pi^2}{l} 0.40225 - \rho A \omega^2 l 1.0576 \right) + c_2 \left(\frac{EA\pi^2}{l} 0.31012 - \rho A \omega^2 l 0.1378 \right) = 0$$

$$c_1 \left(\frac{EA\pi^2}{l} 0.31012 - \rho A \omega^2 l 0.1380 \right) + c_2 \left(\frac{EA\pi^2}{l} 3.0 - \rho A \omega^2 l 1.3333 \right) = 0$$

Frequency equation is

$$\left(0.40225 \frac{EA\pi^2}{l} - \rho A \omega^2 l 1.0576 \right) \left(\frac{EA\pi^2}{l} 3.0 - \rho A \omega^2 l 1.3333 \right) - \left(\frac{EA\pi^2}{l} 0.31012 - \rho A \omega^2 l 0.1380 \right)^2 = 0$$

$$\text{i.e.,} \quad \omega^4 - 25.7090 \omega^2 \left(\frac{E}{\rho l^2} \right) + 77.7692 \left(\frac{E}{\rho l^2} \right)^2 = 0$$

$$\omega^2 = \left(\frac{25.7090 \pm 18.7010}{2} \right) \frac{E}{\rho l^2}$$

$$\therefore \omega_1 = 1.8719 \sqrt{\frac{E}{\rho l^2}}, \quad \omega_2 = 4.7022 \sqrt{\frac{E}{\rho l^2}}$$

8.70

$$U(x) = c_1 \sin \frac{\pi x}{2l} + c_2 \sin \frac{3\pi x}{2l}, \quad \frac{dU}{dx} = \frac{\pi}{2l} \left(c_1 \cos \frac{\pi x}{2l} + 3c_2 \cos \frac{3\pi x}{2l} \right)$$

$$V_{\max} = \frac{1}{2} \int_0^l EA(x) \left(\frac{dU}{dx} \right)^2 dx = \frac{1}{2} \int_0^l 2EA_0 \left(1 - \frac{x}{l} \right) \left(\frac{\pi}{2l} \right)^2 \left(c_1^2 \cos^2 \frac{\pi x}{2l} + 9c_2^2 \cos^2 \frac{3\pi x}{2l} + 6c_1 c_2 \cos \frac{\pi x}{2l} \cos \frac{3\pi x}{2l} \right) dx$$

$$= c_1^2 \frac{EA_0 \pi^2}{16l} \left(1 + \frac{4}{\pi^2} \right) + c_2^2 \frac{9EA_0 \pi^2}{16l} \left(1 + \frac{4}{9\pi^2} \right) + c_1 c_2 \frac{3EA_0 \pi^2}{2l}$$

$$T_{\max} = \frac{\omega^2}{2} \int_0^l m(x) \cdot U^2 \cdot dx = m_0 \omega^2 \int_0^l \left(1 - \frac{x}{l} \right) \left(c_1^2 \sin^2 \frac{\pi x}{2l} + c_2^2 \sin^2 \frac{3\pi x}{2l} + 2c_1 c_2 \sin \frac{\pi x}{2l} \sin \frac{3\pi x}{2l} \right) dx$$

$$= c_1^2 \frac{\omega^2 m_0 l}{4} \left(1 - \frac{4}{\pi^2} \right) + c_2^2 \frac{\omega^2 m_0 l}{4} \left(1 - \frac{4}{9\pi^2} \right)$$

$$+ c_1 c_2 \frac{2\omega^2 m_0 l}{\pi^2}$$

$$X = c_1^2 \frac{EA_0}{l} (0.86685) + c_2^2 \frac{EA_0}{l} (5.80165) + c_1 c_2 \frac{EA_0}{l} (1.5)$$

$$Y = c_1^2 m_0 l (0.14868) + c_2^2 m_0 l (0.23874) + c_1 c_2 m_0 l (0.20264)$$

$$\frac{\partial X}{\partial c_1} = c_1 \frac{EA_0}{l} (1.73370) + c_2 \frac{EA_0}{l} (1.5)$$

$$\frac{\partial X}{\partial c_2} = c_1 \frac{EA_0}{l} (1.5) + c_2 \frac{EA_0}{l} (11.60330)$$

$$\frac{\partial Y}{\partial c_1} = c_1 m_0 l (0.29736) + c_2 m_0 l (0.20264)$$

$$\frac{\partial Y}{\partial c_2} = c_1 m_0 l (0.20246) + c_2 m_0 l (0.47748)$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \quad \text{and} \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \quad \text{give}$$

$$c_1 \left(\frac{EA_0}{l} 1.73370 - \omega^2 m_0 l 0.29736 \right) + c_2 \left(\frac{EA_0}{l} 1.5 - \omega^2 m_0 l 0.20264 \right) = 0$$

$$c_1 \left(\frac{EA_0}{l} 1.5 - \omega^2 m_0 l 0.20246 \right) + c_2 \left(\frac{EA_0}{l} 11.60330 - \omega^2 m_0 l 0.47748 \right) = 0$$

Frequency equation is

$$\left(\frac{EA_0}{l} 1.7337 - \omega^2 m_0 l 0.29736 \right) \left(\frac{EA_0}{l} 11.6033 - \omega^2 m_0 l 0.47748 \right) - \left(\frac{EA_0}{l} 1.5 - \omega^2 m_0 l 0.20264 \right)^2 = 0$$

$$\text{or} \quad \omega^4 - 36.369 \omega^2 \left(\frac{EA_0}{m_0 l^2} \right) + 177.04292 \left(\frac{EA_0}{m_0 l^2} \right)^2 = 0$$

$$\omega^2 = \left(\frac{36.369 \pm 24.7898}{2} \right) \frac{EA_0}{m_0 l^2}$$

$$\therefore \omega_1 = 2.4062 \sqrt{\frac{EA_0}{m_0 l^2}}, \quad \omega_2 = 5.5299 \sqrt{\frac{EA_0}{m_0 l^2}}$$

8.71

For a string, $V_{\max} = \frac{1}{2} \int_0^l \rho \left(\frac{dW}{dx} \right)^2 dx$, $T_{\max} = \frac{\omega^2}{2} \int_0^l \rho W^2 dx$

Here $W(x) = c_1 x(l-x) + c_2 x^2(l-x)^2$

$$\frac{dW}{dx} = c_1 l - 2c_1 x + 2c_2 l^2 x - 6c_2 l x^2 + 4c_2 x^3$$

$$\int_0^l \left(\frac{dW}{dx} \right)^2 dx = c_1^2 \frac{l^3}{3} + c_2^2 \frac{2}{105} l^7 + c_1 c_2 \frac{2}{15} l^5$$

$$\int_0^l W^2 dx = c_1^2 \frac{l^5}{30} + c_2^2 \frac{l^9}{630} + c_1 c_2 \frac{l^7}{70}$$

Equating $T_{\max}$ and $V_{\max}$, we get $\omega^2 = \frac{X}{Y}$

$$\text{where} \quad X = \frac{\rho}{2} \left[c_1^2 \frac{l^3}{3} + c_2^2 \frac{2}{105} l^7 + c_1 c_2 \frac{2}{15} l^5 \right]$$

$$\text{and} \quad Y = \frac{\rho}{2} \left[c_1^2 \frac{l^5}{30} + c_2^2 \frac{l^9}{630} + c_1 c_2 \frac{l^7}{70} \right]$$

$$\frac{\partial(\omega^2)}{\partial c_1} = \frac{\partial(\omega^2)}{\partial c_2} = 0 \quad \text{yield Eqs. (E}_{11}) \text{ and (E}_{12}) \text{ of Example 8.13.}$$

$$\frac{\partial X}{\partial c_1} = \frac{P}{2} \left[c_1 \frac{2}{3} l^3 + c_2 \frac{2}{15} l^5 \right], \quad \frac{\partial Y}{\partial c_1} = \frac{P}{2} \left[c_1 \frac{l^5}{15} + c_2 \frac{l^7}{70} \right]$$

$$\frac{\partial X}{\partial c_2} = \frac{P}{2} \left[c_1 \frac{2}{15} l^5 + c_2 \frac{4}{105} l^7 \right], \quad \frac{\partial Y}{\partial c_2} = \frac{P}{2} \left[c_1 \frac{l^7}{70} + c_2 \frac{l^9}{315} \right]$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \Rightarrow c_1 \left(\frac{Pl^3}{3} - \frac{P\omega^2 l^5}{30} \right) + c_2 \left(\frac{Pl^5}{15} - \frac{P\omega^2 l^7}{140} \right) = 0$$

$$\frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \Rightarrow c_1 \left(\frac{Pl^5}{15} - \frac{P\omega^2 l^7}{140} \right) + c_2 \left(\frac{2Pl^7}{105} - \frac{P\omega^2 l^9}{630} \right) = 0$$

By setting the determinant of the coefficient matrix of c_1 and c_2 to zero, we get the frequency equation as

$$\begin{vmatrix} \left(\frac{1}{3} - \frac{\lambda}{30} \right) & \left(\frac{1}{15} - \frac{\lambda}{140} \right) \\ \left(\frac{1}{15} - \frac{\lambda}{140} \right) & \left(\frac{2}{105} - \frac{\lambda}{630} \right) \end{vmatrix} = \lambda^2 - 112\lambda + 1008 = 0$$

where $\lambda = \frac{Pl^2 \omega^2}{P}$.

$$\lambda_1 = 9.8722, \quad \lambda_2 = 102.4144$$

$$\therefore \omega_1 = 3.142 \sqrt{\frac{P}{Pl^2}}, \quad \omega_2 = 10.12 \sqrt{\frac{P}{Pl^2}}$$

8.72

```

C=====
C
C PROGRAM
C MAIN PROGRAM FOR CALLING THE SUBROUTINE NONEQN
C
C=====
C FOLLOWING LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(2)
      COMMON /ABLOCK/ BETA
      DATA N,NINT,ITER,EPS/2,50,100,0.000001/
      DO 7 I=1,2
        IF (I .EQ. 1) BETA=0.01
        IF (I .EQ. 2) BETA=1.0
        WRITE (12,8) BETA
8      FORMAT (//,2X,7H BETA =,E15.6,/)
        XS=0.001
        XINC=0.1
C END OF PROBLEM-DEPENDENT DATA
      WRITE (12,10) N,XS,XINC,NINT,ITER,EPS
10     FORMAT (//,28H ROOTS OF NONLINEAR EQUATION,/,6H DATA:,,
2      2H N,4X,2H =,14,/,3H XS,3X,2H =,E15.6,/,8H XINC =,E15.6,/,
3      8H NINT =,14,/,8H ITER =,14,/,8H EPS =,E15.6)
      CALL NONEQN (N,X,XS,XINC,NINT,ITER,EPS)
      WRITE (12,20) (X(II),II=1,N)
20     FORMAT (//,7H ROOTS:,,/(E15.6))
7      CONTINUE
      STOP
      END

```

```

=====
C
C FUNCTION F(Y)
C THIS FUNCTION IS PROBLEM-DEPENDENT
C
=====
      FUNCTION F(Y)
      COMMON /ABLOCK/ BETA
      F=(Y/BETA)*TAN(Y)-1.0
      RETURN
      END
-----
      BETA = 0.100000E-01

```

ROOTS OF NONLINEAR EQUATION

```

DATA:
N      = 2
XS     = 0.100000E-02
XINC   = 0.100000E+00
NINT   = 50
ITER   = 100
EPS    = 0.100000E-05

```

```

ROOTS:
0.998336E-01
0.314477E+01

```

BETA = 0.100000E+01

ROOTS OF NONLINEAR EQUATION

```

DATA:
N      = 2
XS     = 0.100000E-02
XINC   = 0.100000E+00
NINT   = 50
ITER   = 100
EPS    = 0.100000E-05

```

```

ROOTS:
0.860334E+00
0.342562E+01

```

8.73

Frequency equation for a fixed-fixed beam is (from Fig. 8.15):

$$\cos \beta l \cosh \beta l - 1 = 0$$

The main program, the function F(Y) and the output are given below.


```

C=====
C
C PROGRAM
C MAIN PROGRAM FOR CALLING THE SUBROUTINE NONEQN
C
C=====
C FOLLOWING LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(5)
      DATA N,NINT,ITER,EPS/5,50,100,0.000001/
      XS=2.0
      XINC=0.5
C END OF PROBLEM-DEPENDENT DATA
      WRITE (12,10) N,XS,XINC,NINT,ITER,EPS
10      FORMAT (//,28H ROOTS OF NONLINEAR EQUATION,//,6H DATA://,
2      2H N,4X,2H =,14,/,3H XS,3X,2H =,E15.6,/,8H XINC =,E15.6,/,
3      8H NINT =,14,/,8H ITER =,14,/,8H EPS =,E15.6)
      CALL NONEQN (N,X,XS,XINC,NINT,ITER,EPS)
      WRITE (12,20) (X(I1),I1=1,N)
20      FORMAT (//,7H ROOTS://,(E15.6))
      STOP
      END
C=====
C
C FUNCTION F(Y)
C THIS FUNCTION IS PROBLEM-DEPENDENT
C
C=====
      FUNCTION F(Y)
      F=COS(Y)*COSH(Y)-1.0
      RETURN
      END

```

ROOTS OF NONLINEAR EQUATION

DATA:

N = 5

XS = 0.200000E+01

XINC = 0.500000E+00

NINT = 50

ITER = 100

EPS = 0.100000E-05

ROOTS:

0.473004E+01
0.785320E+01
0.109956E+02
0.141372E+02
0.172788E+02

8.74 From Example 8.7, the n^{th} mode shape is given by

$$W_n(x) = C_{1n} \left\{ (\cos \beta_n x - \cosh \beta_n x) - \left(\frac{\cos \beta_n l - \cosh \beta_n l}{\sin \beta_n l - \sinh \beta_n l} \right) (\sin \beta_n x - \sinh \beta_n x) \right\}$$

where $\beta_n l = 3.926602, 7.068583, 10.210176, 13.351768$ for $n = 1, 2, 3, 4$.
The program and results are given below.

```

C =====
C
C PROBLEM 8.74
C
C =====

```

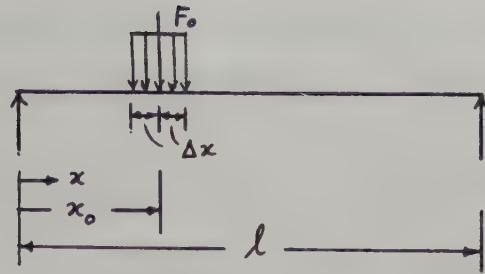
```

      DIMENSION X(4),W(4,9)
      N=4
      M=9
      MD=10
C N      = NUMBER OF MODE SHAPES TO BE COMPUTED
C M      = NUMBER OF STATIONS IN THE BEAM AT WHICH VALUE OF DEFLECTION
C          IS REQUIRED
C MD     = NUMBGR OF SEGMENTS IN THE BEAM
C W(N,M) = MATRIX CONTAINING THE MODE SHAPES IN ROWS
      DATA X/3.926602,7.068583,10.210176,13.351768/
      DO 10 I=1,N
      DO 20 J=1,M
      Y=X(I)*REAL(J)/REAL(MD)
      YI=X(I)
20  W(I,J)=(COS(Y)-COSH(Y))-((COS(YI)-COSH(YI))/(SIN(YI)-SINH(YI)))*
2  (SIN(Y)-SINH(Y))
10  CONTINUE
      WRITE (86,30)
30  FORMAT (//,2X,43H MODE SHAPES OF FIXED-SIMPLY SUPPORTED BEAM,/)
      DO 40 I=1,N
40  WRITE (86,50) I,(W(I,J),J=1,M)
50  FORMAT (/,2X,6H MODE: ,14,4E15.6,/, (12X,4E15.6))
      STOP
      END

```

MODE SHAPES OF FIXED-SIMPLY SUPPORTED BEAM

MODE: 1	-0.133996E+00	-0.455738E+00	-0.848519E+00	-0.120675E+01
	-0.144486E+01	-0.150550E+01	-0.136498E+01	-0.103456E+01
	-0.557241E+00			
MODE: 2	-0.382233E+00	-0.107449E+01	-0.149510E+01	-0.131923E+01
	-0.570351E+00	0.422676E+00	0.119881E+01	0.139349E+01
	0.917175E+00			
MODE: 3	-0.690370E+00	-0.147476E+01	-0.112212E+01	0.204393E+00
	0.130049E+01	0.114192E+01	-0.111633E+00	-0.126025E+01
	-0.120557E+01			
MODE: 4	-0.100204E+01	-0.141423E+01	0.927429E-01	0.139201E+01
	0.539917E+00	-0.114441E+01	-0.107568E+01	0.642578E+00
	0.137500E+01			



Equation of motion is given by Eq. (8.77):

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x, t) \quad (E_1)$$

Representation of load:

Load is $f(x)$ which is zero everywhere except over a length $2\Delta x$ at $x = x_0$, where it is equal to $(F_0/2\Delta x)$. The load $f(x)$ can be expanded into Fourier sine series as

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{l} \quad (E_2)$$

$$\begin{aligned} \text{where} \\ f_n &= \frac{2}{l} \int_0^{x_0 - \Delta x} (0) \sin \frac{n\pi x}{l} dx + \frac{2}{l} \int_{x_0 - \Delta x}^{x_0 + \Delta x} \frac{F_0}{2\Delta x} \sin \frac{n\pi x}{l} dx \\ &\quad + \frac{2}{l} \int_{x_0 + \Delta x}^l (0) \sin \frac{n\pi x}{l} dx = \frac{F_0}{l \Delta x} \int_{x_0 - \Delta x}^{x_0 + \Delta x} \sin \frac{n\pi x}{l} dx \\ &= \frac{2 F_0}{l} \sin \frac{n\pi x_0}{l} \frac{\sin \left(\frac{n\pi \Delta x}{l} \right)}{\left(\frac{n\pi \Delta x}{l} \right)} \quad (E_3) \end{aligned}$$

$$\text{As } \Delta x \rightarrow 0, (E_3) \text{ becomes } f_n = \frac{2 F_0}{l} \sin \frac{n\pi x_0}{l} \quad (E_4)$$

$$\therefore f(x) = \frac{2 F_0}{l} \sum_{n=1}^{\infty} \sin \frac{n\pi x_0}{l} \sin \frac{n\pi x}{l} \quad (E_5)$$

Since F_0 is moving along the bridge with constant velocity v , at time t , F_0 will be at a distance $x_0 = vt$ from left support. Hence the load distribution of Eq. (E5) can be rewritten as

$$f(x, t) = \frac{2 F_0}{l} \left\{ \sin \frac{\pi vt}{l} \sin \frac{\pi x}{l} + \sin \frac{2\pi vt}{l} \sin \frac{2\pi x}{l} + \dots \right\} \quad (E_6)$$

Setting $\omega = \frac{\pi v}{l}$, (E6) can be expressed as

$$f(x, t) = \frac{2 F_0}{l} \left\{ \sin \frac{\pi x}{l} \sin \omega t + \sin \frac{2\pi x}{l} \sin 2\omega t + \dots \right\} \quad (E_7)$$

Equation of motion, (E₁), becomes

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x, t) = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{l} \sin n\omega t \quad (E_8)$$

where $F_n = \frac{2 F_0}{l}$; $n = 1, 2, \dots$

We can find the solution of Eq. (E₈) by superposing the solutions of the individual harmonic components. Thus, we need to find, for the n^{th} harmonic, the solution of the following equation:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = F_n \sin \frac{n\pi x}{l} \sin n\omega t \quad (E_9)$$

The solution (particular integral) of Eq. (E₉) can be taken as

$$w(x, t) = w_n \sin \frac{n\pi x}{l} \sin n\omega t \quad (E_{10})$$

Substitution of (E₁₀) into (E₉) gives

$$w_n = \frac{F_n l^4}{EI (n\pi)^4 \left\{ 1 - \left(\frac{n\omega}{\omega_n} \right)^2 \right\}} \quad (E_{11})$$

where $\omega_n = n^2 \pi^2 \sqrt{\frac{EI}{\rho A l^4}}$ (E₁₂)

Adding the homogeneous solution to the particular integral of Eq. (E₁₀), the general solution can be expressed as

$$w(x, t) = \sum_{i=1}^{\infty} \sin \frac{i\pi x}{l} \left\{ A_i \cos \omega_i t + B_i \sin \omega_i t \right\} + w_n \sin \frac{n\pi x}{l} \sin n\omega t \quad (E_{13})$$

The initial conditions are given by

$$w(x, 0) = 0, \quad \frac{\partial w}{\partial t}(x, 0) = 0 \quad (E_{14})$$

These initial conditions are satisfied if

$$A_i = 0 \text{ for all } i, \quad B_i = 0 \text{ for all } i \neq n$$

$$B_n = -w_n \left(\frac{n\omega}{\omega_n} \right)$$

$$w(x, t) = \frac{F_n l^4}{EI (n\pi)^4 \left\{ 1 - \left(\frac{n\omega}{\omega_n} \right)^2 \right\}} \sin \frac{n\pi x}{l} \left\{ \sin n\omega t - \frac{n\omega}{\omega_n} \sin \omega_n t \right\} \quad (E_{15})$$

Since $\omega_n = n^2 \omega_1$, where ω_1 is the fundamental frequency of the bridge, Eq. (E15) becomes

$$w(x,t) = \frac{F_n l^4}{EI \pi^4 \left\{ n^4 - \left(\frac{n\omega}{\omega_1} \right)^2 \right\}} \sin \frac{n\pi x}{l} \left\{ \sin n\omega t - \frac{n\omega}{n^2 \omega_1} \sin n^2 \omega_1 t \right\} \quad (E16)$$

Thus the total response of the bridge can be expressed as

$$\begin{aligned} w(x,t) = & \frac{2 F_0 l^3}{EI \pi^4} \left[\frac{\sin \frac{\pi x}{l}}{1 - \left(\frac{\omega}{\omega_1} \right)^2} \left\{ \sin \omega t - \left(\frac{\omega}{\omega_1} \right) \sin \omega_1 t \right\} \right. \\ & + \frac{\sin \frac{2\pi x}{l}}{2^4 - \left(\frac{2\omega}{\omega_1} \right)^2} \left\{ \sin 2\omega t - \left(\frac{\omega}{2\omega_1} \right) \sin 4\omega_1 t \right\} \\ & + \frac{\sin \frac{3\pi x}{l}}{3^4 - \left(\frac{3\omega}{\omega_1} \right)^2} \left\{ \sin 3\omega t - \left(\frac{\omega}{3\omega_1} \right) \sin 9\omega_1 t \right\} + \dots \left. \right] \quad (E17) \end{aligned}$$

Chapter 9

Vibration Control

9.1

For static balance :

$$\sum F_x = 0, \quad \sum F_y = 0$$

Here

$$\sum F_x = m_1 r_1 \cos \theta_1 + m_2 r_2 \cos \theta_2 + m_3 r_3 \cos \theta_3 + m_c r_c \cos \theta_c = 0$$

$$\Rightarrow 35(110) \cos 40^\circ + 15(90) \cos 220^\circ + 25(130) \cos 290^\circ + m_c r_c \cos \theta_c = 0$$

$$\Rightarrow 2949.1 - 1034.1 + 1111.5 + m_c r_c \cos \theta_c = 0$$

$$\Rightarrow m_c r_c \cos \theta_c = -3026.5$$

$$\sum F_y = m_1 r_1 \sin \theta_1 + m_2 r_2 \sin \theta_2 + m_3 r_3 \sin \theta_3 + m_c r_c \sin \theta_c = 0$$

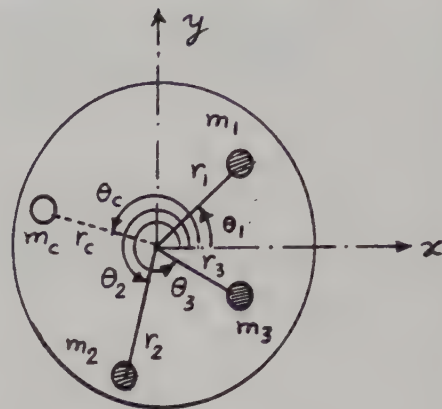
$$\Rightarrow 35(110) \sin 40^\circ + 15(90) \sin 220^\circ + 25(130) \sin 290^\circ + m_c r_c \sin \theta_c = 0$$

$$\Rightarrow 2474.78 - 867.78 - 3054.025 + m_c r_c \sin \theta_c = 0$$

$$\Rightarrow m_c r_c \sin \theta_c = 1447.025$$

$$m_c r_c = \left[(-3026.5)^2 + (1447.025)^2 \right]^{1/2} = 3354.6361 \text{ g-mm}$$

$$\theta_c = \tan^{-1} \left(\frac{1447.025}{-3026.5} \right) = -25.5525^\circ$$



9.2

Unbalance due to hole is proportional to (r.m).

Let m_5 = mass removed from 5th hole

r_5 = radius at which 5th hole is drilled = 5 in.

θ_5 = angle at which 5th hole is drilled

$$\sum F_x = \sum_{i=1}^5 r_i m_i \cos \theta_i = 0$$

$$\Rightarrow 4(4) \cos 0^\circ + 4(4) \cos 60^\circ + 4(5) \cos 120^\circ + 4(5) \cos 180^\circ + r_5 m_5 \cos \theta_5 = 0$$

$$\Rightarrow 16 + 8 - 10 - 20 + 5 m_5 \cos \theta_5 = 0$$

$$\Rightarrow 5 m_5 \cos \theta_5 = 6$$

$$\sum F_y = \sum_{i=1}^5 r_i m_i \sin \theta_i = 0$$

$$\Rightarrow 4(4) \sin 0^\circ + 4(4) \sin 60^\circ + 4(5) \sin 120^\circ + 4(5) \sin 180^\circ + r_5 m_5 \sin \theta_5 = 0$$

$$\Rightarrow 0 + 13.856 + 17.320 + 0 + r_5 m_5 \sin \theta_5 = 0$$

$$\Rightarrow 5 m_5 \sin \theta_5 = -31.176$$

$$\therefore m_5 = \frac{1}{5} \sqrt{(6)^2 + (-31.176)^2} = 6.3496 \text{ oz}$$

$$\theta_5 = \tan^{-1} \left(\frac{-31.176}{6} \right) = -79.1063^\circ$$

9.3

$$\sum F_x = \sum_{i=1}^4 m_i r_i \cos \theta_i = 0$$

Since all r_i are same, we have

$$0.5 \cos 10^\circ + 0.7 \cos 100^\circ + 1.2 \cos 190^\circ + m_4 \cos \theta_4 = 0$$

$$\Rightarrow 0.4924 - 0.12152 - 1.18176 + m_4 \cos \theta_4 = 0$$

$$\Rightarrow m_4 \cos \theta_4 = 0.81088$$

$$\sum F_y = \sum_{i=1}^4 m_i r_i \sin \theta_i = 0$$

$$\Rightarrow 0.5 \sin 10^\circ + 0.7 \sin 100^\circ + 1.2 \sin 190^\circ + m_4 \sin \theta_4 = 0$$

$$\Rightarrow 0.0868 + 0.6894 - 0.2083 + m_4 \sin \theta_4 = 0$$

$$\Rightarrow m_4 \sin \theta_4 = -0.56788$$

$$\therefore m_4 = \left[(0.81088)^2 + (-0.56788)^2 \right]^{1/2} = 0.98996 \text{ oz}$$

$$\theta_4 = \tan^{-1} \left(\frac{-0.56788}{0.81088} \right) = -35.0045^\circ$$

9.4

$$\vec{A}_u = (10 \text{ mils}, 40^\circ \text{ CCW})$$

$$\vec{A}_{u+w} = (19 \text{ mils}, 150^\circ \text{ CCW})$$

$$A_w = \left[A_u^2 + A_{u+w}^2 - 2 A_u A_{u+w} \cos (\phi - \theta) \right]^{1/2}$$

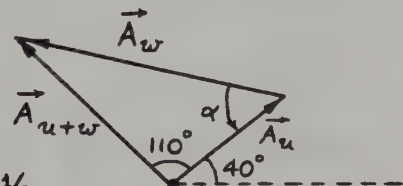
$$= (10^2 + 19^2 - 2(10)(19) \cos 110^\circ)^{1/2} = 18.1943$$

$$W_o = \text{original unbalance} = \left(\frac{A_u}{A_w} \right)' W = \left(\frac{10}{18.1943} \right) 6 = 3.2977 \text{ oz}$$

$$\alpha = \cos^{-1} \left[\frac{A_u^2 + A_w^2 - A_{u+w}^2}{2 A_u A_w} \right] = \cos^{-1} \left[\frac{10^2 + 18.1943^2 - 19^2}{2(10)(18.1943)} \right]$$

$$= \cos^{-1} (0.1925) = 78.9038^\circ \text{ CCW}$$

Grinding wheel will be balanced if a weight of 3.2977 oz is added at 78.9038° clockwise from the position of the trial weight or $(65 + 78.9038) = 143.9038^\circ$ clockwise from the phase mark.



9.5

$$\vec{A}_u = (6.5 \text{ mils}, 15^\circ \text{ CW})$$

$$\vec{A}_{u+w} = (8.8 \text{ mils}, 35^\circ \text{ CCW})$$

$$A_w = [A_u^2 + A_{u+w}^2 - 2 A_u A_{u+w} \cos(\phi - \theta)]^{1/2}$$

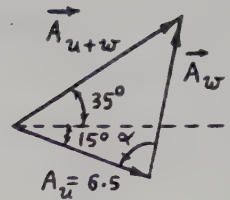
$$= [6.5^2 + 8.8^2 - 2(6.5)(8.8) \cos 50^\circ]^{1/2} = 6.7937 \text{ mils}$$

$$W_0 = \text{original unbalance} = \left(\frac{A_u}{A_w}\right) W = \left(\frac{6.5}{6.7937}\right) 2 = 1.9135 \text{ oz}$$

$$\alpha = \cos^{-1} \left[\frac{A_u^2 + A_w^2 - A_{u+w}^2}{2 A_u A_w} \right] = \cos^{-1} \left[\frac{6.5^2 + 6.7937^2 - 8.8^2}{2(6.5)(6.7937)} \right]$$

$$= \cos^{-1}(0.1241) = 82.8690^\circ \text{ CCW}$$

Flywheel will be balanced if a weight of 1.9135 oz is added at 82.8690° CW from the position of the trial weight or $(-45 + 82.8690) = 37.8690^\circ$ CW from the phase mark.



9.6

$$\vec{A}_u = (4 \text{ mils}, 45^\circ \text{ CCW})$$

$$\vec{A}_{u+w} = (8 \text{ mils}, 145^\circ \text{ CCW})$$

$$A_w = [A_u^2 + A_{u+w}^2 - 2 A_u A_{u+w} \cos(\phi - \theta)]^{1/2}$$

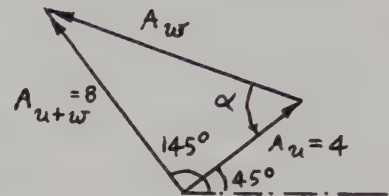
$$= (4^2 + 8^2 - 2(4)(8) \cos 100^\circ)^{1/2} = 9.5453 \text{ mils}$$

$$W_0 = \text{original unbalance} = \left(\frac{A_u}{A_w}\right) W = \left(\frac{4}{9.5453}\right) 4 = 1.6762 \text{ oz}$$

$$\alpha = \cos^{-1} \left[\frac{A_u^2 + A_w^2 - A_{u+w}^2}{2 A_u A_w} \right] = \cos^{-1} \left(\frac{4^2 + 9.5453^2 - 8^2}{2 \times 4 \times 9.5453} \right)$$

$$= \cos^{-1}(0.5646) = 55.6261^\circ \text{ CCW}$$

Grinding wheel will be balanced if a weight of 1.6762 oz is added at 55.6261° CW from the position of trial weight or $(20 + 55.6261) = 75.6261^\circ$ CW from the phase mark.



9.7

Figure 3.11 (b) shows that the phase angle between the displacement and the driving force is 90° at a frequency ratio of $r=1$. The driving force leads the displacement. Since the driving force is due to the centrifugal force of the eccentric mass, the direction of the unbalanced mass is 90° ahead of the displacement. Hence the mass has to be removed at $22.9^\circ + 90^\circ = 319^\circ$ as indicated by the protractor.

9.8

For static balance, sum of all inertia forces must be zero:

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + m_{b1} \vec{r}_{b1} + m_{b2} \vec{r}_{b2} = \vec{0} \quad (E_1)$$

which can be written in scalar form as

$$m_1 r_1 \cos \theta_1 + m_2 r_2 \cos \theta_2 + m_3 r_3 \cos \theta_3 + m_{b1} r_{b1} \cos \theta_{b1} + m_{b2} r_{b2} \cos \theta_{b2} = 0 \quad (E_2)$$

$m_1 r_1 \sin \theta_1 + m_2 r_2 \sin \theta_2 + m_3 r_3 \sin \theta_3 + m_{b1} r_{b1} \sin \theta_{b1} + m_{b2} r_{b2} \sin \theta_{b2} = 0$ (E₃)
For dynamic balance, sum of moments due to inertia forces must be zero about any point. The moments about the point, defined by the intersection of z-axis with plane A, gives

$$l_1 m_1 \omega^2 \vec{r}_1 + l_2 m_2 \omega^2 \vec{r}_2 + l_3 m_3 \omega^2 \vec{r}_3 + l_{b1}^0 m_{b1} \omega^2 \vec{r}_{b1} + l_{b2} m_{b2} \omega^2 \vec{r}_{b2} = \vec{0} \quad (E_4)$$

$$\text{or} \quad l_1 m_1 r_1 \cos \theta_1 + l_2 m_2 r_2 \cos \theta_2 + l_3 m_3 r_3 \cos \theta_3 + l_{b2} m_{b2} r_{b2} \cos \theta_{b2} = 0 \quad (E_5)$$

$$l_1 m_1 r_1 \sin \theta_1 + l_2 m_2 r_2 \sin \theta_2 + l_3 m_3 r_3 \sin \theta_3 + l_{b2} m_{b2} r_{b2} \sin \theta_{b2} = 0 \quad (E_6)$$

Eqs. (E₅) and (E₆) give

$$m_{b2} r_{b2} = \frac{1}{l_{b2}} \left\{ (l_1 m_1 r_1 \cos \theta_1 + l_2 m_2 r_2 \cos \theta_2 + l_3 m_3 r_3 \cos \theta_3)^2 + (l_1 m_1 r_1 \sin \theta_1 + l_2 m_2 r_2 \sin \theta_2 + l_3 m_3 r_3 \sin \theta_3)^2 \right\}^{1/2} \quad (E_7)$$

$$\theta_{b2} = \tan^{-1} \left\{ \frac{l_1 m_1 r_1 \sin \theta_1 + l_2 m_2 r_2 \sin \theta_2 + l_3 m_3 r_3 \sin \theta_3}{l_1 m_1 r_1 \cos \theta_1 + l_2 m_2 r_2 \cos \theta_2 + l_3 m_3 r_3 \cos \theta_3} \right\} \quad (E_8)$$

Once $m_{b2} r_{b2}$ and θ_{b2} are found, Eqs. (E₂) and (E₃) can be used to determine

$$m_{b1} r_{b1} = \left\{ (m_1 r_1 \cos \theta_1 + m_2 r_2 \cos \theta_2 + m_3 r_3 \cos \theta_3 + m_{b2} r_{b2} \cos \theta_{b2})^2 + (m_1 r_1 \sin \theta_1 + m_2 r_2 \sin \theta_2 + m_3 r_3 \sin \theta_3 + m_{b2} r_{b2} \sin \theta_{b2})^2 \right\}^{1/2} \quad (E_9)$$

$$\theta_{b1} = \tan^{-1} \left\{ \frac{m_1 r_1 \sin \theta_1 + m_2 r_2 \sin \theta_2 + m_3 r_3 \sin \theta_3 + m_{b2} r_{b2} \sin \theta_{b2}}{m_1 r_1 \cos \theta_1 + m_2 r_2 \cos \theta_2 + m_3 r_3 \cos \theta_3 + m_{b2} r_{b2} \cos \theta_{b2}} \right\} \quad (E_{10})$$

9.9

Let W_B and W_C be the amounts (weights) of the material removed in planes B and C at angular locations θ_B and θ_C , respectively.

The material removed must balance the weights added

(temporarily) in planes A and D. Measuring angles counter-clockwise from the horizontal (x-axis), we need to satisfy the following equations for static balancing of the rotor:

$$\sum_{i=1}^4 W_i r_i \cos \theta_i + W_B r_B \cos \theta_B + W_C r_C \cos \theta_C = 0 \quad (E_1)$$

$$\sum_{i=1}^4 W_i r_i \sin \theta_i + W_B r_B \sin \theta_B + W_C r_C \sin \theta_C = 0 \quad (E_2)$$

Since $W_1 = W_2 = W_3 = W_4 = 0.2 \text{ lb}$, $\theta_1 = 90^\circ$, $\theta_2 = -60^\circ$, $\theta_3 = 120^\circ$, $\theta_4 = -120^\circ$, $r_1 = r_2 = r_3 = r_4 = 3 \text{ in}$, and $r_B = r_C = 4 \text{ in}$, Eqs.

(E₁) and (E₂) yield

$$0.2(3) \cos 90^\circ + 0.2(3) \cos(-60^\circ) + 0.2(3) \cos 120^\circ + 0.2(3) \cos(-120^\circ) + 4 W_B \cos \theta_B + 4 W_C \cos \theta_C = 0$$

$$\Rightarrow W_B \cos \theta_B + W_C \cos \theta_C = 0.075 \quad (E_3)$$

$$0.2(3) \sin 90^\circ + 0.2(3) \sin(-60^\circ) + 0.2(3) \sin 120^\circ + 0.2(3) \sin(-120^\circ) + 4 W_B \sin \theta_B + 4 W_C \sin \theta_C = 0$$

$$\Rightarrow W_B \sin \theta_B + W_C \sin \theta_C = -0.0201 \quad (E_4)$$

For dynamic balancing (taking moments from plane A),

$$W_3 r_3 \delta_3 \cos \theta_3 + W_4 r_4 \delta_4 \cos \theta_4 + W_B r_B \delta_B \cos \theta_B + W_C r_C \delta_C \cos \theta_C = 0$$

$$W_3 r_3 \delta_3 \sin \theta_3 + W_4 r_4 \delta_4 \sin \theta_4 + W_B r_B \delta_B \sin \theta_B + W_C r_C \delta_C \sin \theta_C = 0$$

i.e.,

$$0.2(3)(24) \cos 120^\circ + 0.2(3)(24) \cos(-120^\circ) + W_B(4)(4) \cos \theta_B + W_C(4)(20) \cos \theta_C = 0$$

$$0.2(3)(24) \sin 120^\circ + 0.2(3)(24) \sin(-120^\circ) + W_B(4)(4) \sin \theta_B + W_C(4)(20) \sin \theta_C = 0$$

i.e.,

$$16 W_B \cos \theta_B + 80 W_C \cos \theta_C = 14.4 \quad (E_5)$$

$$16 W_B \sin \theta_B + 80 W_C \sin \theta_C = 0 \quad (E_6)$$

Eqs. (E₃) and (E₅) give

$$16(0.075) + 64 W_C \cos \theta_C = 14.4$$

$$\text{or } W_C \cos \theta_C = 0.20625 \quad (E_7)$$

Eqs. (E₄) and (E₆) give

$$16(-0.0201) + 64 W_C \sin \theta_C = 0$$

$$\text{or } W_C \sin \theta_C = 0.005025 \quad (E_8)$$

Eqs. (E₇) and (E₈) yield

$$W_C = \left[(0.20625)^2 + (0.005025)^2 \right]^{1/2} = 0.2063 \text{ lb} \quad (E_9)$$

$$\theta_C = \tan^{-1} \left(\frac{0.005025}{0.20625} \right) = 1.3957^\circ$$

Eqs. (E₃) and (E₉) give

$$W_B \cos \theta_B = -0.13124$$

Eqs. (E₄) and (E₉) give

$$W_B \sin \theta_B = -0.025125$$

$$\therefore W_B = \left[(-0.13124)^2 + (-0.025125)^2 \right]^{1/2} = 0.1336 \text{ lb}$$

$$\theta_B = \tan^{-1} \left(\frac{-0.025125}{-0.13124} \right) = 10.8377^\circ$$

$\therefore$ Amount of material to be removed:

0.1336 lb at 10.8377° CCW at radius 4" in plane B
and 0.2063 lb at 1.3957° CCW at radius 4" in plane C.

9.10

$$W_C = 2 \text{ lb}, W_D = 4 \text{ lb}, W_E = 3 \text{ lb}, r_C = 2'', r_D = 3'', r_E = 1'', \\ \theta_C = 90^\circ, \theta_D = 220^\circ, \theta_E = -30^\circ.$$

Let W_A, r_A, θ_A and W_G, r_G, θ_G denote the weights added in planes A and G, respectively.

For static balancing,

$$W_C r_C \cos \theta_C + W_D r_D \cos \theta_D + W_E r_E \cos \theta_E + W_A r_A \cos \theta_A + W_G r_G \cos \theta_G = 0 \\ \Rightarrow W_A r_A \cos \theta_A + W_G r_G \cos \theta_G = 6.594 \quad (E_1)$$

$$W_C r_C \sin \theta_C + W_D r_D \sin \theta_D + W_E r_E \sin \theta_E + W_A r_A \sin \theta_A + W_G r_G \sin \theta_G = 0 \\ \Rightarrow W_A r_A \sin \theta_A + W_G r_G \sin \theta_G = 5.2136 \quad (E_2)$$

For dynamic balancing, we take moments about the left bearing (plane B):

$$W_C r_C \delta_C \cos \theta_C + W_D r_D \delta_D \cos \theta_D + W_E r_E \delta_E \cos \theta_E + W_A r_A \delta_A \cos \theta_A \\ + W_G r_G \delta_G \cos \theta_G = 0 \\ \Rightarrow -16 W_A r_A \cos \theta_A + 88 W_G r_G \cos \theta_G = 201.408 \quad (E_3)$$

$$W_C r_C \delta_C \sin \theta_C + W_D r_D \delta_D \sin \theta_D + W_E r_E \delta_E \sin \theta_E + W_A r_A \delta_A \sin \theta_A \\ + W_G r_G \delta_G \sin \theta_G = 0 \\ \Rightarrow -16 W_A r_A \sin \theta_A + 88 W_G r_G \sin \theta_G = 372.544 \quad (E_4)$$

Eqs. (E₁) and (E₃) yield

$$-16 (6.594) + 104 W_G r_G \cos \theta_G = 201.408 \\ \text{or } W_G r_G \cos \theta_G = 2.9511 \quad (E_5)$$

Eqs. (E₂) and (E₄) give

$$-16 (5.2136) + 104 W_G r_G \sin \theta_G = 372.544 \\ \text{or } W_G r_G \sin \theta_G = 4.3842 \quad (E_6)$$

Eqs. (E₅) and (E₆) give

$$W_G r_G = \left[(2.9511)^2 + (4.3842)^2 \right]^{1/2} = 5.2849 \text{ lb-in} \\ \theta_G = \tan^{-1} \left(\frac{4.3842}{2.9511} \right) = 56.0549^\circ \quad (E_7)$$

Eqs. (E1), (E2) and (E7) yield

$$\left. \begin{aligned} W_A r_A \cos \theta_A &= 6.594 - 5.2849 \cos 56.0549^\circ = 3.6429 \\ W_A r_A \sin \theta_A &= 5.2136 - 5.2849 \sin 56.0549^\circ = 0.9007 \end{aligned} \right\} \quad (E_8)$$

Eqs. (E8) provide

$$\left. \begin{aligned} W_A r_A &= \left[(3.6429)^2 + (0.9007)^2 \right]^{1/2} = 3.7526 \text{ lb-in} \\ \theta_A &= \tan^{-1} \left(\frac{0.9007}{3.6429} \right) = 13.8877^\circ \end{aligned} \right\} \quad (E_9)$$

If the balancing weights are placed at a radial distance of 2" in planes A and G, we have $r_A = r_G = 2''$ and hence

$$W_A = 1.8763 \text{ lb}, \quad \theta_A = 13.8877^\circ; \quad W_G = 2.6425 \text{ lb}, \quad \theta_G = 56.0549^\circ$$

9.11

$$\begin{aligned} \vec{V}_A &= 5 \angle 100^\circ \\ &= -0.8682 + i 4.9240 \end{aligned}$$

$$\begin{aligned} \vec{V}_B &= 4 \angle 180^\circ \\ &= -4.0 - i 0.0 \end{aligned}$$

$$\begin{aligned} \vec{V}_A' &= 6.5 \angle 120^\circ \\ &= -3.25 + i 5.6292 \end{aligned}$$

$$\begin{aligned} \vec{V}_B' &= 4.5 \angle 140^\circ \\ &= -3.4472 + i 2.8925 \end{aligned}$$

$$\begin{aligned} \vec{V}_A'' &= 6.0 \angle 90^\circ \\ &= 0.0 + i 6.0 \end{aligned}$$

$$\begin{aligned} \vec{V}_B'' &= 7.0 \angle 60^\circ \\ &= 3.5 + i 6.0622 \end{aligned}$$

$$\begin{aligned} \vec{W}_L &= 2.0 \angle 30^\circ \\ &= 1.7321 + i 1.0 \end{aligned}$$

$$\begin{aligned} \vec{W}_R &= 2.0 \angle 0^\circ \\ &= 2.0 + i 0.0 \end{aligned}$$

$$\begin{aligned} \vec{A}_{AL} &= \frac{\vec{V}_A' - \vec{V}_A}{\vec{W}_L} \\ &= 1.2420 \angle -46.4914^\circ \end{aligned}$$

$$\begin{aligned} \vec{A}_{AR} &= \frac{\vec{V}_A'' - \vec{V}_A}{\vec{W}_R} \\ &= 0.6913 \angle 51.0984^\circ \end{aligned}$$

$$\begin{aligned} \vec{A}_{BL} &= \frac{\vec{V}_B' - \vec{V}_B}{\vec{W}_L} \\ &= 4.2315 \angle -49.7058^\circ \end{aligned}$$

$$\begin{aligned} \vec{A}_{BR} &= \frac{\vec{V}_B'' - \vec{V}_B}{\vec{W}_R} \\ &= 2.1217 \angle -39.2682^\circ \end{aligned}$$

$$\vec{U}_L = \frac{\vec{A}_{BR} \vec{V}_A - \vec{A}_{AR} \vec{V}_B}{\vec{A}_{BR} \vec{A}_{AL} - \vec{A}_{AR} \vec{A}_{BL}}$$

$$\vec{U}_R = \frac{\vec{A}_{BL} \vec{V}_A - \vec{A}_{AL} \vec{V}_B}{\vec{A}_{BL} \vec{A}_{AR} - \vec{A}_{AL} \vec{A}_{BR}}$$

$$\vec{B}_L = -\vec{U}_L = 4.2315 \angle -49.7058^\circ$$

$$\vec{B}_R = -\vec{U}_R = 2.1217 \angle -39.2682^\circ$$

9.12

$$(a) \omega = 1000 \times \frac{2\pi}{60} = 104.72 \text{ rad/s}$$

Centrifugal forces due to rotating masses (all parallel to yz-plane) are

$$F_1 = m_1 r_1 \omega^2 = (50/1000)(8/100)(104.72)^2 = 43.8651 \text{ N}$$

$$F_2 = m_2 r_2 \omega^2 = (20/1000)(5/100)(104.72)^2 = 10.9663 \text{ N}$$

$$F_3 = m_3 r_3 \omega^2 = (40/1000)(6/100)(104.72)^2 = 26.3191 \text{ N}$$

These can be written in vector form as

$$\vec{F}_1 = F_1 \angle \theta_1 = 43.8651 \angle 0^\circ = 43.8651 \vec{j}$$

$$\vec{F}_2 = F_2 \angle \theta_2 = 10.9663 \angle 120^\circ = -5.4832 \vec{j} + 9.4971 \vec{k}$$

$$\vec{F}_3 = F_3 \angle \theta_3 = 26.3191 \angle 200^\circ = -24.7319 \vec{j} - 9.0017 \vec{k}$$

The moments of these forces taken about the bearing at A must be balanced by the moment of the bearing reaction at B. Hence

$$\sum \vec{M}_A = 0.2 \vec{i} \times 43.8651 \vec{j} + 0.3 \vec{i} \times (-5.4832 \vec{j} + 9.4971 \vec{k}) + 0.9 \vec{i} \times (-24.7319 \vec{j} - 9.0017 \vec{k}) + 1.1 \vec{i} \times \vec{R}_B = \vec{0}$$

where $\vec{R}_B$ = reaction at bearing B and 'x' denotes the cross product. This gives

$$-15.1307 \vec{k} + 5.2524 \vec{j} + 1.1 \vec{i} \times \vec{R}_B = \vec{0} \quad \dots (E_1)$$

Let $\vec{R}_B = (a \vec{j} + b \vec{k})$. Then $1.1 \vec{i} \times (a \vec{j} + b \vec{k}) = 1.1a \vec{k} - 1.1b \vec{j}$
 (E_1) and (E_2) give $a = 13.7552$, $b = 4.7749$ --- (E2)

$$\therefore \vec{R}_B = 13.7552 \vec{j} + 4.7749 \vec{k}$$

Similarly, by taking moments about B,

$$-0.2 \vec{i} \times (-24.7319 \vec{j} - 9.0017 \vec{k}) - 0.6 \vec{i} \times (-5.4832 \vec{j} + 9.4971 \vec{k})$$

$$-0.9 \vec{i} \times (43.8651 \vec{j}) - 1.1 \vec{i} \times \vec{R}_A = \vec{0}$$

where $\vec{R}_A$ = reaction at bearing A = $c \vec{j} + d \vec{k}$.

$$-31.2423 \vec{k} + 3.898 \vec{j} - 1.1c \vec{k} + 1.1d \vec{j} = \vec{0}$$

$$c = -28.4021, \quad d = -3.5436$$

$$\therefore \vec{R}_A = -28.4021 \vec{j} - 3.5436 \vec{k}$$

Note that these are rotating vectors.

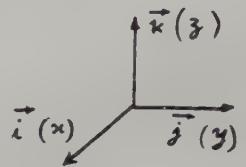
(b) Since the planes L and R pass through the bearings A and B, the balancing forces are given by

$$\vec{B}_R = -\vec{R}_B = -13.7552 \vec{j} - 4.7749 \vec{k} = m_R r \omega^2 (\cos \theta_R \vec{j} + \sin \theta_R \vec{k})$$

$$\text{i.e.} \quad m_R (0.25) (104.72)^2 \cos \theta_R = -13.7552$$

$$m_R (0.25) (104.72)^2 \sin \theta_R = -4.7749$$

$$\text{i.e.} \quad m_R \cos \theta_R = -0.005017, \quad m_R \sin \theta_R = -0.001742$$



$$m_R = \sqrt{(-0.005017)^2 + (-0.001742)^2} = 0.005311 \text{ kg} = 5.311 \text{ g}$$

$$\theta_R = \tan^{-1} \left(\frac{-0.001742}{-0.005017} \right) = 19.1480^\circ + 180^\circ = 199.1480^\circ$$

$$\vec{B}_L = -\vec{R}_A = 28.4021 \vec{j} + 3.5436 \vec{k} = m_L r \omega^2 (\cos \theta_L \vec{j} + \sin \theta_L \vec{k})$$

$$\text{i.e. } m_L \cos \theta_L = \frac{28.4021}{(0.25)(104.72)^2} = 0.01036$$

$$m_L \sin \theta_L = \frac{3.5436}{(0.25)(104.72)^2} = 0.001293$$

$$m_L = \sqrt{(0.01036)^2 + (0.001293)^2} = 0.01044 \text{ kg} = 10.44 \text{ g}$$

$$\theta_L = \tan^{-1} (0.001293 / 0.01036) = 7.1141^\circ$$

Note: Angles are measured clockwise from z-axis while looking from A towards B.

9.13

Stiffness of steel shaft between bearings = $k = \left\{ \frac{48EI}{l^3} \right\}$

$$k = \frac{48(30 \times 10^6)}{(30)^3} \left(\frac{\pi}{64} (1)^4 \right) = 2618 \text{ lb/in}$$



$$(a) \text{ critical speed} = \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2618(386.4)}{100}} = 100.5781 \text{ rad/sec}$$

(b) Vibration amplitude of the rotor (steady state value):

Eg. (9.32) gives the amplitude in x-direction as

$$X = D m \omega^2 e \text{ when damping is zero} = \frac{m \omega^2 e}{|k - m \omega^2|}$$

$$\text{Here } \omega = 1200 \text{ rpm} = 1200 (2\pi) / 60 = 125.664 \text{ rad/sec}$$

$$e = 0.5'' , m \omega^2 = \frac{100}{386.4} (125.664)^2 = 4086.8118$$

$$X = \left(\frac{100}{386.4} \right) (125.664)^2 (0.5) \frac{1}{|2618 - 4086.8118|} = 1.3912''$$

Similarly the amplitude in y-direction is given by

$$Y = X = 1.3912''$$

$$\text{Resultant amplitude of the flywheel} = R = \sqrt{X^2 + Y^2} = 1.9675''$$

(c) Force transmitted to the bearing supports

$$= k R = 2618 (1.9675) = 5150.915 \text{ lb}$$

9.14

Considering bearings as simple supports, the spring constant of the beam is

$$k = \frac{48 EI}{l^3} \quad \text{where } l = \text{distance between bearings.}$$

Let r = variable position of center of mass, and

δ_{st} = static radial displacement of center of mass.

Then equation of motion is

$$mr \omega^2 = k(r - \delta_{st}) \quad \text{or} \quad \frac{k}{k - m\omega^2} = \frac{r}{\delta_{st}} \quad \text{or} \quad r = \frac{k \cdot \delta_{st}}{k - m\omega^2}$$

$$\text{Dynamic force} = F = mr \omega^2 = \frac{m\omega^2 k \delta_{st}}{k - m\omega^2}$$

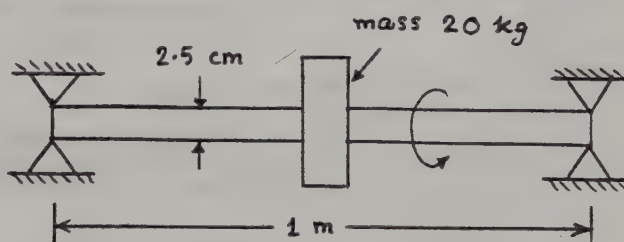
Since F acts at the middle of the beam, $\sigma_{max} = \frac{M \cdot y_{max}}{I} = \frac{F l}{4} \cdot \frac{d}{2I}$
where d = diameter of shaft.

$$\sigma_{max} = \frac{F l d}{8I} = \frac{m \omega^2 k \delta_{st} l d}{8I (k - m\omega^2)}$$

Substituting the expression for k ,

$$\sigma_{max} = \frac{m \omega^2 \delta_{st} l d}{8I} \left(\frac{48 EI}{l^3} \right) \left\{ \frac{1}{\left(\frac{48 EI}{l^3} - m\omega^2 \right)} \right\}$$

9.15



Stiffness of a simply supported beam:

$$k = \frac{48 EI}{l^3} = \frac{48 (207 (10^9)) \left(\frac{\pi}{64} (0.025^4) \right)}{1^3} = 19.0521 (10^4) \text{ N/m}$$

Natural frequency of the system:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{19.0521 (10^4)}{20}} = 97.6014 \text{ rad/sec}$$

$$\text{Frequency of rotor (speed of shaft): } \omega = \frac{6000}{60} (2\pi) = 628.32 \text{ rad/sec}$$

$$\text{Whirl amplitude of the disc: } A = \frac{a r^2}{\sqrt{(1 - r^2)^2 + (2 \zeta r)^2}}$$

(a) At operating speed:

$$r = \frac{\omega}{\omega_n} = \frac{628.32}{97.6014} = 6.4376$$

$$A = \frac{(0.005) (6.4376^2)}{\sqrt{(1 - 6.4376^2)^2 + (2 (0.01) (6.4376))^2}} = 0.005124 \text{ m}$$

(b) At critical speed (Eq. 9.34):

Critical speed:

$$\omega = \omega_{\text{cri}} = \frac{\omega_n}{\left\{1 - \frac{1}{2} \left(\frac{c}{\omega_n}\right)^2\right\}^{\frac{1}{2}}} = \frac{97.6014}{\left\{1 - \frac{1}{2} \left(\frac{39.0406}{97.6014}\right)^2\right\}^{\frac{1}{2}}} = 101.7565 \text{ rad/sec}$$

$$\text{where } c = 2 \sqrt{k m \zeta} = 2 \sqrt{(19.0521 (10^4)) (20) (0.01)} = 39.0406 \text{ N-s/m.}$$

$$r = \frac{\omega}{\omega_n} = \frac{101.7565}{97.6014} = 1.0426$$

$$A = \frac{(0.005) (1.0426^2)}{\sqrt{(1 - 1.0426^2)^2 + (2 (0.01) (1.0426))^2}} = 0.06074 \text{ m}$$

(c) At 1.5 times critical speed:

$$r = \frac{1.5 \omega_{\text{cri}}}{\omega_n} = \frac{152.6347}{97.6014} = 1.5638$$

$$A = \frac{(0.005) (1.5638^2)}{\sqrt{(1 - 1.5638^2)^2 + (2 (0.01) (1.5638))^2}} = 0.008457 \text{ m}$$

9.16

(a) At operating speed:

$r = 6.4376$; $\omega = 628.32 \text{ rad/sec}$. Deflection of mass center:

$$R = a \left\{ \frac{1 + (2 \zeta r)^2}{(1 - r^2)^2 + (2 \zeta r)^2} \right\}^{\frac{1}{2}} \\ = (0.005) \left\{ \frac{1 + (2 (0.01) (6.4376))^2}{(1 - 6.4376^2)^2 + (2 (0.01) (6.4376))^2} \right\}^{\frac{1}{2}} = 1.2465 (10^{-4}) \text{ m}$$

$$\text{Centrifugal force: } m \omega^2 R = (20) (628.32^2) (1.2465 (10^{-4})) = 984.2015 \text{ N}$$

$$\text{Bearing reactions: } R_1 = R_2 = \frac{m \omega^2 R}{2} = 492.1007 \text{ N}$$

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64} (0.025^4) = 1.9175 (10^{-8}) \text{ m}^4$$

Maximum bending stress:

$$\frac{(R_1 \frac{\ell}{2}) \frac{d}{2}}{I} = \frac{492.1007 \left(\frac{1}{2}\right) \left(\frac{0.025}{2}\right)}{1.9175 (10^{-8})} = 1.6040 (10^8) \text{ N/m}^2$$

(b) At critical speed:

$$r = 1.0426 ; \omega = 101.7565 \text{ rad/sec}$$

$$R = (0.005) \left\{ \frac{1 + (2 (0.01) (1.0426))^2}{(1 - 1.0426^2)^2 + (2 (0.01) (1.0426))^2} \right\}^{\frac{1}{2}} = 0.05589 \text{ m}$$

$$\text{Centrifugal force: } m \omega^2 R = (20) (101.7565^2) (0.05589) = 11574.1319 \text{ N}$$

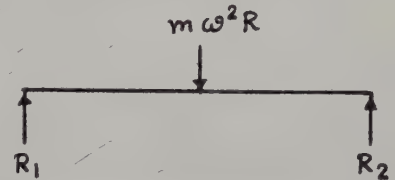
$$R_1 = R_2 = 5787.0659 \text{ N}$$

Maximum bending stress:

$$\frac{R_1 \left(\frac{\ell}{2}\right) \left(\frac{d}{2}\right)}{I} = \frac{5787.0659 \left(\frac{1}{2}\right) \left(\frac{0.025}{2}\right)}{1.9175 (10^{-8})} = 18.8627 (10^8) \text{ N/m}^2$$

(c) At 1.5 times critical speed:

$$r = 1.5638 ; \omega = 152.6347 \text{ rad/sec}$$



$$R = (0.005) \left\{ \frac{1 + (2 (0.01) (1.5638))^2}{(1 - 1.5638^2)^2 + (2 (0.01) (1.5638))^2} \right\}^{\frac{1}{2}} = 0.003460 \text{ m}$$

Centrifugal force:

$$m \omega^2 R = (20) (152.6347^2) (0.003460) = 1612.1767 \text{ N}$$

$$R_1 = R_2 = 806.0884 \text{ N}$$

Maximum bending stress:

$$\frac{(806.0884) \left(\frac{1}{2}\right) \left(\frac{0.025}{2}\right)}{1.9175 (10^{-8})} = 2.6274 (10^8) \text{ N/m}^2$$

9.17

Stiffness of beam (k):

$$k = \frac{48 EI}{\ell^3} = \frac{48 (71 (10^9)) (1.9175 (10^{-8}))}{1^3} = 65347.7344 \text{ N/m}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{65347.7344}{20}} = 57.1611 \text{ rad/sec}$$

(a) At operating speed:

$$\omega = \frac{6000}{60} (2\pi) = 628.32 \text{ rad/sec}; r = \frac{\omega}{\omega_n} = \frac{628.32}{57.1611} = 10.9921$$

$$\begin{aligned} \text{Whirl amplitude of rotor: } A &= \frac{a r^2}{\left\{ (1 - r^2)^2 + (2 \zeta r)^2 \right\}^{\frac{1}{2}}} \\ &= \frac{(0.005) (10.9921^2)}{\left\{ (1 - 10.9921^2)^2 + (2 (0.01) (10.9921))^2 \right\}^{\frac{1}{2}}} = 0.005041 \text{ m} \end{aligned}$$

(b) At critical speed:

$$c = 2 \sqrt{k m \zeta} = 2 \sqrt{(65347.7344) (20) (0.01)} = 22.8644 \text{ N-s/m}$$

$$\omega_{\text{cri}} = \omega = \frac{\omega_n}{\left\{ 1 - \frac{1}{2} \left(\frac{c}{\omega_n} \right)^2 \right\}^{\frac{1}{2}}} = \frac{57.1611}{\left\{ 1 - \frac{1}{2} \left(\frac{22.8644}{57.1611} \right)^2 \right\}^{\frac{1}{2}}} = 59.5946 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = \frac{59.5946}{57.1611} = 1.0426$$

$$A = \frac{(0.005) (1.0426^2)}{\left\{ (1 - 1.0426^2)^2 + (2 (0.01) (1.0426))^2 \right\}^{\frac{1}{2}}} = 0.06074 \text{ m}$$

(c) At 1.5 times critical speed:

$$r = \frac{1.5 \omega_{\text{cri}}}{\omega_n} = \frac{89.3919}{57.1611} = 1.5638$$

$$A = \frac{(0.005) (1.5638^2)}{\left\{ (1 - 1.5638^2)^2 + (2 (0.01) (1.5638))^2 \right\}^{\frac{1}{2}}} = 0.008457 \text{ m}$$

9.18

(a) At operating speed:

$r = 10.9921$; $\omega = 628.32 \text{ rad/sec}$. Deflection of mass center of disc:

$$R = \frac{(0.005) \left\{ 1 + (2 (0.01) (10.9921))^2 \right\}^{\frac{1}{2}}}{\left\{ (1 - 10.9921^2)^2 + (2 (0.01) (10.9921))^2 \right\}^{\frac{1}{2}}} = 0.4272 (10^{-4}) \text{ m}$$

$$\text{Centrifugal force: } m \omega^2 R = 20 (628.32^2) (0.4272 (10^{-4})) = 337.3052 \text{ N}$$

$$\text{Bearing reactions: } R_1 = R_2 = \frac{m \omega^2 R}{2} = 168.6526 \text{ N}$$

$$\text{Maximum bending stress: } \frac{R_1 \left(\frac{\ell}{2}\right) \left(\frac{d}{2}\right)}{I} = \frac{(168.6526) \left(\frac{1}{2}\right) \left(\frac{0.025}{2}\right)}{1.9175 (10^{-8})} = 0.5497 (10^8) \text{ N/m}^2$$

(b) At critical speed:

$$r = 1.0426 ; \omega = 59.5946 \text{ rad/sec}$$

$$R = \frac{(0.005) \left\{ 1 + (2 (0.01) (1.0426))^2 \right\}^{\frac{1}{2}}}{\left\{ (1 - 1.0426^2)^2 + (2 (0.01) (1.0426))^2 \right\}^{\frac{1}{2}}} = 0.05589 \text{ m}$$

$$\text{Centrifugal force: } m \omega^2 R = (20) (59.5946^2) (0.05589) = 3969.8850 \text{ N}$$

$$R_1 = R_2 = 1984.9425 \text{ N}$$

Maximum bending stress:

$$\frac{R_1 \left(\frac{\ell}{2}\right) \left(\frac{d}{2}\right)}{I} = \frac{(1984.9425) \left(\frac{1}{2}\right) \left(\frac{0.025}{2}\right)}{1.9175 (10^{-8})} = 6.4698 (10^8) \text{ N/m}^2$$

(c) At 1.5 times critical speed:

$$r = 1.5638 ; \omega = 89.3919 \text{ rad/sec}$$

$$R = \frac{(0.005) \left\{ 1 + (2 (0.01) (1.5638))^2 \right\}^{\frac{1}{2}}}{\left\{ (1 - 1.5638^2)^2 + (2 (0.01) (1.5638))^2 \right\}^{\frac{1}{2}}} = 0.003460 \text{ m}$$

$$\text{Centrifugal force: } m \omega^2 R = (20) (89.3919^2) (0.003460) = 552.9711 \text{ N}$$

$$R_1 = R_2 = 276.4855 \text{ N}$$

Maximum bending stress:

$$\frac{R_1 \left(\frac{\ell}{2}\right) \left(\frac{d}{2}\right)}{I} = \frac{(276.4855) \left(\frac{1}{2}\right) \left(\frac{0.025}{2}\right)}{1.9175 (10^{-8})} = 0.9012 (10^8) \text{ N/m}^2$$

9.19 $k = 3.75 (10^6) \text{ N/m}$; $\zeta = 0.05$; $\omega = \frac{3600}{60} (2\pi) = 376.992 \text{ rad/sec}$
 $m = 60 \text{ kg}$; $a = 2000 (10^{-6}) \text{ m}$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3.75 (10^6)}{60}} = 250.0 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = \frac{376.992}{250.0} = 1.5080$$

(a) Steady state whirl amplitude:

$$A = \frac{a r^2}{\left\{ (1 - r^2)^2 + (2 \zeta r)^2 \right\}^{\frac{1}{2}}} \quad (1)$$

$$= \frac{(2000 (10^{-6})) (1.5080^2)}{\left\{ (1 - 1.5080^2)^2 + (2 (0.05) (1.5080))^2 \right\}^{\frac{1}{2}}} = 0.003545 \text{ m}$$

(b) During start-up and stopping conditions, rotor passes through the natural frequency of the system. Thus, using $r = 1$ in Eq. (1), we obtain the whirl amplitude as

$$A|_{r=1} = \frac{a}{2 \zeta} = \frac{0.005}{2 (0.05)} = 0.05 \text{ m} \quad (2)$$

9.20 Let $t = 0$
 Unbalanced forces:

$$F_{xp} = m r \omega^2 (\cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3 + \cos \alpha_4)$$

$$= m r \omega^2 (\cos 0^\circ + \cos 180^\circ + \cos 180^\circ + \cos 0^\circ) = 0$$

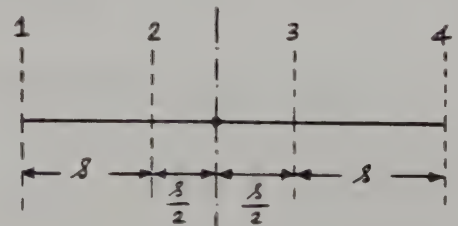
$$F_{xs} = m \frac{r^2 \omega^2}{\ell} \sum_{i=1}^4 \cos 2\alpha_i = \frac{m r^2 \omega^2}{\ell} (\cos 0^\circ + \cos 360^\circ + \cos 360^\circ + \cos 0^\circ)$$

$$= \frac{4 m r^2 \omega^2}{\ell} = \frac{4}{10} \left(\frac{2}{386.4} \right) (4)^2 \left(\frac{3000 \times 2\pi}{60} \right)^2 = 3269.4495 \text{ lb}$$

Unbalanced moments:

$$M_{zp} = F_{xp1} \left(\frac{3s}{2} \right) + F_{xp2} \left(\frac{s}{2} \right)$$

$$- F_{xp3} \left(\frac{s}{2} \right) - F_{xp4} \left(\frac{3s}{2} \right)$$



$$= \frac{mr\omega^2 s}{2} (3 \cos 0^\circ + \cos 180^\circ - \cos 180^\circ - 3 \cos 0^\circ)$$

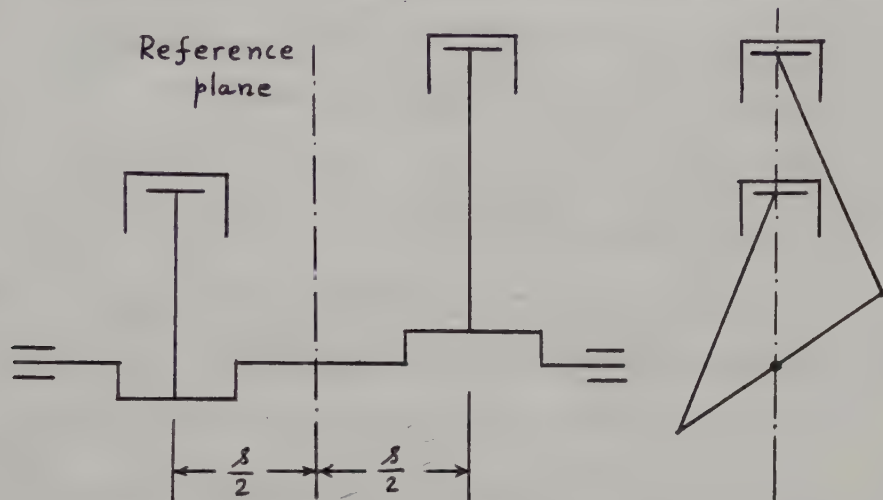
$$= 0$$

$$M_{zs} = F_{xs1} \left(\frac{3s}{2}\right) + F_{xs2} \left(\frac{s}{2}\right) - F_{xs3} \left(\frac{s}{2}\right) - F_{xs4} \left(\frac{3s}{2}\right)$$

$$= \frac{mr^2\omega^2 s}{2l} (3 \cos 0^\circ + \cos 360^\circ - \cos 360^\circ - 3 \cos 0^\circ)$$

$$= 0$$

9.21



Let the cylinders be separated axially by a distance s .

$$F_{xp} = mr\omega^2 (\cos \alpha_1 + \cos \alpha_2) = mr\omega^2 (\cos 0^\circ + \cos 180^\circ) = 0$$

$$F_{xs} = \frac{mr^2\omega^2}{l} (\cos 2\alpha_1 + \cos 2\alpha_2) = \frac{mr^2\omega^2}{l} (\cos 0^\circ + \cos 360^\circ) = \frac{2mr^2\omega^2}{l}$$

Moments about the reference plane:

$$M_{zp} = F_{xp1} \cdot \frac{s}{2} - F_{xp2} \cdot \frac{s}{2} = \frac{s mr \omega^2}{2} (\cos 0^\circ - \cos 180^\circ) = s mr \omega^2$$

$$M_{zs} = F_{xs1} \cdot \frac{s}{2} - F_{xs2} \cdot \frac{s}{2} = \frac{s mr^2 \omega^2}{2l} (\cos 0^\circ - \cos 360^\circ) = 0$$

$\therefore$ Secondary forces and primary couple are unbalanced.

9.22

$r = 3''$, $mg = 3 \text{ lb}$, $l = 10''$, $\omega = 1500 \text{ rpm} = 157.08 \text{ rad/sec}$,

$\alpha_1 = 0^\circ$, $\alpha_2 = 180^\circ$, $\alpha_3 = 90^\circ$, $\alpha_4 = 270^\circ$

Assume $m_c = 0$ and $m_p \cdot g = m \cdot g = 3 \text{ lb}$.

Consider the vertical and horizontal components of the inertia forces at $t = 0$. This gives

$$F_{xp} = (m_p + m_c) r \omega^2 \sum_{i=1}^4 \cos \alpha_i$$

$$= (m_p + m_c) r \omega^2 (\cos 0^\circ + \cos 180^\circ + \cos 90^\circ + \cos 270^\circ) = 0$$

$$\begin{aligned}
 F_{yp} &= -m_c r \omega^2 \sum_{i=1}^4 \sin \alpha_i \\
 &= -m_c r \omega^2 (\sin 0^\circ + \sin 180^\circ + \sin 90^\circ + \sin 270^\circ) = 0 \\
 F_{xs} &= m_p \frac{r^2 \omega^2}{l} \sum_{i=1}^4 \cos 2\alpha_i \\
 &= \frac{m_p r^2 \omega^2}{l} (\cos 0^\circ + \cos 360^\circ + \cos 180^\circ + \cos 540^\circ) = 0
 \end{aligned}$$

Primary and secondary forces are balanced.

Moments about the reference plane:

$$\begin{aligned}
 M_{zp} &= F_{xp1}(6) + F_{xp2}(2) - F_{xp3}(2) - F_{xp4}(6) \\
 &= (m_p + m_c) r \omega^2 [6 \cos 0^\circ + 2 \cos 180^\circ - 2 \cos 90^\circ - 6 \cos 270^\circ] \\
 &= 4 r \omega^2 (m_p + m_c) = 4(3)(157.08)^2 \left(\frac{3}{386.4}\right) = 2298.8316 \text{ lb-in} \\
 &= \text{unbalanced primary couple}
 \end{aligned}$$

$$\begin{aligned}
 M_{zs} &= F_{xs1}(6) + F_{xs2}(2) - F_{xs3}(2) - F_{xs4}(6) \\
 &= \frac{m_p r^2 \omega^2}{l} [6 \cos 2\alpha_1 + 2 \cos 2\alpha_2 - 2 \cos 2\alpha_3 - 6 \cos 2\alpha_4] \\
 &= \frac{m_p r^2 \omega^2}{l} (16) = \left(\frac{3}{386.4}\right) (3)^2 (157.08)^2 (16) \cdot \frac{1}{10} \\
 &= 2758.5980 \text{ lb-in} \\
 &= \text{unbalanced secondary couple}
 \end{aligned}$$

$$\begin{aligned}
 M_{xp} &= F_{yp1}(6) + F_{yp2}(2) - F_{yp3}(2) - F_{yp4}(6) \\
 &= -m_c r \omega^2 (6 \sin \alpha_1 + 2 \sin \alpha_2 - 2 \sin \alpha_3 - 6 \sin \alpha_4) \\
 &= -m_c r \omega^4 (4) = 0 \quad \text{since } m_c = 0.
 \end{aligned}$$

9.23

Primary unbalanced forces are given by

$$F_{xp} = \sum_{i=1}^6 (F_x)_{pi} = \sum (m_p + m_c)_i r \omega^2 \cos(\omega t + \alpha_i) \quad (E_1)$$

$$F_{yp} = \sum_{i=1}^6 (F_y)_{pi} = \sum -(m_c)_i r \omega^2 \sin(\omega t + \alpha_i) \quad (E_2)$$

Secondary unbalanced force is given by

$$F_{xs} = \sum_{i=1}^6 (F_x)_{si} = \sum (m_p)_i \frac{r^2 \omega^2}{l} \cos(2\omega t + 2\alpha_i) \quad (E_3)$$

Primary and secondary unbalanced moments are given by

$$(M_z)_r = \sum_{i=2}^6 (F_x)_{pi} l_i \quad (E_4)$$

$$(M_z)_s = \sum_{i=2}^6 (F_x)_{si} l_i \quad (E_5)$$

$$(M_x)_p = \sum_{i=2}^6 (F_y)_{pi} l_i \quad (E_6)$$

Eg. (E₁) gives

$$(m_p + m_c) r \omega^2 \sum_{i=1}^6 \cos \alpha_i = (m_p + m_c) r \omega^2 (2 \cos 0^\circ + 2 \cos 120^\circ + 2 \cos 240^\circ) = 0$$

Eg. (E₂) gives

$$-m_c r \omega^2 \sum_{i=1}^6 \sin \alpha_i = -m_c r \omega^2 (2 \sin 0^\circ + 2 \sin 120^\circ + 2 \sin 240^\circ) = 0$$

Eg. (E₃) gives

$$\frac{m_p r^2 \omega^2}{l} \sum_{i=1}^6 \cos 2\alpha_i = \frac{m_p r^2 \omega^2}{l} (2 \cos 0^\circ + 2 \cos 240^\circ + 2 \cos 480^\circ) = 0$$

Eg. (E₄) gives

$$(m_p + m_c) r \omega^2 \sum_{i=2}^6 l_i \cos \alpha_i = (m_p + m_c) r \omega^2 a (\cos 120^\circ + 2 \cos 240^\circ + 3 \cos 240^\circ + 4 \cos 120^\circ + 5 \cos 0^\circ) = 0$$

Eg. (E₅) gives

$$\frac{m_p r^2 \omega^2}{l} \sum_{i=2}^6 l_i \cos 2\alpha_i = \frac{m_p r^2 \omega^2 a}{l} [\cos 240^\circ + 2 \cos 480^\circ + 3 \cos 480^\circ + 4 \cos 240^\circ + 5 \cos 0^\circ] = 0$$

Eg. (E₆) gives

$$-m_c r \omega^2 \sum_{i=2}^6 l_i \sin \alpha_i = -m_c r \omega^2 a (\sin 120^\circ + 2 \sin 240^\circ + 3 \sin 240^\circ + 4 \sin 120^\circ + 5 \sin 0^\circ) = 0$$

$\therefore$ Engine is completely force and moment balanced.

9.24 (a) $\omega_n = \sqrt{\frac{k}{M}} \approx 0$; $\omega = 600 \text{ rpm} = 600 \times \frac{2\pi}{60} = 62.832 \text{ rad/s}$

$$\frac{\omega}{\omega_n} \approx \infty, c = 0$$

$$F_0 = m \omega^2 r = (2.5) (62.832)^2 \left(\frac{7.5}{100} \right) = 740.2238 \text{ N}$$

$$\text{Eg. (9.83)} \Rightarrow X = \frac{F_0}{(k - m \omega^2)} = \frac{F_0}{k \left(1 - \frac{\omega^2}{\omega_n^2} \right)} = 0 \text{ for } \frac{\omega}{\omega_n} \approx \infty$$

$$(b) \quad \omega_n = \infty, \quad \frac{\omega}{\omega_n} = 0$$

$$\text{Eq. (9.86)} \Rightarrow F_T = \frac{F_0 k}{k - m\omega^2} = \frac{F_0}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = F_0 = 740.2238$$

9.25

$$\omega = 25 \text{ Hz to } 35 \text{ Hz} = 157.08 \text{ rad/sec to } 219.912 \text{ rad/sec}$$

$$m = 85/9.81 = 8.6646 \text{ kg}$$

Transmissibility of an undamped isolator is given by Eq. (9.87):

$$T_r = \frac{1}{\left| 1 - \left(\frac{\omega}{\omega_n}\right)^2 \right|} \quad (E_1)$$

For 80% vibration isolation, Eq. (E1) gives

$$0.2 = \frac{1}{\left| 1 - \left(\frac{\omega}{\omega_n}\right)^2 \right|} \quad \text{i.e.,} \quad \left| 1 - \left(\frac{\omega}{\omega_n}\right)^2 \right| = 5$$

$$\text{or} \quad \frac{\omega}{\omega_n} = \sqrt{6} = 2.4495$$

$$\text{At } \omega = 25 \text{ Hz,} \quad \omega_n = 157.08/2.4495 = 64.1274 \text{ rad/sec}$$

$$\text{At } \omega = 35 \text{ Hz,} \quad \omega_n = 219.912/2.4495 = 89.7783 \text{ rad/sec}$$

$$\text{But } \omega_n = \sqrt{k/m} = \sqrt{kg/w} = \sqrt{g/\delta_{st}} = \sqrt{9.81/\delta_{st}}$$

$$\text{or} \quad \delta_{st} = \frac{9.81}{\omega_n^2}$$

$$\text{At } \omega = 25 \text{ Hz,} \quad \delta_{st} = 9.81/(64.1274)^2 = 0.002385 \text{ m}$$

$$\text{At } \omega = 35 \text{ Hz,} \quad \delta_{st} = 9.81/(89.7783)^2 = 0.001217 \text{ m}$$

$\therefore$ Select static deflection of isolator as 0.002385 m.

checking the performance at $\omega = 35 \text{ Hz}$:

$$\omega_n = 64.1274 \text{ rad/sec. At } \omega = 35 \text{ Hz,}$$

$$T_r = \frac{1}{\left| 1 - \left(219.912/64.1274\right)^2 \right|} = 0.0850$$

$\Rightarrow 91.5\%$ isolation, better than the required amount.

$$\therefore \delta_{st} \text{ of isolator} = 0.2385 \text{ mm}$$

9.26

$$mg = 800 \text{ N,} \quad \omega_o = 600 \text{ rpm} = \text{operating speed} = 62.832 \text{ rad/sec}$$

$$T_r = 2.5 \text{ at } \omega = \omega_n$$

Eq. (9.87) gives

$$T_r^2 = \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \quad \text{with } r = \frac{\omega}{\omega_n} \quad (E_1)$$

$$\text{At } r=1, \quad T_r^2 = \frac{1 + 4\zeta^2}{4\zeta^2} ; \quad 6.25 = \frac{1 + 4\zeta^2}{4\zeta^2} \Rightarrow \zeta = 0.2182$$

At operating speed, $T_r = 0.1$ and $\zeta = 0.2182$; $E_g(E_1)$ gives

$$(0.1)^2 = \frac{1 + 4(0.2182)^2 r^2}{(1 - r^2)^2 + 4(0.2182)^2 r^2}$$

which, upon simplification, becomes

$$r^4 - 20.8595 r^2 - 99 = 0 \Rightarrow r^2 = 24.8443$$

$$\text{or } r = \frac{\omega_0}{\omega_n} = 4.9844$$

$$\text{Since } \omega_0 = 62.832 \text{ rad/sec, } \omega_n = 12.6057 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

$$k = \omega_n^2 m = (12.6057)^2 \left(\frac{800}{9.81} \right) = 12958.5054 \text{ N/m}$$

$\therefore$ Isolator is defined by

$$k = 12958.5054 \text{ N/m}$$

$$c = 2m\omega_n\zeta = 2 \left(\frac{800}{9.81} \right) (12.6057) (0.2182) = 448.6139 \frac{\text{N-s}}{\text{m}}$$

9.27

$$M = 500 \text{ kg, } m_e = 50 \text{ kg-cm, } \omega = 300 \text{ rpm} = 31.416 \text{ rad/sec}$$

$$F_0 = \text{steady state force magnitude} = m_e \omega^2, \quad \omega_n = \sqrt{k/M}$$

$$\text{static deflection of compressor} = \delta_{st} = \frac{F_0}{k} = \frac{m_e \omega^2}{k} \quad (E_1)$$

$$\text{Transmission ratio} = T_r = \frac{F_t}{F_0} = \left\{ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right\}^{1/2} \quad (E_2)$$

$$\text{with } r = \omega/\omega_n.$$

Amplitude of vibration of compressor

$$= X = F_0 / \left[(k - M\omega^2)^2 + \omega^2 c^2 \right]^{1/2} = \frac{m_e}{M} \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \quad (E_3)$$

For a good design, T_r must be small. Also X should be small for smaller dynamic stress.

Isolator with κ

Eg. (E_2):

$$T_r = \frac{1}{|1 - r^2|}$$

For small T_r , $r = \frac{\omega}{\omega_n}$ should be large or ω_n small

Let $T_r = 0.1$ so that $|1 - r^2| = 10$.

$$r = \frac{\omega}{\omega_n} = \sqrt{11} = 3.3166$$

$$\omega_n = \omega / 3.3166 = 31.416 / 3.3166 = 9.4724 \text{ rad/sec}$$

$$= \sqrt{\frac{\kappa}{500}}$$

$$\kappa = (9.4724)^2 (500) = 44863.1809 \text{ N/m}$$

$$\delta_{st} = \frac{m e \omega^2}{\kappa} = \frac{\left(\frac{50}{100}\right) (31.416)^2}{44863.1809} = 0.0110 \text{ m}$$

$$r^2 = 11, (1 - r^2)^2 = 100$$

$$(2\zeta r)^2 = (0.2r)^2 = (0.2 \times 3.3166)^2 = 0.44$$

Eg. (E_3):

$$X = \frac{\left(\frac{50}{100}\right)}{500} \cdot \frac{11}{\sqrt{100 + 0.44}}$$

$$= 0.001098 \text{ m}$$

Shock absorber with ζ and κ

Eg. (E_2) for $T_r = 0.1$ and $\zeta = 0.1$:

$$0.1 = \left\{ \frac{1 + (0.2r)^2}{(1 - r^2)^2 + (0.2r)^2} \right\}^{1/2}$$

$$\text{or } r^4 - 5.96 r^2 - 99 = 0$$

$$\text{or } r^2 = 13.3665$$

$$\text{or } r = \omega / \omega_n = 3.6560$$

$$\omega_n = 31.416 / 3.656$$

$$= 8.5929 \text{ rad/sec}$$

$$\kappa = (8.5929)^2 (500)$$

$$= 36919.2174 \text{ N/m}$$

$$\delta_{st} = \frac{m e \omega^2}{\kappa} = \frac{\left(\frac{50}{100}\right) (31.416)^2}{36919.2174}$$

$$= 0.0134 \text{ m}$$

$$r^2 = 13.3665, (1 - r^2)^2 = 152.9303$$

$$(2\zeta r)^2 = (0.2 \times 3.656)^2 = 0.5347$$

Eg. (E_3):

$$X = \frac{\left(\frac{50}{100}\right)}{500} \cdot \frac{13.3665}{\sqrt{152.9303 + 0.5347}}$$

$$= 0.001079 \text{ m}$$

Since X is smaller in the case of shock absorber, it is to be preferred. In this case, a smaller value of κ will be sufficient; this leads to a cheaper design.

9.28

$$m = 200 \text{ kg}, k = 10000 \text{ N/m}, \zeta = 0.15, \omega_n = \sqrt{\frac{10^4}{200}} = 7.0711 \text{ rad/s}$$

$$(a) \frac{F_t}{F_0} > 1:$$

$$\text{From Eq. (9.87), } \left\{ \frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right\}^{\frac{1}{2}} > 1; r = \frac{\omega}{\omega_n}$$

$$\text{i.e. } 1 + (2\zeta r)^2 > (1-r^2)^2 + (2\zeta r)^2$$

$$\text{i.e. } 1 > (1-r^2)^2$$

$$\text{This gives } 1 > 1 + r^4 - 2r^2; \quad r^4 - 2r^2 < 0$$

$$r^2(r^2 - 2) < 0$$

$$r^2 < 0 \quad \text{or} \quad r^2 < 2$$

$$\begin{aligned} \text{Physically possible solution is } \omega &< \sqrt{2} \omega_n \\ &< \sqrt{2} (7.0711) = 10 \text{ rad/s} \\ &< 95.4927 \text{ rpm} \end{aligned}$$

$$(b) \frac{F_t}{F_0} < 0.1:$$

$$\text{From Eq. (9.87), } 1 + (2\zeta r)^2 < 0.01 \{ (1-r^2)^2 + (2\zeta r)^2 \}$$

$$1 + 0.09 r^2 < 0.01 + 0.01 r^4 - 0.02 r^2 + 0.0009 r^2$$

$$0.01 r^4 - 0.1091 r^2 - 0.99 > 0$$

$$(r^2 - 16.8021)(r^2 + 5.8920) > 0$$

$$\text{i.e. } r^2 - 16.8021 > 0, \quad r^2 + 5.8920 > 0 \quad \text{or} \quad r^2 - 16.8021 < 0, \quad r^2 + 5.8920 < 0$$

$$\text{i.e. } r^2 > 16.8021, \quad r^2 > -5.892 \quad \text{or} \quad r^2 < 16.8021, \quad r^2 < -5.892$$

(not possible)

$$\therefore r^2 > 16.8021, \quad r > 4.0990$$

$$\omega > 4.0990(7.0711) = 28.9844 \text{ rad/s} = 276.7803 \text{ rpm}$$

9.29

For undamped system, transmission ratio is:

$$T_r = \frac{F_T}{F_0} = \frac{k}{k - \omega^2}$$

Since isolation is 60%, we have

$$\frac{F_t}{F_0} = 0.4 = \pm \left[\frac{k}{k - \left(\frac{150}{386.4} \right) \left(\frac{300}{60} (2\pi) \right)^2} \right] = \pm \left(\frac{k}{k - 383.1386} \right)$$

$$\text{or } 0.4k - 153.2554 = -k \quad \text{or} \quad k = 109.4682 \text{ lb/in}$$

Thus k has to be less than 109.4682 lb/in to provide more than 60% isolation.

$$k = \frac{m g}{\delta_{st}} \quad \text{or} \quad (\delta_{st})_{\min} = \frac{m g}{k_{\max}} = \frac{150}{109.4682} = 1.3703 \text{ inch.}$$

$$\textcircled{9.30} \quad m = 50 \text{ kg} ; \omega = \frac{1200}{60} (2 \pi) = 125.664 \text{ rad/sec} ; \zeta = 0.07$$

$$T_r = \frac{F_T}{F_0} = \left\{ \frac{1 + (2 \zeta r)^2}{(1 - r^2)^2 + (2 \zeta r)^2} \right\}^{\frac{1}{2}} \quad (1)$$

For 75% isolation, Eq. (1) gives

$$0.25^2 = \frac{1 + (2 (0.07) r)^2}{(1 - r^2)^2 + (2 (0.07) (r))^2} \quad \text{or} \quad 0.0625 r^4 - 0.143375 r^2 - 0.9375 = 0 \quad (2)$$

The solution of Eq. (2) is given by:

$$r^2 = 5.186255 \text{ (positive value)} \quad \text{or} \quad r = \frac{\omega}{\omega_n} = 2.2773$$

$$\text{This gives } \omega_n = \frac{\omega}{2.2773} = \frac{125.664}{2.2773} = 55.1803 \text{ rad/sec}$$

$$\text{Maximum stiffness: } k = m \omega_n^2 = (50) (55.1803^2) = 152,243.1865 \text{ N/m}$$

$$\textcircled{9.31} \quad m = 80 \text{ kg} ; \omega = \frac{1000}{60} (2 \pi) = 104.72 \text{ rad/sec}$$

$$(a) \quad \frac{F_T}{F_0} = \frac{2000}{10000} = 0.2 = \pm \frac{k}{k - m \omega^2} = \pm \frac{k}{k - (80) (104.72^2)} \quad (1)$$

Using the negative sign in Eq. (1), we find

$$0.2 k - 17.5460 (10^4) = -k$$

$$\text{Maximum stiffness} = k_{\max} = \frac{17.5460 (10^4)}{1.2} = 146217.0453 \text{ N/m}$$

(b) Steady state amplitude:

$$X = \left| \frac{F_0}{k - m \omega^2} \right| = \left| \frac{10000}{146217.0453 - 87.7302 (10^4)} \right| = 0.01368 \text{ m}$$

(c) Maximum amplitude of fan during start-up:

$$X = \left| \frac{F_0}{k - m \omega^2} \right| = \left| \frac{F_0}{k \left(1 - \frac{\omega^2}{\omega_n^2} \right)} \right|$$

$$\text{where } \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{146217.0453}{80}} = 42.7518 \text{ rad/sec}$$

Using $\omega = \omega_n$, $X \rightarrow \infty$. Hence an undamped isolator must pass through resonance very quickly to avoid damage.

9.32

$$m = 300 \text{ kg} ; \omega = \frac{3000}{60} (2 \pi) = 314.16 \text{ rad/sec} ; F_0 = 30,000 \text{ N}$$

Requirements:

$$1. \quad \delta_{st} = \frac{m g}{k} = \text{small} \rightarrow k = \text{large} \quad (1)$$

$$2. \quad X = \frac{F_0}{\left\{ (k - m \omega^2)^2 + \omega^2 c^2 \right\}^{\frac{1}{2}}} \leq 2.5 (10^{-3}) \text{ m} \quad (2)$$

$$3. \quad X|_{r=1} = \frac{F_0}{k \left\{ \left(1 - r^2 \right)^2 + \left(\frac{\omega^2 c^2}{k^2} \right) \right\}^{\frac{1}{2}}} |_{r=1} < 2 (10^{-2}) \text{ m} \quad (3)$$

$$\text{where } r = \frac{\omega}{\omega_n}$$

$$4. \quad \frac{F_T}{F_0} = \left\{ \frac{k^2 + \omega^2 c^2}{(k - m \omega^2)^2 + \omega^2 c^2} \right\}^{\frac{1}{2}} \leq \frac{10000}{30000} = \frac{1}{3} \quad (4)$$

5. For achieving isolation, $r > \sqrt{2}$. Hence

$$k < \frac{m \omega^2}{2} \quad \text{or} \quad k < 14.8045 (10^6) \text{ N/m} \quad (5)$$

Since four inequalities, Eqs. (2) to (5), are to be satisfied, in general, we need to use an iterative process. From Eq. (3), we obtain:

$$\frac{F_0}{k \left(\frac{\omega c}{k} \right)} = \frac{F_0}{\omega c} = \frac{30000}{314.16 c} \leq 0.02 \quad \text{or} \quad c \geq 4774.6371 \text{ N-s/m} \quad (6)$$

We assume $c = 5000.0 \text{ N-s/m}$ and $k = 6 (10^6) \text{ N/m}$ as trial values. These values satisfy Eqs. (5) and (6) and give:

$$\begin{aligned} \left\{ (k - m \omega^2)^2 + \omega^2 c^2 \right\}^{\frac{1}{2}} &= \left\{ (6 (10^6) - (300) (314.16^2))^2 + (314.16 (5000))^2 \right\}^{\frac{1}{2}} \\ &= 23.6611 (10^6) \end{aligned}$$

and the left hand side of Eq. (2) becomes:

$$\frac{30000.0}{23.6611 (10^6)} = 0.001268 \text{ m} < 0.0025 \text{ m}$$

and hence Eq. (2) is satisfied. The numerator on the left hand side of Eq. (4) is:

$$\left\{ k^2 + \omega^2 c^2 \right\}^{\frac{1}{2}} = \left\{ 36 (10^{12}) + (314.16 (5000.0))^2 \right\}^{\frac{1}{2}} = 6.1844 (10^6)$$

and the left hand side of Eq. (4) thus becomes:

$$\frac{6.1844 (10^6)}{23.6611 (10^6)} = 0.2614$$

which is less than the value on the right hand side. Thus Eq. (4) is satisfied. The final design is given by $k = 6.0 (10^6) \text{ N/m}$ and $c = 5000.0 \text{ N-s/m}$.

9.33

$$m = 120 \text{ kg} ; m_e = 0.2 \text{ kg-m} ; k = 0.5 (10^6) \text{ N/m} ; \zeta = 0.06$$

$$\omega_n = \left\{ \frac{k}{m} \right\}^{\frac{1}{2}} = \left\{ \frac{0.5 (10^6)}{120} \right\}^{\frac{1}{2}} = 64.5497 \text{ rad/sec} ; F_T < 2500 \text{ N}$$

Eq. (9.92) gives:

$$r^2 \left\{ \frac{1 + (2 (0.06) (r))^2}{(1 - r^2)^2 + (2 (0.06) (r))^2} \right\}^{\frac{1}{2}} = \frac{F_T}{m_e \omega_n^2} < \frac{2500}{(0.2) (64.5497^2)} < 3$$

$$\text{or } \left\{ \frac{1 + 0.0144 r^2}{1 + r^4 - 2 r^2 + 0.0144 r^2} \right\} < \frac{9}{r^4} \quad (1)$$

Setting the left hand side of Eq. (1) equal to $\frac{8}{r^4}$, we obtain

$$r^6 - 486.1111 r^4 + 1103.1111 r^2 - 555.5555 = 0 \quad (2)$$

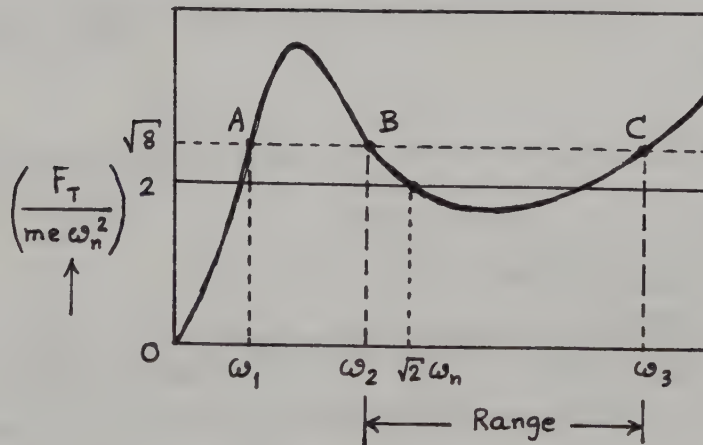
The roots of this cubic equation are given by (found using the Program 7):

$$r_1^2 = 0.751358 ; r_1 = 0.866809 ; \omega_1 = 55.9523 \text{ rad/sec}$$

$$r_2^2 = 1.52760 ; r_2 = 1.23596 ; \omega_2 = 79.7808 \text{ rad/sec}$$

$$r_3^2 = 483.834 ; r_3 = 21.9962 ; \omega_3 = 1419.8481 \text{ rad/sec}$$

From Fig. 3.16, the values of ω_1 , ω_2 and ω_3 can be interpreted as shown in the following figure. It can be seen that the force transmitted to the foundation will be less than 2500 N (actually, less than $2500 \sqrt{8/3} = 2357.0226 \text{ N}$) over the frequency range $\omega_2 - \omega_3$ (i.e., 79.7808 rad/sec - 1419.8481 rad/sec).



9.34

$m e = 1.0 \text{ kg-m}$; $\omega = 800 \text{ to } 2000 \text{ rpm} = 83.776 \text{ to } 209.44 \text{ rad/sec}$

$F_0 = 7018 \text{ N at } 800 \text{ rpm and } 43865 \text{ N at } 2000 \text{ rpm}$

$F_T \leq 6000 \text{ N over the speed range ; } \zeta = 0.08$

To find k .

$$\begin{aligned} \text{Relation be satisfied: } \frac{F_T}{m e \omega^2} &= \left\{ \frac{1 + (2 \zeta r)^2}{(1 - r^2)^2 + (2 \zeta r)^2} \right\}^{\frac{1}{2}} \leq \frac{6000}{m e \omega^2} \\ \text{or } \left\{ \frac{1 + 0.0256 r^2}{(1 - r^2)^2 + 0.0256 r^2} \right\}^{\frac{1}{2}} &\leq \frac{6000}{7018} = 0.8549 \text{ at } \omega = 800 \text{ rpm} \\ \text{and } &\leq \frac{6000}{43865} = 0.1368 \text{ at } \omega = 2000 \text{ rpm} \end{aligned} \quad (1)$$

Equating the left side of Eq. (1) to 0.85 at $\omega = 800 \text{ rpm}$, we obtain

$$\begin{aligned} \frac{1 + 0.0256 r_1^2}{1 + r_1^4 - 2 r_1^2 + 0.0256 r_1^2} &= 0.7225 \\ \text{or } r_1^4 - 2.00983 r_1^2 - 0.3841 &= 0 \quad \text{or } r_1^2 = 2.1856 \text{ (positive root)} \\ \text{or } r_1 &= 1.4784 \end{aligned}$$

Equating the left side of Eq. (1) to 0.135 at $\omega = 2000 \text{ rpm}$, we obtain

$$\begin{aligned} \frac{1 + 0.0256 r_2^2}{1 + r_2^4 - 2 r_2^2 + 0.0256 r_2^2} &= 0.018225 \\ \text{or } r_2^4 - 3.3791 r_2^2 - 53.8697 &= 0 \quad \text{or } r_2^2 = 9.2211 \text{ (positive root)} \\ \text{or } r_2 &= 3.0366 \end{aligned}$$

By selecting $r_2 = 3.0366$, we obtain $\omega_n = \frac{\omega}{r_2} = \frac{209.44}{3.0366} = 68.9713 \text{ rad/sec}$. If $r_1 = 1.4784$ is selected, we obtain $\omega_n = \frac{\omega}{r_1} = \frac{83.776}{1.4784} = 56.6667 \text{ rad/sec}$. Thus $\omega_n = 56.6667 \text{ rad/sec}$ satisfies the transmitted force requirement at both ends of the operating speed.

Verification:

$$\text{At the speed } 2000 \text{ rpm, the value of } r = \frac{\omega}{\omega_n} \text{ is: } r = \frac{209.44}{56.6667} = 3.6960$$

This gives $r^2 = 13.6604$ and

$$\left\{ \frac{1 + 0.3497}{(1 - 13.6604)^2 + 0.3497} \right\}^{\frac{1}{2}} = 0.09166 < 0.1368 \text{ of Eq. (1).}$$

Stiffness of the isolator:

$$k = M \omega_n^2 = (200) (56.6667^2) = 64.2223 (10^4) \text{ N/m}$$

9.35

$$m = 100 \text{ kg}; \omega = \frac{600}{60} (2\pi) = 62.832 \text{ rad/sec}; \text{Isolation} = 90\%$$

$$0.1 = \frac{F_T}{F_0} = \left| \frac{k}{k - m \omega^2} \right| = \left| \frac{k}{k - 100 (62.832^2)} \right|$$

$$\text{or } 0.1 k - 39478.6022 = -k \quad \text{or } k = \frac{39478.6022}{1.1} = 35889.6384 \text{ N/m}$$

$$\text{Static deflection of isolator: } \delta_{st} = \frac{m g}{k} = \frac{100 (9.81)}{35889.6384} = 0.02733 \text{ m}$$

9.36

$$m = 300 \text{ kg}; \omega = \frac{1800}{60} (2\pi) = 188.496 \text{ rad/sec. Unbalance} = m e = 1 \text{ kg-m}$$

$$F_T = \text{maximum permissible force transmitted to floor} = 8000 \text{ N}$$

$$\text{Force transmissibility: } T_r = \frac{F_T}{m e \omega^2} = \frac{8000}{1 (188.496^2)} = 0.2251$$

$$= \left\{ \frac{1 + (2 \zeta r)^2}{(1 - r^2)^2 + (2 \zeta r)^2} \right\}^{\frac{1}{2}} \quad (1)$$

Frequency ratio (r) that satisfies Eq. (1) can be found as

$$(0.2251^2) = \frac{1 + (2 (0.05) (r))^2}{(1 - r^2)^2 + (2 (0.05) (r))^2}$$

$$\text{or } r^4 - 2.1872 r^2 - 18.7239 = 0 \quad \text{or } r^2 = 5.5568 \text{ (positive root)}$$

$$\text{or } r = \frac{\omega}{\omega_n} = 2.3573$$

$$\text{Necessary natural frequency: } \omega_n = \frac{\omega}{r} = \frac{188.496}{2.3573} = 79.9633 \text{ rad/sec}$$

Possible solutions:

(a) If the available isolator is used,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1 (10^5)}{300}} = 57.7350 \text{ rad/sec}$$

(smaller than the necessary value of 79.9633 rad/sec).

If two identical isolators are used in parallel, $k_{eq} = 2 (10^5) \text{ N/m}$ and

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{2 (10^5)}{300}} = 81.6496 \text{ rad/sec.}$$

(approximately equal to the necessary value of 79.9633 rad/sec).

(b) If the isolator is available in the form of a helical spring, it can be cut into two halves and one of them can be used for isolation to achieve a value of $\omega_n = 81.6496 \text{ rad/sec}$.

9.37

Mass of engine = $m = 500$ kg

Force transmitted with out isolator = $F_T = (18000 \cos 300 t + 3600 \cos 600 t)$ N

Maximum magnitude of force transmitted:

$$F_{01} = 18000 \text{ N at } \omega = 300 \text{ rad/sec}$$

$$F_{02} = 3600 \text{ N at } \omega = 600 \text{ rad/sec}$$

The maximum possible force transmitted will be the sum of the magnitudes of the two harmonics:

$$F_0 = F_{01} + F_{02} = 18000 + 3600 = 21600 \text{ N}$$

Since $F_T = 12000$ N, we use the relation

$$\frac{F_T}{F_0} = \frac{12000}{21600} = \left| \frac{1}{1 - r^2} \right| \quad \text{or} \quad 0.5556 = \frac{1}{r^2 - 1} \quad \text{or} \quad r = 1.6733$$

At $\omega = 300$ rad/sec:

$$\omega_n = \frac{\omega}{r} = \frac{300}{1.6733} = 179.2843 \text{ rad/sec} = \sqrt{\frac{k}{500}}$$

$$\text{or } k = m (\omega_n^2) = (500) (179.2843^2) = 16.0714 (10^6) \text{ N/m}$$

With this value of k , the value of $\frac{F_{T1}}{F_{01}}$ at 300 rad/sec is:

$$\frac{F_{T1}}{F_{01}} = \left| \frac{1}{1 - r_1^2} \right| = \left| \frac{1}{1 - \left(\frac{300}{179.2843} \right)^2} \right| = 0.5556$$

$$\text{or } F_{T1} = 0.5556 (F_{01}) = 0.5556 (18000) = 10000 \text{ N}$$

The value of $\frac{F_{T2}}{F_{02}}$ at $\omega = 600$ rad/sec is:

$$\frac{F_{T2}}{F_{02}} = \left| \frac{1}{1 - r_2^2} \right| = \left| \frac{1}{1 - \left(\frac{600}{179.2843} \right)^2} \right| = 0.0980$$

$$\text{or } F_{T2} = 0.0980 (F_{02}) = 0.0980 (3600) = 352.8 \text{ N}$$

Since $F_{T1} + F_{T2} = 10000 + 352.8 = 10352.8 \text{ N} < 12000 \text{ N}$ (permitted value), the stiffness of the isolator can be taken as $k = 16.0714 (10^6) \text{ N/m}$.

At $\omega = 600$ rad/sec:

$$\omega_n = \frac{\omega}{r} = \frac{600}{1.6733} = 358.5729 \text{ rad/sec}$$

$$k = m \omega_n^2 = (500) (358.5729^2) = 64.2872 (10^6) \text{ N/m} \quad (1)$$

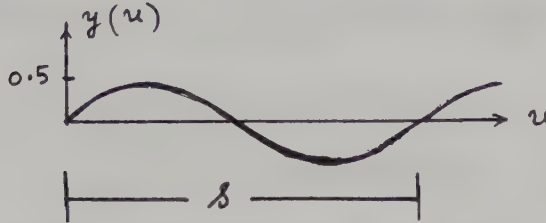
With this value of k , the value of $\frac{F_{T1}}{F_{01}}$ at $\omega = 300$ rad/sec is:

$$\frac{F_{T1}}{F_{01}} = \left| \frac{1}{1 - r_1^2} \right| = \left| \frac{1}{1 - \left(\frac{300}{358.5729} \right)^2} \right| = 3.3331$$

This corresponds to a larger value of F_{T1} than F_{01} and hence the value of k given by Eq. (1) is not suitable for isolation.

9.38 Speed range: 40 to 80 mph. Since $1 \text{ mph} = \frac{1 (5280)}{60 (60)} = 1.4667 \text{ ft/sec}$, speed range = 58.6667 to 117.3333 ft/sec.

Road surface is given by: $y(u) = 0.5 \sin 2 u \text{ ft}$ (1)



where $u = \text{horizontal distance (ft)} = v t$, $v = \text{velocity (ft/sec)}$ and $t = \text{time (sec)}$. Since $\omega = 2 \pi f = \frac{2 \pi v}{s}$ where $\omega = \text{frequency of road waviness in rad/sec}$ and $s = \text{one wave length (ft)}$, Eq. (1) can be expressed as

$$y(t) = 0.5 \sin 2 u \equiv Y \sin \Omega t \quad (2)$$

where $Y = 0.5 \text{ ft}$ and $\Omega = 2 v$.

Steady state response of the system subjected to the base excitation, $y(t) = Y \sin \Omega t$, is given by Eq. (3.67):

$$x_p(t) = X \sin (\Omega t - \phi) \quad (3)$$

$$\text{where } X = Y \left\{ \frac{1 + (2 \zeta r)^2}{(1 - r^2)^2 + (2 \zeta r)^2} \right\}^{\frac{1}{2}} \quad (4)$$

Maximum acceleration of mass (driver) is given by

$$\begin{aligned} \ddot{x}_p(t) |_{\max} &= \left| -\Omega^2 X \right| \\ &= \Omega^2 Y \left\{ \frac{1 + (2 \zeta r)^2}{(1 - r^2)^2 + (2 \zeta r)^2} \right\}^{\frac{1}{2}} = 2 (32.2) = 64.4 \text{ ft/sec}^2 \end{aligned} \quad (5)$$

At 40 mph:

$\Omega = 2 v = 2 (58.6667) = 117.3334 \text{ rad/sec}$, and Eq. (5) gives:

$$\begin{aligned} \Omega^4 Y^2 \left\{ \frac{1 + (2 \zeta r)^2}{(1 - r^2)^2 + (2 \zeta r)^2} \right\} &= 64.4^2 \\ \text{or } (117.3334^4) (0.5^2) \left\{ \frac{1 + (2 (0.05) (r))^2}{(1 - r^2)^2 + (2 (0.05) (r))^2} \right\} &= 64.4^2 \end{aligned}$$

$$\text{or } r^4 - 116.24 r^2 - 1.1424 (10^4) = 0 \quad (6)$$

The solution of Eq. (6) gives (with positive value of r^2)

$$r = 13.4083 = \frac{\Omega}{\omega_n}$$

$$\omega_n = \frac{\Omega}{13.4083} = \frac{117.3334}{13.4083} = 8.7508 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

$$\text{Stiffness of isolator (suspension)} = k = m \omega_n^2 = \frac{1500}{32.2} (8.7508^2) = 3567.2175 \text{ lb/ft.}$$

Check for acceleration at 80 mph:

$$\begin{aligned} \Omega_2 = 2 v_2 &= 2 (117.3334) = 234.6668 \text{ rad/sec, } \omega_n = 8.7508 \text{ rad/sec and} \\ r = \frac{\Omega_2}{\omega_n} &= \frac{234.6668}{8.7508} = 26.8166, r_2^2 = 719.1306, \text{ and } (2 \zeta r_2)^2 = (2 (0.05) (26.8166))^2 = \\ &7.1913. \end{aligned}$$

$$\begin{aligned} x_p(t) |_{\max} &= \Omega_2^2 Y \left\{ \frac{1 + (2 \zeta r_2)^2}{(1 - r_2^2)^2 + (2 \zeta r_2)^2} \right\}^{\frac{1}{2}} \\ &= (234.6668^2) (0.5) \left\{ \frac{1 + 7.1913}{(1 - 719.1306)^2 + 7.1913} \right\}^{\frac{1}{2}} = 109.724 \text{ ft/sec}^2 > 2 g \end{aligned}$$

Hence $k = 3567.2175 \text{ lb/ft}$ is not acceptable.

At 80 mph:

$\Omega = 2 v = 2 (117.3334) = 234.6668 \text{ rad/sec}$, and Eq. (5) gives:

$$\begin{aligned} \Omega^4 (Y^2) \left\{ \frac{1 + (2 \zeta r)^2}{(1 - r^2)^2 + (2 \zeta r)^2} \right\}^{\frac{1}{2}} &= 64.4^2 \\ \text{or } (234.6668^4) (0.5^2) \left\{ \frac{1 + 0.01 r^2}{(1 - r^2)^2 + 0.01 r^2} \right\} &= 64.4^2 \end{aligned}$$

$$\text{or } r^4 - 1829.98 r^2 - 182798.0 = 0 \quad (7)$$

The solution of Eq. (7) gives (with positive value of r^2): $r = 43.8742$. Hence

$$\omega_n = \frac{\Omega}{43.8742} = \frac{234.6668}{43.8742} = 5.3486 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

Stiffness of the isolator (suspension) is:

$$k = m \omega_n^2 = \frac{1500}{32.2} (5.3486^2) = 1332.6646 \text{ lb/ft}$$

Check for acceleration at 40 mph:

$$\Omega_1 = 2 v_1 = 117.3334 \text{ rad/sec, } \omega_n = 5.3486 \text{ rad/sec, and}$$

$$r_1 = \frac{\Omega_1}{\omega_n} = 21.9372$$

$$r_1^2 = 481.2415, (2 \zeta r_1)^2 = 4.8124, (1 - r_1^2)^2 = 230631.8983, \text{ and hence}$$

$$x_p(t) \big|_{\max} = \Omega_1^2 Y \left\{ \frac{1 + (2 \zeta r_1)^2}{(1 - r_1^2)^2 + (2 \zeta r_1)^2} \right\}^{\frac{1}{2}}$$

$$= (6883.5634) \left\{ \frac{5.8124}{230636.7107} \right\}^{\frac{1}{2}} = 34.5563 \text{ ft/sec}^2 < 2 \text{ g}$$

Hence $k = 1332.6646 \text{ lb/ft}$ is acceptable.

9.39

Force transmitted to base in case of Coulomb damping can be found using the equivalent viscous damping constant:

$$F_T = \left\{ (k x)^2 + (c_{eq} \dot{x})^2 \right\}^{\frac{1}{2}} = X \left\{ k^2 + \omega^2 c_{eq}^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ k^2 + \omega^2 c_{eq}^2 \right\}^{\frac{1}{2}} \frac{F_0}{k} \left[\frac{1 - \left(\frac{4 \mu N}{\pi F_0} \right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2} \right]^{\frac{1}{2}}$$

$$\text{with } c_{eq} = \left(\frac{4 \mu N}{\pi \omega X} \right) \text{ and } \frac{\omega^2 c_{eq}^2}{k^2} = \frac{(4 \mu N)^2}{\pi^2 F_0^2} \frac{(1 - r^2)^2}{\left\{ 1 - \left(\frac{4 \mu N}{\pi F_0} \right)^2 \right\}}$$

$$\text{Thus } F_T = F_0 \left\{ \frac{1 + \frac{(1 - r^2)^2 (4 \mu N)^2}{\pi^2 F_0^2} \left\{ 1 - \left(\frac{4 \mu N}{\pi F_0} \right)^2 \right\}}{\pi^2 F_0^2 \left\{ 1 - \left(\frac{4 \mu N}{\pi F_0} \right)^2 \right\}} \right\}^{\frac{1}{2}} \left[\frac{1 - \left(\frac{4 \mu N}{\pi F_0} \right)^2}{(1 - r^2)^2} \right]^{\frac{1}{2}}$$

$$= F_0 \left\{ \frac{1 + r^2 (r^2 - 2) \left(\frac{4 \mu N}{\pi F_0} \right)^2}{(1 - r^2)^2} \right\}^{\frac{1}{2}}$$

Thus the force transmissibility is given by

$$T_r = \frac{F_T}{F_0} = \left\{ \frac{1 + r^2 (r^2 - 2) \left(\frac{4 \mu N}{\pi F_0} \right)^2}{(1 - r^2)^2} \right\}^{\frac{1}{2}}$$

9.40

Under base excitation, the displacement transmissibility is given by (similar to that of a viscously damped system, Eq. 3.68):

$$\frac{X}{Y} = \left[\frac{1 + \left(\frac{\omega^2 c_{eq}^2}{k^2} \right)}{\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left(\frac{\omega^2 c_{eq}^2}{k^2} \right)} \right]^{\frac{1}{2}} \quad (1)$$

$$\text{But } \frac{\omega^2 c_{eq}^2}{k^2} = \left(\frac{4 \mu N}{k \pi} \right)^2 \frac{1}{X^2} \quad (2)$$

Substituting Eq. (2) into (1), we obtain

$$\frac{X}{Y} = \left[\frac{1 + \left(\frac{4 \mu N}{\pi k X} \right)^2}{(1 - r^2)^2 + \left(\frac{4 \mu N}{\pi k X} \right)^2} \right]^{\frac{1}{2}} \quad (3)$$

Relative displacement transmissibility is given by (similar to Eq. 3.77):

$$\frac{Z}{Y} = \frac{m \omega^2}{\left\{ (k - m \omega^2)^2 + \omega^2 c_{eq}^2 \right\}^{\frac{1}{2}}} = \frac{\left(\frac{\omega}{\omega_n} \right)^2}{\left\{ \left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left(\frac{\omega^2 c_{eq}^2}{k^2} \right) \right\}^{\frac{1}{2}}} \quad (4)$$

Using Eq. (2), (4) can be written as

$$\frac{Z}{Y} = \frac{r^2}{\left\{ (1 - r^2)^2 + \left(\frac{4 \mu N}{\pi k X} \right)^2 \right\}^{\frac{1}{2}}} \quad (5)$$

9.41

$$M = 200 \text{ kg} ; m_e = 0.02 \text{ kg-m} ; \delta_{st} = \frac{M g}{k} = 0.005 \text{ m}$$

$$k = \frac{M g}{\delta_{st}} = \frac{(200)(9.81)}{0.005} = 39.24 (10^4) \text{ N/m}$$

$$\omega = \frac{1200}{60} (2 \pi) = 125.664 \text{ rad/sec}$$

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{39.24 (10^4)}{200}} = 44.2944 \text{ rad/sec}$$

(a) Assume $\zeta = 0$ for the isolator.

$$r = \frac{\omega}{\omega_n} = \frac{125.664}{44.2944} = 2.8370$$

Amplitude of washing machine (from Eq. 3.82):

$$X = \left(\frac{m e}{M} \right) \frac{r^2}{\left\{ (1 - r^2)^2 \right\}^{\frac{1}{2}}} = \frac{0.02}{200} \frac{(2.8370^2)}{|1 - 2.8370^2|} = 11.4188 (10^{-5}) \text{ m}$$

(b) Force transmitted to foundation (given by Eq. 9.92):

$$F_T = m e \omega_n^2 r^2 \frac{1}{|1 - r^2|} = (0.02) (44.2944^2) \frac{(2.8370^2)}{|1 - 2.8370^2|} = 44.8069 \text{ N}$$

9.42

$$M = 60 \text{ kg} ; m e = 0.002 \text{ kg-m} ; \omega = \frac{3000}{60} (2 \pi) = 314.16 \text{ rad/sec}$$

$$T_r = \frac{F_T}{m e \omega^2} < 0.25$$

Let $\zeta = 0$ for the isolator. From the relation:

$$T_r = \frac{F_T}{m e r^2 \omega_n^2} = \frac{1}{|1 - r^2|} < 0.25$$

$$\text{we obtain } |1 - r^2| > 4 \text{ or } r > 2.2361$$

Let $r = 0.25$.

$$(a) r = \frac{\omega}{\omega_n} = 2.5 ; \omega_n = \frac{314.16}{2.5} = 125.664 \text{ rad/sec.}$$

$$k = M \omega_n^2 = (60) (125.664^2) = 9.4749 (10^5) \text{ N/m.}$$

$$(b) X = \frac{m e}{M} \frac{r^2}{|1 - r^2|} \text{ (given by Eq. 3.82)} \\ = \left(\frac{0.002}{60} \right) \frac{2.5^2}{|1 - 2.5^2|} = 39.6825 (10^{-6}) \text{ m}$$

$$(c) F_T = m e \omega_n^2 \frac{r^2}{|1 - r^2|} = (0.002) (125.664^2) \frac{6.25}{5.25} = 37.5987 \text{ N}$$

9.43

$$m_1 = 200 \text{ kg}, \omega_1 = 1200 (2\pi/60) = 40 \pi \text{ rad/s} = \sqrt{\frac{k_1}{m_1}}$$

$$k_1 = \text{equivalent spring constant of air compressor} = \omega_1^2 m_1 = 3.1583 \text{ MN/m}$$

Let the absorber be tuned so that $\frac{\omega_2}{\omega_1} = 1$.

Natural frequencies of the combined system are given by the roots of Eq. (9.127), which for $(\omega_2/\omega_1) = 1$ becomes

$$\left(\frac{\omega}{\omega_2}\right)^4 - \left(2 + \frac{m_2}{m_1}\right)\left(\frac{\omega}{\omega_2}\right)^2 + 1 = 0$$

$$\text{or, } r^4 - \left(2 + \frac{m_2}{m_1}\right)r^2 + 1 = 0 \quad \dots (E_1)$$

$$\text{Now } r_1 = \frac{\omega_1}{\omega_2} = 0.8. \quad E_2.(E_1) \text{ gives}$$

$$(0.8)^4 - \left(2 + \frac{m_2}{m_1}\right)(0.8)^2 + 1 = 0 \Rightarrow \frac{m_2}{m_1} = 0.2025$$

$$m_2 = 40.5 \text{ kg}$$

$$\text{As } \frac{k_2}{k_1} = \frac{m_2}{m_1} \left(\frac{\omega_2}{\omega_1}\right)^2 = \frac{m_2}{m_1}, \quad k_2 = k_1(0.2025) = 0.6396 \text{ MN/m}$$

$$\text{If we use } r_2 = 1.2, E_2.(E_1) \text{ gives}$$

$$(1.2)^4 - \left(2 + \frac{m_2}{m_1}\right)(1.2)^2 + 1 = 0 \Rightarrow \frac{m_2}{m_1} = 0.1344$$

Since this value of $\frac{m_2}{m_1}$ is smaller, we have to use the values of m_2 and k_2 given by $r_1 = 0.8$.

9.44

$$\text{Beam: } \omega_1 = \omega_n = 1500 \left(\frac{2\pi}{60}\right) = 157.08 \text{ rad/sec}$$

$$m_1 = M = 300 \text{ kg} = \text{mass of motor}$$

$$k_1 = k_{\text{beam}} = \omega_1^2 m_1 = (157.08)^2 (300) = 7.4022 \times 10^6 \text{ N/m}$$

$$\omega_1 = \sqrt{k_1/m_1} = \omega_2 = \sqrt{k_2/m_2}$$

$$\therefore k_2 = m_2 \omega_2^2 = 24674.1264 m_2 \quad (E_1)$$

(a) Beam with absorber

$$\omega_1 = 0.75 \omega_2 \quad \text{or} \quad r_1 = \omega_1/\omega_2 = 0.75$$

For a tuned absorber, $\omega_1/\omega_2 = 1$, and

$$r_1^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1}$$

$$\text{or } \mu = \left(\frac{r_1^4 + 1}{r_1^2}\right) - 2 = \frac{0.3164 + 1}{0.5625} - 2 = 0.3403 = \frac{m_2}{m_1}$$

$$\therefore \text{Ratio of absorber mass to the mass of the motor} \\ = \mu = 0.3403$$

$$(b) \text{ Mass of the absorber } = m_2 = \mu m_1 = (0.3403)(300) = 102.09 \text{ kg}$$

$$\text{Stiffness of the absorber } = k_2 = m_2 \omega_2^2 \text{ from } E_2.(E_1)$$

$$k_2 = 24674.1264 (102.09) = 2.519 \times 10^6 \text{ N/m}$$

(c) Amplitude of vibration of the absorber mass (X_2):

$$\text{Eq. (9.124) gives } X_2 = -\frac{F_0}{k_2} = -\frac{m e \omega^2}{k_2} = -\frac{\left(\frac{2}{100}\right)(157.08)^2}{2.519 \times 10^6}$$

$$= -0.1959 \text{ mm}$$

9.45

Forcing frequency = $800(2\pi)/60 = 83.776 \text{ rad/sec} = \omega_1 = \omega_2$

$$m'_1 = 1 \text{ kg}$$

$$\text{For } \omega_2/\omega_1 = 1, \quad r_1^2, r_2^2 = \left(1 + \frac{\mu}{2}\right) \mp \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1}$$

$$\text{where } r_1 = \omega_1/\omega_2, \quad r_2 = \omega_2/\omega_1, \quad \mu = m'_2/m_1$$

Here $\omega_1 = 750 \text{ rpm} = 78.54 \text{ rad/sec}$, $\omega_2 = 1000 \text{ rpm} = 104.72 \text{ rad/sec}$,

$$r_1 = 750/800 = 0.9375, \text{ and } r_2 = 1000/800 = 1.2500.$$

$$r_1^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} \Rightarrow \left(1 + \frac{\mu}{2}\right)^2 - 1 = \left\{\left(1 + \frac{\mu}{2}\right) - r_1^2\right\}^2$$

$$\Rightarrow 1 + \frac{\mu}{2} = (1 + r_1^4)/2r_1^2$$

$$\Rightarrow \mu = (r_1^4 + 1)/r_1^2 - 2$$

$$= \frac{0.7725 + 1}{0.8789} - 2 = 0.0167$$

$$\therefore m_1 = \frac{m'_2}{0.0167} = 59.8861 \text{ kg.}$$

New required value of ω_1 is $700 \text{ rpm} = 73.304 \text{ rad/sec}$

which corresponds to $r_1 = 700/800 = 0.875$

$$\mu = \frac{r_1^4 + 1}{r_1^2} - 2 = \frac{1.5862}{0.7656} - 2 = 0.07182$$

$$m_2 = \mu m_1 = 0.07182 (59.8861) = 4.3010 \text{ kg}$$

With these values, ω_2 can be found as

$$r_2 = \left(1 + \frac{\mu}{2}\right) + \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} = (1 + 0.03591) + 0.2704 = 1.3063$$

$$\omega_2 = r_2 \omega_1 = 1.3063 (83.776) = 109.437 \text{ rad/sec} = 1045.04 \text{ rpm}$$

This is larger than the desired upper value of 1040 rpm .

$$\text{Spring stiffness of the absorber} = k_2 = m_2 \omega_2^2$$

$$= (4.301)(83.776)^2 = 30186.2166 \text{ lb/in.}$$

9.46

Original system: $m_1 = 2000/386.4 = 5.176 \text{ lb-sec}^2/\text{in}$

$$\omega_1 = \omega_n = 600(2\pi)/60 = 62.832 \text{ rad/sec} = \sqrt{k_1/m_1}$$

$$k_1 = m_1 \omega_1^2 = 5.176 (62.832)^2 = 20434.1245 \text{ lb/in}$$

(a) Absorber: For tuned absorber, $\omega_2 = \omega_1 = 62.832 \text{ rad/sec}$

$$k_2 = 5000 \text{ lb/in}, \quad m_2 = k_2/\omega_2^2 = 5000/(62.832)^2 = 1.2665 \frac{\text{lb-sec}^2}{\text{in}}$$

$$\text{weight of the absorber} = 489.379 \text{ lb}$$

(b) New system:

Natural frequencies of the new system, ω_1 , and ω_2 , are given by (for $\omega_2/\omega_1 = 1$):

$$r_1^2, r_2^2 = \left(1 + \frac{\mu}{2}\right) \mp \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1}$$

where $r_1 = \omega_1/\omega_2$, $r_2 = \omega_2/\omega_2$ and $\mu = m_2/m_1$.

Here $\mu = 1.2665/5.176 = 0.2447$, and hence --

$$r_1^2 = 0.6127, \quad r_1 = 0.7827$$

$$r_2^2 = 1.6319, \quad r_2 = 1.2775$$

$$\omega_1 = r_1 \omega_2 = 0.7827 (62.832) = 49.1818 \text{ rad/sec} = 469.6509 \text{ rpm}$$

$$\omega_2 = r_2 \omega_2 = 1.2775 (62.832) = 80.2653 \text{ rad/sec} = 766.4750 \text{ rpm}$$

9.47

Natural frequencies of the combined system are

$$\omega_1 = 0.7 \omega_2 = 0.7 (62.832) = 43.9824 \text{ rad/sec}$$

$$\omega_2 = 1.3 \omega_2 = 1.3 (62.832) = 81.6816 \text{ rad/sec}$$

$$r_1 = 0.7, \quad r_2 = 1.3, \quad \sqrt{k_2/m_2} = \omega_2 = \omega_1 = 62.832 \text{ rad/sec}$$

$$k_2 = m_2 \omega_2^2 = (62.832)^2 m_2 \dots (E_1) \text{ (for tuned absorber)}$$

$$r_1^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} \quad \text{or} \quad \mu = \frac{r_1^4 + 1}{r_1^2} - 2$$

$$\mu = \frac{(0.7)^4 + 1}{(0.7)^2} - 2 = 0.5308 = \frac{m_2}{m_1} \Rightarrow m_2 = 0.5308 (5.176) = 2.7474 \text{ lb-sec}^2/\text{in.}$$

$$\text{From Eq. (E1), } k_2 = (62.832)^2 (2.7474) = 10846.3512 \text{ lb/in.}$$

Verification of r_2 :

$$r_2^2 = \left(1 + \frac{\mu}{2}\right) + \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} = \left(1 + \frac{0.5308}{2}\right) + \sqrt{\left(1 + \frac{0.5308}{2}\right)^2 - 1} = 2.0408$$

$$\text{or } r_2 = 1.4286 > 1.3 \text{ (desired value)}$$

$$\text{Hence: } m_2 = 2.7474 \text{ lb-sec}^2/\text{in}$$

$$(\text{weight of absorber} = m_2 g = 1061.5954 \text{ lb})$$

$$k_2 = 10846.3512 \text{ lb/in.}$$

9.48

$$G = 11.5 \times 10^6 \text{ psi}, \quad J = \frac{\pi d^4}{32}$$

For shaft 1:

$$k_{t1} = \frac{\pi G}{32l} (D^4 - d^4)$$

$$= \frac{\pi (11.5 \times 10^6)}{32 \times 30} (2^4 - 1.5^4)$$

$$= 0.4116 \times 10^6 \text{ lb-in/rad}$$

$$J_1 = \frac{100}{386.4} \frac{(15)^2}{8} = 7.2787 \text{ lb-in-sec}^2$$

$$\omega_n = \omega_1 = \sqrt{k_{t1}/J_1} = (0.4116 \times 10^6 / 7.2787)^{\frac{1}{2}} = 237.7994 \text{ rad/sec}$$

For shaft 2:

$$k_{t2} = \frac{\pi (11.5 \times 10^6)}{32 \times 30} (D^4 - d^4)$$

Assuming $\frac{D}{d} = \frac{4}{3}$ (D and d in inches),

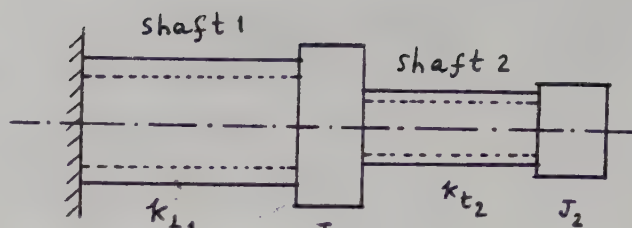
$$k_{t2} = \frac{\pi (11.5 \times 10^6)}{32 \times 30} d^4 \left\{ \left(\frac{D}{d} \right)^4 - 1 \right\} = 0.122 \times 10^6 d^4 \text{ lb-in/rad}$$

$$J_2 = \frac{20}{386.4} \left(\frac{6^2}{8} \right) = 0.2329 \text{ lb-in-sec}^2$$

For $\omega_2 = \omega_1$, and $\omega_2 = \sqrt{k_{t2}/J_2}$, we get

$$237.7994 = \sqrt{\frac{0.122 \times 10^6 d^4}{0.2329}} = 723.761 d^2$$

$$\therefore d = 0.5732'' \text{ and } D = 0.7643''$$



9.49

$$J_1 = 15 \text{ kg-m}^2, \quad k_{t1} = 0.6 \times 10^6 \text{ N-m/rad}$$

$$\omega_1 = \sqrt{k_{t1}/J_1} = \sqrt{0.6 \times 10^6 / 15} = 200 \text{ rad/sec}$$

Absorber:

$$k_{t2}, J_2, \omega_2 = \sqrt{k_{t2}/J_2}$$

For the combined system with tuned absorber, the natural frequencies ω_1 and ω_2 are given by an equation similar to Eq. (9.128) as

$$r_1^2, r_2^2 = \left(1 + \frac{\mu}{2} \right) \mp \sqrt{\left(1 + \frac{\mu}{2} \right)^2 - 1} \quad (E_1)$$

where $\mu = J_2/J_1$, $r_1 = \omega_1/\omega_2$ and $r_2 = \omega_2/\omega_1$.

Let ω_1 be 25% less than ω_1 . Then $r_1 = \frac{150}{200} = 0.75$

$\therefore$ Eq. (E₁) gives

$$(0.75)^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} \Rightarrow \mu = 0.3403$$

$$\therefore J_2 = \mu J_1 = 5.1045 \text{ kg-m}^2$$

$$\text{Since } \omega_2 = \sqrt{k_{t2}/J_2} = 200, \quad k_{t2} = 4 \times 10^4 J_2 = 0.2042 \times 10^6 \text{ N-m/rad}$$

Since ω_2 has to be at least 20% greater than ω_1 ,

$$r_2 = \frac{\omega_2}{\omega_1} \geq 1.2$$

Eq. (E₁) gives, for $\mu = 0.3403$,

$$r_2^2 = \left(1 + \frac{0.3403}{2}\right) + \sqrt{\left(1 + \frac{0.3403}{2}\right)^2 - 1} = 1.7779 \Rightarrow r_2 = 1.3333$$

Hence $J_2 = 5.1045 \text{ kg-m}^2$ and $k_{t2} = 0.2042 \text{ MN-m/rad}$ are acceptable.

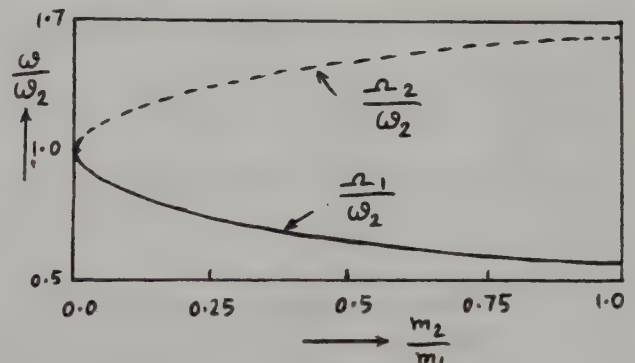
9.50 $\left(\frac{\omega}{\omega_2}\right)^4 - (2+\mu)\left(\frac{\omega}{\omega_2}\right)^2 + 1 = 0$ where $\mu = \frac{m_2}{m_1}$

or, $r^4 - (2+\mu)r^2 + 1 = 0$ where $r = \frac{\omega}{\omega_2} = \frac{\omega_1}{\omega_2}$ or $\frac{\omega_2}{\omega_1}$

$$\therefore r^2 = \left\{ \frac{(2+\mu) \pm \sqrt{(2+\mu)^2 - 4}}{2} \right\}$$

μ	$r_1 = \frac{\omega_1}{\omega_2}$	$r_2 = \frac{\omega_2}{\omega_1}$
0	1.0	1.0
0.1	0.8543	1.1705
0.2	0.8011	1.2483
0.3	0.7630	1.3107
0.4	0.7326	1.3650
0.5	0.7071	1.4142
0.6	0.6851	1.4597
0.7	0.6656	1.5023
0.8	0.6482	1.5427
0.9	0.6325	1.5811
1.0	0.6180	1.6180

Plot:



9.51

$$\left| \frac{X_1}{\delta_{st}} \right| \leq 0.5, \quad \frac{\omega_2}{\omega_1} = 1, \quad \frac{m_2}{m_1} = 0.1$$

Eq. (9.122) gives, for $\frac{\omega_2}{\omega_1} = 1$, $\frac{m_2}{m_1} = 0.1$ and $\frac{k_2}{k_1} = \frac{m_2}{m_1} \left(\frac{\omega_2}{\omega_1}\right)^2 = 0.1$,

$$\pm 0.5 = \frac{1 - \left(\frac{\omega}{\omega_2}\right)^2}{\left[1 + 0.1 - \left(\frac{\omega}{\omega_1}\right)^2\right] \left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right] - 0.1}$$

But $\frac{\omega}{\omega_1} = \frac{\omega}{\omega_2} \cdot \frac{\omega_2}{\omega_1} = \frac{\omega}{\omega_2} = r_2$ (say)

$$\therefore \pm 0.5 = \frac{1 - r_2^2}{(1.1 - r_2^2)(1 - r_2^2) - 0.1} = \frac{1 - r_2^2}{r_2^4 - 2.1 r_2^2 + 1}$$

For $+0.5$, we get $r_2^4 - 2.1 r_2^2 + 1 = 2 - 2 r_2^2$
i.e. $r_2^2 = 1.05125$ (+ value) ; $r_2 = 1.0253$

For -0.5 , we get $r_2^4 - 2.1 r_2^2 + 1 = -2 + 2 r_2^2$
i.e. $r_2^2 = 0.9534, 3.1466$
 $r_2 = 0.9764, 1.7739$ (above the upper resonance point)

operating range is $0.9764 \leq \frac{\omega}{\omega_2} \leq 1.05125$

9.52 $\omega_1 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{10^5}{40}} = 50 \text{ rad/sec}$

Assuming that $\omega_2 = \omega_1$, we obtain $\omega_2 = \left\{ \frac{k_2}{m_2} \right\}^{\frac{1}{2}} = \left\{ \frac{k}{30} \right\}^{\frac{1}{2}} = 50$

$k_2 = 75000 \text{ N/m}$, and

$$X_2 = -\frac{F_0}{m_2 \omega^2} \approx -\frac{F_0}{m_2 \omega_1^2} = -\frac{300}{30 (50^2)} = -0.004 \text{ m}$$



Motor: $m_1 = 20 \text{ kg}$, $\omega = 1350 \text{ rpm} = 141.372 \text{ rad/sec}$, $m_e = 0.1 \text{ kg-m}$.
From Appendix B, we have

$$y_A = \frac{P b^2 a^2 (3 a \ell - a (3 a + b))}{6 E I \ell^3}$$

where $I = \frac{1}{12} w d^3 = \frac{1}{12} (0.15) (0.012^3) = 2.16 (10^{-8}) \text{ m}^4$

Hence $k_1 = \frac{P}{y_A} = \frac{(207 (10^9)) (2.16 (10^{-8})) (1.5^3)}{(0.5^2) (1^2) [3 (1) (1.5) - 1 (3 (1) + 0.5)]} = 362167.2 \text{ N/m}$

Natural frequency of the motor on the beam:

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{362167.2}{20}} = 134.5678 \text{ rad/sec}$$

Natural frequency of the absorber:

$$\omega_2 = \sqrt{\frac{k_2}{m_2}} = \omega = 141.372 \text{ rad/sec}$$

Selecting $m_2 = 10 \text{ kg}$, we obtain

$$k_2 = 10 (141.372^2) = 19.9860 (10^4) \text{ N/m}$$

Amplitude of the absorber at forcing frequency ω :

$$X_2 = -\frac{F_0}{m_2 \omega^2}$$

where F_0 is the amplitude of the forcing function $= m e \omega^2$. Hence

$$X_2 = -\frac{m e \omega^2}{m_2 \omega^2} = -\frac{m e}{m_2} = -\frac{0.1}{10} = -0.01 \text{ m}$$

9.54

$m_1 = 15,000 \text{ kg}$; $k_1 = 2 (10^6) \text{ N/m}$; $F_1(t) = 600 \cos \omega t$

Assume that the forcing frequency coincides with the natural frequency of the bridge.

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{2 (10^6)}{15000}} = 11.5470 \text{ rad/sec}$$

$$\text{For a tuned absorber, } \omega_2 = \sqrt{\frac{k_2}{m_2}} = \omega_1 = 11.5470 \text{ rad/sec}$$

Choose $m_2 = 10 \text{ kg}$. This gives

$$k_2 = m_2 \omega_2^2 = 10 (11.5470^2) = 1333.3333 \text{ N/m}$$

Amplitude of bridge will be zero at the forcing frequency, $\omega = 11.5470 \text{ rad/sec}$.

9.55

$\omega_1 = 100 \text{ rad/sec}$. To suppress the vibration of the motor, the absorber should have the natural frequency:

$$\omega_2 = \sqrt{\frac{k_2}{m_2}} = 80 \text{ rad/sec (operating frequency)}$$

$$\text{or } k_2 = m_2 (80^2) = \left(\frac{10}{386.4} \right) (80^2) = 165.6315 \text{ lb/in}$$

Equations of motion:

$$\sum F = m \ddot{x} \quad m \ddot{x} + c \dot{x} + (k + K_2) x - K_2 R \theta = F_0 \sin \omega t \quad (1)$$

$$\sum M_0 = I \ddot{\theta} \quad (I + M R^2) \ddot{\theta} + (K_1 + K_2) R^2 \theta - K_2 R x = 0 \quad (2)$$

Treating the forcing function as the imaginary component of $F_0 e^{i \omega t}$, we assume the solution as:

$$x = X e^{i(\omega t - \phi)} \quad (3)$$

$$\theta = \Theta e^{i(\omega t - \phi)} \quad (4)$$

Substitution of Eqs. (3) and (4) into (1) and (2) gives

$$\left[-m \omega^2 + (k + K_2) + i c \omega \right] X - (K_2 R) \Theta = F_0 \quad (5)$$

$$- (K_2 R) X + \left[- (I + M R^2) \omega^2 + R^2 (K_1 + K_2) \right] \Theta = 0 \quad (6)$$

Solution of Eqs. (5) and (6) yields:

$$\Theta = \left\{ \frac{-K_2 R F_0}{(-K_2 R)^2 - \left[m \omega^2 + (k + K_2) + i \omega c \right] \left[-I_0 \omega^2 + R^2 (K_1 + K_2) \right]} \right\} \quad (7)$$

$$X = \left\{ \frac{\left[-I_0 \omega^2 + R^2 (K_1 + K_2) \right] F_0}{\left[-m \omega^2 + (k + K_2) + i \omega c \right] \left[-I_0 \omega^2 + R^2 (K_1 + K_2) \right] - (-K_2 R)^2} \right\} \quad (8)$$

$$\text{where } I_0 = I + M R^2 \quad (9)$$

Equation (8) shows that the steady state displacement of mass m (X) will be zero if

$$I_0 \omega^2 = R^2 (K_1 + K_2) \quad (10)$$

The main program and output are shown below:

```

C =====
C
C PROGRAM 18
C TWO-PLANE BALANCING
C
C =====
      DIMENSION VA(2), VB(2), VAP(2), VBP(2), VAPP(2), VBPP(2), WL(2), WR(2),
2     BL(2), BR(2)
C FOLLOWING 8 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DATA VA/8.5,60.0/
      DATA VAP/6.0,125.0/
      DATA WL/10.0,270.0/
      DATA VB/6.5,205.0/
      DATA VBP/4.5,230.0/
      DATA VAPP/6.0,35.0/
      DATA VBPP/10.5,160.0/
      DATA WR/12.0,180.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL BALAN (VA,VB,VAP,VBP,VAPP,VBPP,WL,WR,BL,BR)
      PRINT 10
10     FORMAT (//,31H RESULTS OF TWO-PLANE BALANCING)
      PRINT 20, BL(1),BL(2)
20     FORMAT (//,28H LEFT-PLANE BALANCING WEIGHT,/,11H MAGNITUDE=,
2     E15.8,/,7H ANGLE=,4X,E15.8)
      PRINT 30, BR(1),BR(2)
30     FORMAT (//,29H RIGHT-PLANE BALANCING WEIGHT,/,
2     11H MAGNITUDE=,E15.8,/,7H ANGLE=,4X,E15.8)
      STOP
      END

RESULTS OF TWO-PLANE BALANCING

LEFT-PLANE BALANCING WEIGHT

MAGNITUDE= 0.10056143E+02
ANGLE=     -0.34445206E+02

RIGHT-PLANE BALANCING WEIGHT

MAGNITUDE= 0.58773575E+01
ANGLE=     0.68255920E+02

```

9.58

The main program and output are given below:

```

C =====
C
C PROGRAM 18
C TWO-PLANE BALANCING
C
C =====
      DIMENSION VA(2), VB(2), VAP(2), VBP(2), VAPP(2), VBPP(2), WL(2), WR(2),
2     BL(2), BR(2)
C FOLLOWING 8 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DATA VA/5. 0, 100. 0/
      DATA VB/4. 0, 180. 0/
      DATA WL/2. 0, 30. 0/
      DATA VAP/6. 5, 120. 0/
      DATA VBP/4. 5, 140. 0/
      DATA WR/2. 0, 0. 0/
      DATA VAPP/6. 0, 90. 0/
      DATA VBPP/7. 0, 60. 0/
C END OF PROBLEM-DEPENDENT DATA
      CALL BALAN (VA, VB, VAP, VBP, VAPP, VBPP, WL, WR, BL, BR)
      PRINT 10
10     FORMAT (//, 31H RESULTS OF TWO-PLANE BALANCING)
      PRINT 20, BL(1), BL(2)
20     FORMAT (//, 28H LEFT-PLANE BALANCING WEIGHT, //, 11H MAGNITUDE=,
2     E15. 8, //, 7H ANGLE=, 4X, E15. 8)
      PRINT 30, BR(1), BR(2)
30     FORMAT (//, 29H RIGHT-PLANE BALANCING WEIGHT, //,
2     11H MAGNITUDE=, E15. 8, //, 7H ANGLE=, 4X, E15. 8)
      STOP
      END

```

RESULTS OF TWO-PLANE BALANCING

LEFT-PLANE BALANCING WEIGHT

MAGNITUDE= 0.42315292E+01

ANGLE= -0.49705841E+02

RIGHT-PLANE BALANCING WEIGHT

MAGNITUDE= 0.21217260E+01

ANGLE= -0.39268173E+02

9.59

From Eq. (9.133),

$$\frac{x_2}{x_1} = \frac{k_2 + i\omega c_2}{k_2 - m_2 \omega^2 + i\omega c_2} = \left\{ \frac{k_2^2 + \omega^2 c_2^2}{(k_2 - m_2 \omega^2)^2 + \omega^2 c_2^2} \right\}^{1/2}$$

i.e.

$$\frac{x_2}{\delta_{st}} = \frac{x_1}{\delta_{st}} \left\{ \frac{(f^2)^2 + (2\gamma g)^2}{(f^2 - g^2)^2 + (2\gamma g)^2} \right\}^{1/2}$$

 $\frac{x_1}{\delta_{st}}$ is given by Eq. (9.134).

The program for generating the values of $\frac{x_1}{\delta_{st}}$ and $\frac{x_2}{\delta_{st}}$ for $\frac{m_2}{m_1} = \mu = 1/20$, $f = 1$, $\gamma = 0.1, 0, \infty$ as $\frac{\omega}{\omega_n} = g$ varies between 0.6 and 1.3 is given below.

```
C =====
C
C PROBLEM 9.59
C
C =====
```

```
      DIMENSION X(2)
      REAL MU
      MU=0.05
      DO 100 II=1,3
      G=0.4
      DO 90 JJ=1,15
      G=G+0.1
      F=1.0
      IF (II .EQ. 1) ZETA=0.0
      IF (II .EQ. 2) ZETA=0.1
      IF (II .EQ. 3) ZETA=10.0
      XN=((2.0*ZETA*G)**2+(G**2-F**2)**2
      XD=((2.0*ZETA*G)**2)*((G**2-1.0+MU*G*G)**2)+
      2 (MU*F*F*G*G-(G*G-1.0)*(G*G-F*F))**2
      X(1)=SQRT(XN/XD)
      XN=(F**4)+(2.0*ZETA*G)**2
      XD=(F*F-G*G)**2+(2.0*ZETA*G)**2
      X(2)=X(1)*SQRT(XN/XD)
      PRINT 50, MU,ZETA,G,F
50    FORMAT (/ ,2X,7H MU      =,E15.8,2X,7H ZETA =,E15.8,/,
      2 2X,7H G      =,E15.8,2X,7H F      =,E15.8)
      PRINT 60, X(1),X(2)
60    FORMAT (2X,7H X(1) =,E15.8,2X,7H X(2) =,E15.8)
90    CONTINUE
100   CONTINUE
      STOP
      END

MU      = 0.50000001E-01      ZETA = 0.00000000E+00
G       = 0.50000000E+00      F      = 0.10000000E+01
X(1)    = 0.13636364E+01      X(2)   = 0.18181819E+01

MU      = 0.50000001E-01      ZETA = 0.00000000E+00
G       = 0.60000002E+00      F      = 0.10000000E+01
X(1)    = 0.16343209E+01      X(2)   = 0.25536263E+01
```

MU	= 0.50000001E-01	ZETA	= 0.00000000E+00
G	= 0.70000005E+00	F	= 0.10000000E+01
X(1)	= 0.21646802E+01	X(2)	= 0.42444835E+01
:			
MU	= 0.50000001E-01	ZETA	= 0.10000000E+02
G	= 0.17000003E+01	F	= 0.10000000E+01
X(1)	= 0.49167660E+00	X(2)	= 0.49113098E+00
MU	= 0.50000001E-01	ZETA	= 0.10000000E+02
G	= 0.18000003E+01	F	= 0.10000000E+01
X(1)	= 0.41646856E+00	X(2)	= 0.41582504E+00
MU	= 0.50000001E-01	ZETA	= 0.10000000E+02
G	= 0.19000003E+01	F	= 0.10000000E+01
X(1)	= 0.35850242E+00	X(2)	= 0.35778359E+00

9.60

Crane location: $x_c(t) = A_c e^{-\omega_c \zeta_c t} \sin \omega_c t$ (E₁)

Forging press location: $x_f(t) = A_f \sin \omega_f t$ (E₂)

Air compressor location: $x_a(t) = A_a \sin \omega_a t$ (E₃)

$A_c = 20 \mu\text{m}$, $\omega_c = 10 \text{ Hz}$, $\zeta_c = 0.1$

Attenuation law:

$A_r = A_0 e^{-0.005 r}$ where A_0 = amplitude at source (E₄)
and r = distance from source

Application of Eq. (E₄) gives

$A_c = 20 \mu\text{m}$ reduces to $20 e^{-0.005(60)} = 14.8164 \mu\text{m}$

$A_f = 30 \mu\text{m}$ reduces to $30 e^{-0.005(80)} = 20.1096 \mu\text{m}$

$A_a = 25 \mu\text{m}$ reduces to $25 e^{-0.005(40)} = 20.4683 \mu\text{m}$

Disturbances at site of milling machine are

$\tilde{x}_c(t) = 14.8164 e^{-2\pi t} \sin 20\pi t \mu\text{m}$ (E₅)

$\tilde{x}_f(t) = 20.1096 \sin 30\pi t \mu\text{m}$ (E₆)

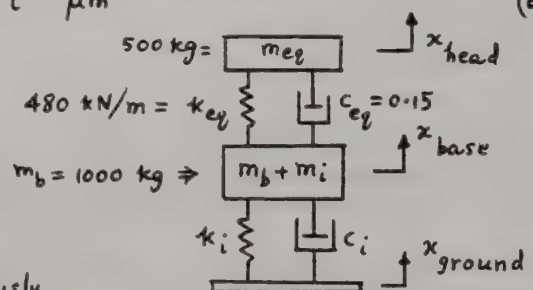
$\tilde{x}_a(t) = 20.4683 \sin 40\pi t \mu\text{m}$ (E₇)

Find m_i , k_i and c_i

such that

$|x_{\text{cutter}}|_{\text{max}} \leq 2.5 \mu\text{m}$

when $x_{\text{ground}} = \tilde{x}_c + \tilde{x}_f + \tilde{x}_a$
all acting simultaneously



Equations of motion :

$$(m_b + m_i) \ddot{x}_b - k_i (x_g - x_b) + k_{eq} (x_b - x_h) - c_i (\dot{x}_g - \dot{x}_b) + c_{eq} (\dot{x}_b - \dot{x}_h) = 0 \quad (E_8)$$

$$m_{eq} \ddot{x}_h - k_{eq} (x_b - x_h) - c_{eq} (\dot{x}_b - \dot{x}_h) = 0 \quad (E_9)$$

i.e.,

$$\begin{bmatrix} (m_b + m_i) & 0 \\ 0 & m_{eq} \end{bmatrix} \begin{Bmatrix} \ddot{x}_b \\ \ddot{x}_h \end{Bmatrix} + \begin{bmatrix} c_i + c_{eq} & -c_{eq} \\ -c_{eq} & c_{eq} \end{bmatrix} \begin{Bmatrix} \dot{x}_b \\ \dot{x}_h \end{Bmatrix} + \begin{bmatrix} k_i + k_{eq} & -k_{eq} \\ -k_{eq} & k_{eq} \end{bmatrix} \begin{Bmatrix} x_b \\ x_h \end{Bmatrix} = \begin{Bmatrix} k_i x_g + c_i \dot{x}_g \\ 0 \end{Bmatrix} \quad (E_{10})$$

The right hand side of E_8 (E_{10}) gives, for each type of disturbance,

When $x_g = \tilde{x}_c(t)$:

$$k_i x_g + c_i \dot{x}_g = k_i (14.8164 e^{-2\pi t} \sin 20\pi t \times 10^{-6}) + c_i \{14.8164 (-2\pi) e^{-2\pi t} \sin 20\pi t + 14.8164 (20\pi) e^{-2\pi t} \cos 20\pi t\} \times 10^{-6}$$

When $x_g = \tilde{x}_f(t)$: (E_{11})

$$k_i x_g + c_i \dot{x}_g = k_i (20.1096 \sin 30\pi t) + c_i (20.1096 \times 30\pi \times \cos 30\pi t) \quad (E_{12})$$

When $x_g = \tilde{x}_a(t)$:

$$k_i x_g + c_i \dot{x}_g = k_i (20.4683 \sin 40\pi t) + c_i (20.4683 \times 40\pi \times \cos 40\pi t) \quad (E_{13})$$

Procedure:

$$\text{Let } \zeta_i = \frac{c_i}{2m_i \omega_n} = \frac{c_i \sqrt{m_i}}{2m_i \sqrt{k_i}} = \frac{c_i}{2\sqrt{m_i k_i}} = 0.1 \quad \text{or} \quad c_i = 0.2 \sqrt{m_i k_i}$$

- (1) Assume m_i
- (2) Assume k_i
- (3) Find $c_i = 0.2 \sqrt{m_i k_i}$
- (4) Solve E_8 (E_{10}) numerically with $x_g = \tilde{x}_c(t)$. Find $\max \tilde{x}_{hc}(t)$.
- (5) Solve E_8 (E_{10}) numerically with $x_g = \tilde{x}_f(t)$. Find $\max \tilde{x}_{hf}(t)$.

- (6) Solve Eq. (E₁₀) numerically with $x_g = \tilde{x}_a(t)$. Find $\max \tilde{x}_{ha}(t)$.
 - (7) Find $\max x_h(t) = \max |\tilde{x}_{hc}(t)| + \max |\tilde{x}_{hf}(t)| + \max |\tilde{x}_{ha}(t)|$
 - (8) If $\max x_h(t) \leq 2.5 \mu\text{m}$, current values of m_i , k_i and c_i constitute the desired design.
 - (9) Otherwise, increment m_i and/or k_i , and go to step (3).
-

Chapter 10

Vibration Measurement and Applications

10.1

Voltage sensitivity = $v = 0.098$ volt-meter/Newton
thickness = $t = 2 \text{ mm} = \frac{2}{1000} \text{ m}$

output voltage = 220 volts, pressure applied = $p_x = ?$

$$E = vt p_x \Rightarrow 220 = (0.098) \left(\frac{2}{1000} \right) p_x$$

$$\therefore p_x = 1.1224 \times 10^6 \text{ N/m}^2$$

10.2

$m = 0.5 \text{ kg}$, $k = 10000 \text{ N/m}$, $c \approx 0$

amplitude = $Y = 4/1000 \text{ m}$

total displacement of mass = $x = 12/1000 \text{ m}$

relative displacement = $z = x - y = 8/1000 \text{ m}$

$$Z = \frac{r^2 Y}{1 - r^2} \quad \text{i.e.,} \quad \frac{8}{1000} = \left\{ \frac{(4/1000) r^2}{1 - r^2} \right\} \Rightarrow r^2 = 2/3$$

$$r = \frac{\omega}{\omega_n} = 0.8165$$

$$\omega = r \omega_n = 0.8165 \sqrt{\frac{10000}{0.5}} = 115.4705 \text{ rad/sec} = 18.3777 \text{ Hz}$$

10.3

$$Z = \frac{r^2 Y}{\sqrt{(1 - r^2)^2 + (2 \zeta r)^2}} \quad \text{where} \quad \zeta = \frac{c}{c_{\text{cri}}} = \frac{1}{\sqrt{2}}$$

$$= \frac{r^2 Y}{\sqrt{(1 - r^2)^2 + (\sqrt{2} r)^2}} = \frac{r^2 Y}{\sqrt{1 + r^4}}$$

10.4

speed range:

500 rpm = 52.36 rad/sec - 1500 rpm = 157.08 rad/sec

$x = X \cos \omega t$;

$$Z = \frac{r^2 Y}{\sqrt{(1 - r^2)^2 + (2 \zeta r)^2}} \quad (E_1)$$

CASE (i): Let $\zeta = 0$

Eg. (E₁) gives, for 2% error,

$$\frac{Z}{Y} = \frac{r^2}{|r^2 - 1|} = 1.02 \Rightarrow r = \frac{\omega}{\omega_n} = 7.1414$$

$$\omega_n = \frac{\omega}{7.1414} = \frac{500}{7.1414} \quad \text{or} \quad \frac{1500}{7.1414} = 70.0143 \quad \text{or} \quad 210.0428 \text{ rpm}$$

$$= 1.1669 \quad \text{or} \quad 3.5007 \text{ Hz} = 7.3319 \quad \text{or} \quad 21.9957 \text{ rad/sec}$$

$$\therefore \omega_n = 1.1669 \text{ Hz}$$

CASE (ii): Let $\zeta = 0.6$

Eg. (E₁) gives $\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = E = 1.02$

$$\Rightarrow 0.0404 r^4 - 0.5826 r^2 + 1.0404 = 0$$

which gives $r = 2.2562, 3.0546$

Since the quantity E attains maximum at

$$r = \frac{1}{\sqrt{1-2\zeta^2}} = 1.8898 \quad \text{for} \quad \zeta = 0.6$$

we use $r = 2.2562$.

$$\therefore \text{Maximum } \omega_n = \omega/r = 500/(2.2562 \times 60) = 3.6935 \text{ Hz}$$

10.5

Error factor for vibrometer is

$$E = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Maximum of E occurs at

$$r^* = \frac{1}{\sqrt{1-2\zeta^2}}$$

For $\zeta = 0$, $E = \frac{r^2}{|1-r^2|}$ and $r^* = 1$

Since the range is $4 \leq r < \infty$, we use $r = 4$ for maximum error.

$$E|_{r=4} = \frac{4^2}{|1-4^2|} = 1.0667$$

$$\text{Percent error} = (E-1)100 = 6.67\%$$

10.6

Error factor = $E = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

E attains maximum at $r^* = 1/\sqrt{1-2\zeta^2}$

For $\zeta = 0.67$, $r^* = 3.1281$ and $E|_{r=r^*, \zeta=0.67} = 1.0053$

$$\text{Percent error} = 0.53\%$$

10.7

Select the vibrometer on the basis of lowest frequency being measured.

(a) $\zeta = 0$:

From Eq. (10.19), $\frac{Z}{Y} = 1.03 = \frac{r^2}{|r^2 - 1|}$, $r^2 = \frac{1.03}{0.03} = 34.3333$

$$r = \frac{\omega}{\omega_n} = \frac{2\pi(500)}{60 \omega_n} = 5.8595$$

$$\omega_n = 8.9359 \text{ rad/sec} = 1.4222 \text{ Hz}$$

(b) $\zeta = 0.6$:

From Eq. (10.19), $\frac{Z}{Y} = 1.03 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = E$

$$(1.03)^2 \{1 + r^4 - 2r^2 + 4r^2\zeta^2\} = r^4$$

i.e., $0.057404 r^4 - 0.56 r^2 + 1 = 0$

i.e., $r^2 = 2.3535, 7.4018$

i.e., $r = 1.5341^\dagger, 2.7206$

By selecting $r = 2.7206$,

$$\omega_n = \frac{1000\pi}{60(2.7206)}$$

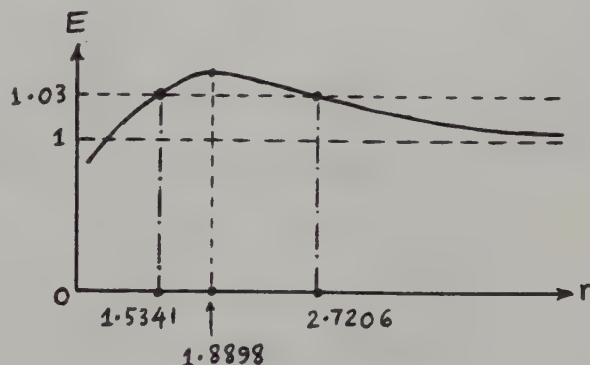
$$= 19.2458 \text{ rad/sec}$$

$$= 3.0630 \text{ Hz}$$

† The quantity E attains maximum at

$$r = \frac{1}{\sqrt{1 - 2\zeta^2}} = 1.8898$$

for $\zeta = 0.6$. Hence we have to take $r = 2.7206$ to avoid peak of E .



10.8

$$\delta_{st} = \frac{10}{1000} \text{ m}, \quad \omega = 4000 \text{ rpm} = 418.88 \text{ rad/sec}$$

$$\omega_n = \sqrt{g/\delta_{st}} = \sqrt{9.81 \left(\frac{1000}{10} \right)} = 31.3209 \text{ rad/sec}$$

$$r = \omega/\omega_n = 418.88/31.3209 = 13.3738$$

Let $\zeta = 0$:

$$\text{Error factor} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \bigg|_{\zeta=0} = \frac{r^2}{|1-r^2|} = \frac{13.3738^2}{|1-13.3738^2|} = 1.0056$$

- (i) Maximum displacement = $Y = Z/1.0056 = 1/1.0056 = 0.9944 \text{ mm}$
(ii) Maximum velocity = $\omega Y = (418.88)(0.9944) = 416.5473 \text{ mm/sec}$
(iii) Maximum acceleration = $\omega^2 Y = (418.88)^2(0.9944) = 174483.35 \text{ mm/sec}^2$

10.9

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad \text{--- (E}_1\text{)}$$

Maximum of $\frac{Z}{Y}$ occurs when $r = r^* = \frac{1}{\sqrt{1-2\zeta^2}}$ (see Eg. (3.84))

For $\zeta = 0.5$, $r^* = \frac{1}{\sqrt{0.5}} = 1.4142$, $(r^*)^2 = 2$

$$\left. \frac{Z}{Y} \right|_{r^*} = \frac{2}{\sqrt{(1-2)^2 + (2 \times 0.5 \times 1.4142)^2}} = \frac{2}{\sqrt{3}} = 1.1547$$

When error is one percent, $\frac{Z}{Y} = 1.01$ or $\frac{Y}{Z} = 0.9901$

Eg. (E₁) can be rewritten as:

$$\left| \frac{Y}{Z} \right|^2 = \frac{(1-r^2)^2 + 4\zeta^2 r^2}{r^4}$$

$$r^4 \left(1 - \left| \frac{Y}{Z} \right|^2 \right) - r^2 (2 - 4\zeta^2) + 1 = 0 \quad \text{--- (E}_2\text{)}$$

For $\frac{Y}{Z} = 0.9901$ and $\zeta = 0.5$, (E₂) becomes

$$0.0197 r^4 - r^2 + 1 = 0$$

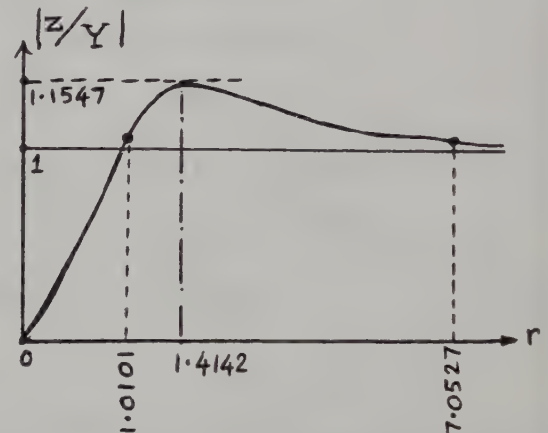
$$r^2 = 1.0203, \quad 49.7411$$

$$r = \frac{\omega}{\omega_n} = 1.0101, \quad 7.0527$$

Lowest frequency for
one percent accuracy

$$= 7.0527 (5)$$

$$= 35.2635 \text{ Hz}$$



10.10

Frequency range $> 100 \text{ Hz}$, maximum error = 2%

$k = 4000 \text{ N/m}$, $c = 0 \Rightarrow \zeta = 0$, $m = ?$

For vibrometer with $\zeta = 0$,
$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \Big|_{\zeta=0}$$

$$= \frac{r^2}{|1-r^2|} = 1.02$$

or $r^2/(1-r^2) = -1.02$ since r must be greater than one
for higher frequencies

or $r^2 = 51$ i.e., $r = 7.1414$

Minimum impressed frequency = $\omega = 10 \text{ Hz}$

$$\text{Since } r = \omega/\omega_n = 100/\omega_n = 7.1414, \quad \omega_n = 14.0029 \text{ Hz} = 87.9827 \frac{\text{rad}}{\text{sec}} \\ = \sqrt{k/m}$$

$$\therefore m = k/\omega_n^2 = 4000/(87.9827)^2 = 0.5167 \text{ kg}$$

10.11

$$\omega_n = 10 \text{ Hz}, \quad \omega_d = 8 \text{ Hz} = \omega_n \sqrt{1-\zeta^2} = 10 \sqrt{1-\zeta^2} \Rightarrow \zeta = 0.6$$

Let the lowest frequency = $\omega_0 \Rightarrow r_0 = \omega_0/\omega_n$

Error = 2% in the range $r_0 \leq r < \infty$

$$\text{Error} = E = 1.02 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2 \times 0.6 r)^2}}$$

$$\Rightarrow 0.0404 r^4 - 0.5826 r^2 + 1.0404 = 0$$

$$\Rightarrow r = 2.2562, 3.0546$$

Let $r_0 = 2.2562$:

$$\omega_0 = r_0 \omega_n = 2.2562(10) = 22.562 \text{ Hz}$$

Let $r_0 = 3.0546$:

$$\omega_0 = r_0 \omega_n = 3.0546(10) = 30.546 \text{ Hz}$$

$$\therefore \text{Lowest frequency} = 22.562 \text{ Hz} = 141.7616 \text{ rad/sec}$$

10.12

$$\text{Error factor for accelerometer} = E = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Maximum of E occurs at $r^* = \sqrt{1-2\zeta^2}$

$$\text{When } \zeta = 0, \quad r^* = 1 \text{ and } E = \frac{1}{|1-r^2|}$$

Since the range is $0 \leq r \leq 0.65$, we use $r = 0.65$:

$$E|_{r=0.65} = \frac{1}{|1-0.65^2|} = 1.7316$$

$$\text{Percent error} = (E-1)100 = 73.16\%$$

10.13

$$\text{Error factor} = E = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Value of r at which E attains maximum is $r^* = \sqrt{1-2\zeta^2}$

$$\text{When } \zeta = 0.75, \quad E = \frac{1}{\sqrt{(1-r^2)^2 + 2.25 r^2}} \quad \text{and } r^* = \sqrt{1-1.125} \\ = \text{imaginary}$$

Since the range is $0 \leq r \leq 0.6$, we use $r = 0.6$

$$E|_{r=0.6} = \frac{1}{\sqrt{(1-0.36)^2 + 2.25(0.6)^2}} = 0.9055$$

$$\text{Percent error} = (E-1)100 = -9.45\%$$

10.14

$m = 0.05 \text{ kg}$, max error = 3% over frequency range of 0 to 100 Hz
Find k and c .

For accelerometer, error factor = $E = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

E attains maximum at $r = r^* = \sqrt{1-2\zeta^2}$

and $E_{\max} = E|_{r^*} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

(i) Consideration of maximum error:

let error = $e = E - 1 = 0.03 = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$

upon rearrangement, this leads to $\zeta^4 - \zeta^2 + 0.23565 = 0$

or $\zeta = 0.6164, 0.7874$

r^* at $\zeta = 0.6164 = \sqrt{1-2(0.6164)^2} = 0.49$

r^* at $\zeta = 0.7874 = \sqrt{1-2(0.7874)^2} = \text{imaginary}$

(ii) Consideration of minimum error:

let error = $e = E - 1 = -0.03 = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} - 1$ (E₁)

with $\zeta = 0.6164$, Eq. (E₁) can be simplified as

$r^4 - 0.4802 r^2 - 0.0628 = 0$

$\Rightarrow r^2 = 0.5871, -0.1069$

or $r = 0.7662$

At the maximum frequency,

$\omega = 2\pi(100) = 628.32 \text{ rad/sec}$

$\omega_n = \omega/r = 628.32/0.7662$

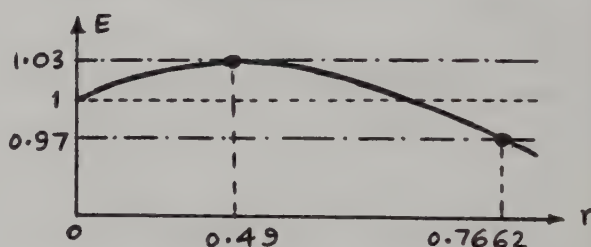
$= 820.0470 \text{ rad/sec}$

$k = m\omega_n^2 = 0.05(820.047)^2$

$= 33623.8541 \text{ N/m}$

$c_c = 2m\omega_n = \frac{c}{\zeta} \Rightarrow c = 2m\omega_n\zeta = 2(0.05)(820.047)(0.6164)$

$= 50.5477 \text{ N-s/m}$



10.15

$m = 0.1 \text{ kg}$, $k = 10000 \text{ N/m}$, $c = 0 \Rightarrow \zeta = 0$

$\omega_n = \sqrt{k/m} = \sqrt{10000/0.1} = 316.2278 \text{ rad/sec}$

Engine speed = $\omega = 1000 \text{ rpm} = 104.72 \text{ rad/sec}$

$r = \omega/\omega_n = 104.72/316.2278 = 0.3312$

peak-to-peak travel of mass = 10 mm

Find: Y , ωY , $\omega^2 Y$.

We have, from Eq. (10.19),

$$\frac{Z}{Y} \bigg|_{\zeta=0} = \frac{r^2}{|1-r^2|} = \frac{0.3312^2}{|1-0.3312^2|} = 0.1232$$

Since peak-to-peak travel of mass = 10 mm, $Z = 5$ mm

$$Y = Z/0.1232 = 5/0.1232 = 40.5844 \text{ mm}$$

Max displacement of foundation = $Y = 40.5844$ mm

Max velocity of foundation = $\omega Y = 4249.9984$ mm/sec

Max acceleration of foundation = $\omega^2 Y = 445059.8291$ mm/sec²

10.16

Maximum speed = 3000 rpm = 50 Hz ; $r = \frac{\omega}{\omega_n} = \frac{50}{100} = 0.5$

For accelerometer, $\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 1 + \text{error} = 0.9$ (E₁)

$$\text{Here } c = 20 \text{ N-s/m ; } \zeta = \frac{c}{2m\omega_n} = \frac{20}{2m(100 \times 2\pi)} = \frac{0.015915}{m}$$

For $r=0.5$, Eq. (E₁) gives $\zeta = 0.8198$

$$c_c = \frac{c}{\zeta} = 20/0.8198 = 24.3962 \text{ N-s/m}$$

$$m = \frac{c_c}{2\omega_n} = \frac{24.3962}{2(100 \times 2\pi)} = 0.01941 \text{ kg} = 19.41 \text{ grams}$$

$$k = m\omega_n^2 = 0.01941 (100 \times 2\pi)^2 = 7622.7967 \text{ N/m}$$

10.17

$$\omega_n = 2\pi(0.5) = \pi \text{ rad/sec ; } \omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$\sqrt{1-\zeta^2} = \omega_d/\omega_n = 0.48/0.5 \Rightarrow \zeta = 0.28$$

$$r_1 = \omega_1/\omega_n = 4\pi/\pi = 4 ; \quad r_2 = \omega_2/\omega_n = 8 ; \quad r_3 = \omega_3/\omega_n = 12$$

$$\phi_i = \tan^{-1} \left(\frac{2\zeta r_i}{1-r_i^2} \right)$$

$$\phi_1 = \tan^{-1} \left(\frac{2(0.28)4}{1-16} \right) = -8.4934^\circ$$

$$\phi_2 = \tan^{-1} \left(\frac{2 \times 0.28 \times 8}{1-64} \right) = -4.0675^\circ$$

$$\phi_3 = \tan^{-1} \left(\frac{2 \times 0.28 \times 12}{1-144} \right) = -2.6905^\circ$$

$$\frac{r_1^2}{\sqrt{(1-r_1^2)^2 + (2\zeta r_1)^2}} \times 20 = 21.0994$$

$$\frac{r_2^2}{\sqrt{(1-r_2^2)^2 + (2\zeta r_2)^2}} \times 10 = 10.1331$$

$$\frac{r_3^2}{\sqrt{(1-r_3^2)^2 + (2\zeta r_3)^2}} \times 5 = 5.0294$$

Record indicated by vibrometer is given by

$$z(t) = 21.0994 \sin(4\pi t + 8.4934^\circ) + 10.1331 \sin(8\pi t + 4.0675^\circ) + 5.0294 \sin(12\pi t + 2.6905^\circ) \text{ mm}$$

10.18

$$x(t) = 20 \sin 50t + 5 \sin 150t \text{ mm} \quad (E_1)$$

$$\ddot{x}(t) = -20(50)^2 \sin 50t - 5(150)^2 \sin 150t \text{ mm/sec}^2$$

$$= -50000 \sin 50t - 112500 \sin 150t \text{ mm/sec}^2 \quad (E_2)$$

$$\omega_n = 100 \text{ rad/sec}, \quad \omega_d = \omega_n \sqrt{1-\zeta^2} = 80 \Rightarrow \zeta = 0.6$$

$$r_1 = \frac{\omega_1}{\omega_n} = \frac{50}{100} = 0.5; \quad r_2 = \frac{\omega_2}{\omega_n} = \frac{150}{100} = 1.5$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r_1}{1-r_1^2}\right) = \tan^{-1}\left(\frac{2 \times 0.6 \times 0.5}{1-0.25}\right) = 38.6598^\circ$$

$$\phi_2 = \tan^{-1}\left(\frac{2\zeta r_2}{1-r_2^2}\right) = \tan^{-1}\left(\frac{2 \times 0.6 \times 1.5}{1-2.25}\right) = -55.2222^\circ$$

$$\frac{50000}{\sqrt{(1-r_1^2)^2 + (2\zeta r_1)^2}} = 52057.9206$$

$$\frac{112500}{\sqrt{(1-r_2^2)^2 + (2\zeta r_2)^2}} = 51335.6229$$

output of the accelerometer is given by

$$\ddot{z}(t) = -52057.9206 \sin(50t - 38.6598^\circ) - 51335.6229 \sin(150t + 55.2222^\circ) \text{ mm/sec}^2 \quad (E_3)$$

It can be seen that E_3 is substantially different from E_2 .

10.19

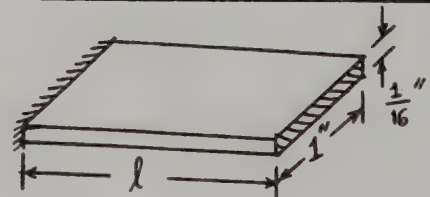
For given beam,

$$A = 1\left(\frac{1}{16}\right) = 0.0625 \text{ in}^2$$

$$I = \frac{1}{12}(1)\left(\frac{1}{16}\right)^3 = 20.35 \times 10^{-6} \text{ in}^4$$

$$l = 2'' \text{ to } 10''$$

For a cantilever beam, Fig. 8.15 gives



$$\omega_n = (\beta_n l)^2 \left(\frac{EI}{\rho A l^4} \right)^{1/2}$$

where $(\beta_1 l)^2 = (1.875104)^2 = 3.516015$

$$(\beta_2 l)^2 = (4.694091)^2 = 22.03449$$

$$(\beta_3 l)^2 = (7.854757)^2 = 61.69721$$

$$(\beta_4 l)^2 = (10.995541)^2 = 120.90192$$

For spring steel, $E = 30 \times 10^6$ psi, $\rho g = 0.283$ lb/in³

$$\left(\frac{EI}{\rho A l^4} \right)^{1/2} = \left\{ \frac{30 \times 10^6 (20.35 \times 10^{-6})}{\left(\frac{0.283}{386.4} \right) (0.0625) l^4} \right\}^{1/2} = \frac{3652.0}{l^2}$$

$$\omega_n = (\beta_n l)^2 \left(\frac{3652.0}{l^2} \right)$$

The first four frequencies are given below:

	l^2	ω_1	ω_2	ω_3	ω_4
$l = 2''$	4	3210.1020	20117.4894	56329.5527	110383.4530
$l = 10''$	100	128.4049	804.6996	2253.1821	4415.3381

Hence the range of frequencies that can be measured is given by $\omega > 128.4049$ rad/sec.

However, for first mode only (which is easiest to excite), the range of frequencies is $128.4049 \frac{\text{rad}}{\text{sec}} \leq \omega \leq 3210.102 \frac{\text{rad}}{\text{sec}}$.

10.20

$$\frac{k X_R}{F_0} = \frac{1 - r^2}{(1 - r^2)^2 + (2 \zeta r)^2} = \frac{N}{D} \text{ (assume)}$$

$$\frac{d}{dr} \left(\frac{k X_R}{F_0} \right) = \frac{D \frac{dN}{dr} - N \frac{dD}{dr}}{D^2} = 0 \text{ i.e., } D \frac{dN}{dr} - N \frac{dD}{dr} = 0$$

$$\text{or } \left\{ (1 - r^2)^2 + (2 \zeta r)^2 \right\} (-2r) - (1 - r^2) \left\{ 2(1 - r^2)(-2r) + 2(2 \zeta r)(2 \zeta) \right\} = 0$$

This equation can be simplified as

$$r^4 - 2r^2 + (1 - 4\zeta^2) = 0$$

and its solution is given by

$$r^2 = 1 \pm 2\zeta \text{ or } r = \sqrt{1 + 2\zeta}; \sqrt{1 - 2\zeta}$$

Since

$$\frac{k X_R}{F_0} \Big|_{r = \sqrt{1 + 2\zeta}} = -\frac{1}{4\zeta(1 + \zeta)} \quad (1)$$

and

$$\frac{k X_R}{F_0} \Big|_{r = \sqrt{1-2\zeta}} = \frac{1}{4\zeta(1-\zeta)} \quad (2)$$

we note that $r = R_1 = \sqrt{1-2\zeta}$ corresponds to a maximum and $r = R_2 = \sqrt{1+2\zeta}$ corresponds to a minimum of X_R .

10.21

$$\frac{k X_I}{F_0} = \frac{-2\zeta r}{(1-r^2)^2 + 4\zeta^2 r^2}$$

$$\frac{d}{dr} \left(\frac{k X_I}{F_0} \right) = \frac{D \frac{dN}{dr} - N \frac{dD}{dr}}{D^2} = 0 \quad (1)$$

$$\text{where } \frac{dN}{dr} = -2\zeta \quad \text{and} \quad \frac{dD}{dr} = -2(1-r^2)(2r) + 8\zeta^2 r$$

By setting the numerator of Eq. (1) equal to zero, we obtain

$$\left\{ 1 + r^4 - 2r^2 + 4\zeta^2 r^2 \right\} (-2\zeta) - (-2\zeta r) \left\{ -4r + 4r^3 + 8\zeta^2 r \right\} = 0$$

which can be simplified to obtain

$$3r^4 + (4\zeta^2 - 2)r^2 - 1 = 0 \quad (2)$$

The roots of Eq. (2) are given by

$$r^2 = \frac{1 - 2\zeta^2 - 2\sqrt{\zeta^4 - \zeta^2 + 1}}{3} ; \frac{1 - 2\zeta^2 + 2\sqrt{\zeta^4 - \zeta^2 + 1}}{3} \quad (3)$$

Since it is difficult to determine, from Eq. (3), the correct value of r that corresponds to the minimum of X_I , we use a numerical computation. For $\zeta = 0.1$, for example, Eq. (3) gives

$$r^2 = -0.0030 ; 0.6583$$

This shows that

$$r = \left\{ \frac{1 - 2\zeta^2 + 2\sqrt{\zeta^4 - \zeta^2 + 1}}{3} \right\}^{\frac{1}{2}} \quad (4)$$

corresponds to the minimum of X_I . For small values of ζ , $\zeta^2 \ll 1$ and Eq. (4) gives

$$r \approx 1 \quad (5)$$

Thus X_I attains its minimum value close to $r = 1$.

10.22

Response of a single d.o.f. system with hysteretic damping is given by Eq. (3.106):

$$\frac{X}{F_0} = \frac{1}{k - m\omega^2 + i k \beta}$$

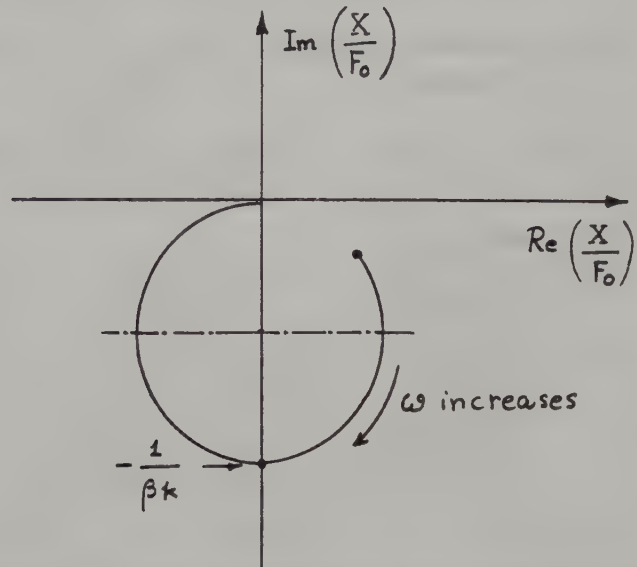
$$\operatorname{Re} \left(\frac{X}{F_0} \right) = \frac{k - m \omega^2}{(k - m \omega^2)^2 + (k \beta)^2} \quad \operatorname{Im} \left(\frac{X}{F_0} \right) = \frac{-k \beta}{(k - m \omega^2)^2 + (k \beta)^2}$$

$$\left[\operatorname{Re} \left(\frac{X}{F_0} \right) \right]^2 + \left[\operatorname{Im} \left(\frac{X}{F_0} \right) \right]^2 = \frac{1}{(k - m \omega^2)^2 + (k \beta)^2} \quad (1)$$

It can be verified that Eq. (1) can be rewritten as

$$\left[\operatorname{Re} \left(\frac{X}{F_0} \right) \right]^2 + \left[\operatorname{Im} \left(\frac{X}{F_0} \right) + \frac{1}{2 \beta k} \right]^2 = \left(\frac{1}{2 \beta k} \right)^2 \quad (2)$$

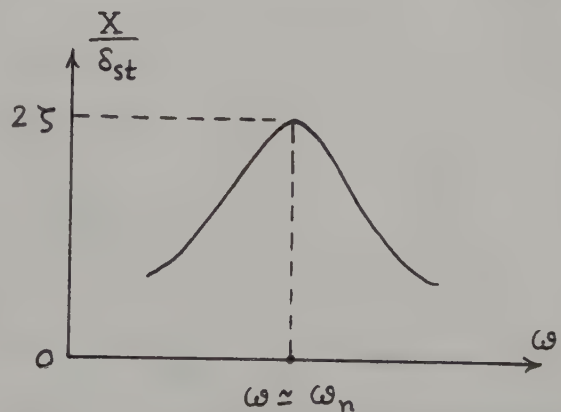
Eq. (2) shows that the locus of $\left(\frac{X}{F_0} \right)$ as ω increases from zero is part of a circle, with center $\left(0, -\frac{1}{2 k \beta} \right)$ and radius $\frac{1}{2 k \beta}$, as shown in the following figure.



10.23 The peak of Bode diagram is equal to $\approx \frac{1}{2 \zeta}$. In the present case, peak-to-peak value is plotted; hence $X \approx \frac{0.45}{2}$ mil = 0.225 mil.

$$\frac{X}{\delta_{st}} = \frac{0.225}{0.05} = 4.5 = \frac{1}{2 \zeta}$$

or $\zeta = 0.1111$.



10.24

Reduction in amplitude from 6.8 in/sec to 0.8 in/sec in 7 cycles or 22 milliseconds.

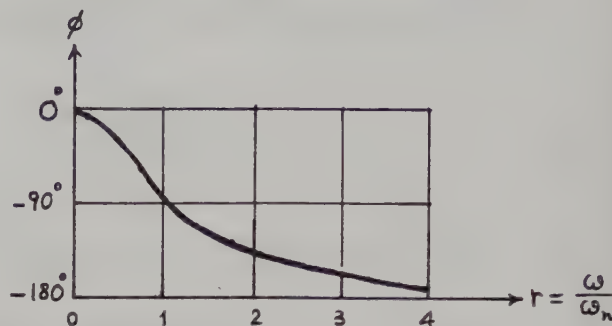
Eq. (2.92) gives:

$$\frac{1}{7} \ln \left(\frac{x_1}{x_8} \right) = \delta \quad \text{or} \quad \delta = \frac{1}{7} \ln \left(\frac{6.8}{0.8} \right) = 0.3057$$

$$\text{Hence } \zeta = \frac{\delta}{2\pi} = 0.04866.$$

10.25

Typical Bode plot of phase angle is shown in the figure.



(a) $\phi = 90^\circ$ at $r = \frac{\omega}{\omega_n} \approx 1$. Hence the value of ω_n can be determined from the value of r corresponding to $\phi = 90^\circ$.

(b) Since

$$\phi = \tan^{-1} \left(-\frac{2\zeta r}{1-r^2} \right) = \tan^{-1} \left(-\frac{c\omega}{k-m\omega^2} \right)$$

we find

$$-\frac{c\omega_1}{k-m\omega_1^2} = -1 \quad \text{and} \quad -\frac{c\omega_2}{k-m\omega_2^2} = 1$$

where ω_1 and ω_2 correspond to the half power points. Hence, by finding the values of ω corresponding to $\phi = -45^\circ$ and $\phi = -135^\circ$, we obtain ω_1 and ω_2 . From these values, the damping ratio can be found using the relation:

$$\zeta = \frac{\omega_2 - \omega_1}{2\omega_n}$$

10.26

10.27

10.28

Characteristic	Problem 10.26	Problem 10.27	Problem 10.28
n	16	18	18
N	750 rpm	1000 rpm	1500 rpm
d	15 mm	2 cm	10 mm
D	100 mm	15 cm	80 mm
α	30°	20°	40°
$\frac{d}{D} \cos \alpha$	0.1299	0.1253	0.09576
Dominant frequency of vibration			
Inner race defect	6779.4 cycles/min (1078.97 Hz)	10127.7 cycles/min (1611.87 Hz)	14792.8 cycles/min (2354.33 Hz)
Outer race defect	5220.6 cycles/min (830.88 Hz)	7872.3 cycles/min (1252.91 Hz)	12207.2 cycles/min (1942.84 Hz)
Ball or roller defect	1214.6 cycles/min (193.31 Hz)	1761.7 cycles/min (280.39 Hz)	2188.1 cycles/min (348.25 Hz)
Cage defect	326.3 cycles/min (51.93 Hz)	437.3 cycles/min (69.61 Hz)	678.2 cycles/min (107.93 Hz)

10.29

$$f(x) = \frac{1}{4} ; 1 \leq x \leq 5$$

$$\bar{x} = \text{mean value of } x = \int_1^5 f(x) x \, dx = \frac{1}{4} \left(\frac{x^2}{2} \right)_1^5 = 3$$

$$\sigma^2 = (\text{standard deviation})^2 = \int_1^5 (x - \bar{x})^2 f(x) \, dx$$

$$= \frac{1}{4} \int_1^5 (x - 3)^2 \, dx = \frac{1}{4} \left(\frac{x^3}{3} - 3x^2 + 9x \right)_1^5 = \frac{4}{3}$$

$$k = \text{kurtosis} = \frac{1}{\sigma^4} \int_{-\infty}^{\infty} (x - \bar{x})^4 f(x) \, dx = \frac{9}{16} \int_1^5 (x - 3)^4 \left(\frac{1}{4} \right) \, dx$$

Let $y = x - 3$ so that $dy = dx$. This gives

$$k = \frac{9}{64} \int_{y=-2}^2 y^4 \, dy = \frac{9}{5}$$

10.30

$$\bar{x} = \text{mean value of } x = \sum_i f(x_i) x_i$$

$$= \frac{1}{32} (1) + \frac{3}{32} (2) + \frac{3}{16} (3) + \frac{6}{16} (4) + \frac{3}{16} (5) + \frac{3}{32} (6) + \frac{1}{32} (7) = 4$$

$$\sigma^2 = (\text{standard deviation})^2 = \sum_i (x_i - \bar{x})^2 f(x_i)$$

$$= (1-4)^2 \frac{1}{32} + (2-4)^2 \frac{3}{32} + (3-4)^2 \frac{3}{16} + (4-4)^2 \frac{6}{16} \\ + (5-4)^2 \frac{3}{16} + (6-4)^2 \frac{3}{32} + (7-4)^2 \frac{1}{32} = \frac{27}{16} = 1.6875$$

$$\sum_i (x_i - \bar{x})^4 f(x_i) = (1-4)^4 \frac{1}{32} + (2-4)^4 \frac{3}{32} + (3-4)^4 \frac{3}{16} \\ + (4-4)^4 \frac{6}{16} + (5-4)^4 \frac{3}{16} + (6-4)^4 \frac{3}{32} + (7-4)^4 \frac{1}{32} = \frac{135}{16} = 8.4375$$

$$k = \text{kurtosis} = \frac{1}{\sigma^4} \sum_i (x_i - \bar{x})^4 f(x_i) = \frac{8.4375}{1.6875^2} = 2.9630$$

10.31

Range of $\omega = 62.832$ to 314.16 rad/sec = 600 to 3000 rpm

Max acceleration level = $10g = 98.1$ m/sec²

Max weight of specimen = 10 N

Max vibration amplitude = 0.0025 m

Frequency range:

variable speed electric motor can be used to obtain the frequency range (for a mechanical shaker).

Vibration amplitude:

$$\text{If } y(t) = A \sin \omega t, \quad (E_1)$$

$$\text{acceleration} = A\omega^2.$$

At $\omega = 314.16$ rad/sec, amplitude needed to achieve the maximum acceleration is:

$$\text{amplitude } (A) = \text{acceleration} / \omega^2 \\ = 98.1 / (314.16)^2 = 0.9939 \times 10^{-3} \text{ m}$$

At $\omega = 62.832$ rad/sec, amplitude needed to achieve the maximum acceleration is:

$$\begin{aligned}\text{amplitude } (A) &= \text{acceleration} / \omega^2 \\ &= 98.1 / (62.832)^2 = 0.02485 \text{ m}\end{aligned}$$

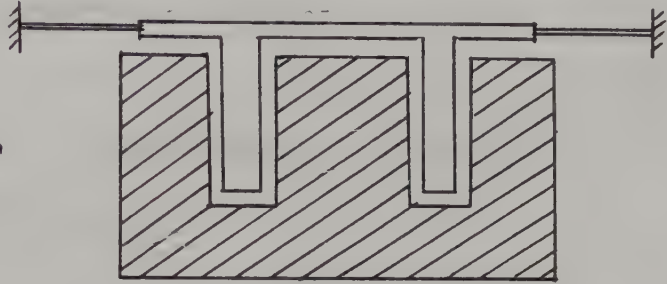
(amplitude is too high; hence direct application of $y(t)$ is not permitted).

Mechanical shaker of the type shown in Fig. 10.16 can be used. Electrodynamic shaker of the type shown in Fig. 10.17(a) can also be used.

Electromagnetic shaker:

Max force ($F_{\max}$) available depends on:

- (a) magnetic field strength
- (b) number of turns
- (c) coil diameter
- (d) current flowing



Limitations are: (a) material strength, (b) cooling provided.

$$\text{Max acceleration} = \left(\frac{F_{\max}}{m_s + m_t} \right)$$

where m_s = mass of specimen and m_t = mass of shaker table.

10.32

Speed range: 300 - 600 rpm

Frequency range: 31.416 - 62.832 rad/sec

Number of reeds = 12

Uniform spacing of frequencies give the reed frequencies as:

$$\Omega_1, \dots, \Omega_{12} = 31.416, 34.272, 37.128, 39.984, 42.840,$$

$$45.696, 48.552, 51.408, 54.264, 57.120, 59.976, 62.832 \text{ rad/sec}$$

Let each reed be considered as a cantilever beam of cross section $a \times b$ inches. Let lengths of all reeds be same and the material be aluminum for light weight. The fundamental natural frequency of a reed is given by (Fig. 8.15):

$$\begin{aligned}\omega_1 &= (\beta_1 \ell)^2 \left\{ \frac{EI}{\rho A \ell^4} \right\}^{\frac{1}{2}} = (1.8751^2) \left\{ \frac{EI}{\rho A \ell^4} \right\}^{\frac{1}{2}} \\ &= 3.516 \left\{ \frac{(10^7) I}{(0.1/386.4) A \ell^4} \right\}^{\frac{1}{2}} = 69.1142 (10^4) \left\{ \frac{I}{A \ell^4} \right\}^{\frac{1}{2}} \quad (1)\end{aligned}$$

By equating ω_1 given by Eq. (1) to $\Omega_1, \dots, \Omega_{12}$, in turn, the proper value of $\left\{ \frac{I}{A \ell^4} \right\}^{\frac{1}{2}}$

needed for different reeds can be computed. By selecting a common value of ℓ for all reeds, the cross section of any reed can then be found to achieve the required value of $\left\{ \frac{I}{A \ell^4} \right\}^{\frac{1}{2}}$.

10.33

Iterative process is to be used.

1. Select trial values of the design parameters (material of the beam and its dimensions).
2. Model the beam as a spring-mass system with:
 $m = \text{end mass} = 50\% \text{ of mass of beam:}$

$$m = \frac{1}{2} \rho A \ell \quad (1)$$

$k = \text{stiffness of a cantilever beam:}$

$$k = \frac{3 E I}{\ell^3} \quad (2)$$

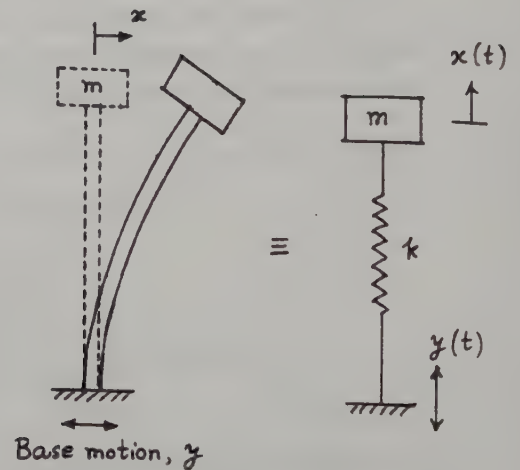
3. Equations of motion:

$$m \ddot{x} + k(x - y) = 0$$

$$\text{or } m \ddot{z} + k z = -m \ddot{y} \quad (3)$$

where $z = \text{relative displacement of end mass.}$

4. Since $\ddot{y}_{\max} = 0.2 \text{ g}$, assume a constant force of $-m(0.2 \text{ g})$ on the right hand side of Eq. (3) and solve the equation to find $z(t)$.
5. From the known $z_{\max}$ value, compute the maximum stress ($\sigma_{\max}$) induced in the beam. If $\sigma_{\max}$ is less than the yield stress of the material, the design is complete. Otherwise, go to step 1 and change one or more design parameters and repeat the procedure until a satisfactory design is found.



Chapter 11

Numerical Integration Methods in Vibration Analysis

$$11.1 \quad \left. \frac{d^2 x}{dt^2} \right|_i = \frac{\left. \frac{dx}{dt} \right|_{i+1} - \left. \frac{dx}{dt} \right|_i}{\Delta t} = \frac{\left(\frac{x_{i+2} - x_{i+1}}{\Delta t} \right) - \left(\frac{x_{i+1} - x_i}{\Delta t} \right)}{\Delta t} = \frac{x_{i+2} - 2x_{i+1} + x_i}{(\Delta t)^2}$$

$$\begin{aligned} \left. \frac{d^3 x}{dt^3} \right|_i &= \frac{\left. \frac{d^2 x}{dt^2} \right|_{i+1} - \left. \frac{d^2 x}{dt^2} \right|_i}{\Delta t} = \frac{\left(\frac{x_{i+3} - 2x_{i+2} + x_{i+1}}{(\Delta t)^2} \right) - \left(\frac{x_{i+2} - 2x_{i+1} + x_i}{(\Delta t)^2} \right)}{\Delta t} \\ &= \frac{x_{i+3} - 3x_{i+2} + 3x_{i+1} - x_i}{(\Delta t)^3} \end{aligned}$$

$$\begin{aligned} \left. \frac{d^4 x}{dt^4} \right|_i &= \frac{\left. \frac{d^3 x}{dt^3} \right|_{i+1} - \left. \frac{d^3 x}{dt^3} \right|_i}{\Delta t} = \frac{\left(\frac{x_{i+4} - 3x_{i+3} + 3x_{i+2} - x_{i+1}}{(\Delta t)^3} \right) - \left(\frac{x_{i+3} - 3x_{i+2} + 3x_{i+1} - x_i}{(\Delta t)^3} \right)}{\Delta t} \\ &= \frac{x_{i+4} - 4x_{i+3} + 6x_{i+2} - 4x_{i+1} + x_i}{(\Delta t)^4} \end{aligned}$$

$$11.2 \quad \left. \frac{d^2 x}{dt^2} \right|_i = \frac{\left. \frac{dx}{dt} \right|_i - \left. \frac{dx}{dt} \right|_{i-1}}{\Delta t} = \frac{\left(\frac{x_i - x_{i-1}}{\Delta t} \right) - \left(\frac{x_{i-1} - x_{i-2}}{\Delta t} \right)}{\Delta t} = \frac{x_i - 2x_{i-1} + x_{i-2}}{(\Delta t)^2}$$

$$\begin{aligned} \left. \frac{d^3 x}{dt^3} \right|_i &= \frac{\left. \frac{d^2 x}{dt^2} \right|_i - \left. \frac{d^2 x}{dt^2} \right|_{i-1}}{\Delta t} = \frac{\left(\frac{x_i - 2x_{i-1} + x_{i-2}}{(\Delta t)^2} \right) - \left(\frac{x_{i-1} - 2x_{i-2} + x_{i-3}}{(\Delta t)^2} \right)}{\Delta t} \\ &= \frac{x_i - 3x_{i-1} + 3x_{i-2} - x_{i-3}}{(\Delta t)^3} \end{aligned}$$

$$\begin{aligned} \left. \frac{d^4 x}{dt^4} \right|_i &= \frac{\left. \frac{d^3 x}{dt^3} \right|_i - \left. \frac{d^3 x}{dt^3} \right|_{i-1}}{\Delta t} = \frac{\left(\frac{x_i - 3x_{i-1} + 3x_{i-2} - x_{i-3}}{(\Delta t)^3} \right) - \left(\frac{x_{i-1} - 3x_{i-2} + 3x_{i-3} - x_{i-4}}{(\Delta t)^3} \right)}{\Delta t} \\ &= \frac{x_i - 4x_{i-1} + 6x_{i-2} - 4x_{i-3} + x_{i-4}}{(\Delta t)^4} \end{aligned}$$

$$11.3 \quad \left. \frac{d^2 x}{dt^2} \right|_i = \frac{x_{i+1} - 2x_i + x_{i-1}}{(\Delta t)^2}$$

$$\left. \frac{d^4 x}{dt^4} \right|_i = \frac{\left. \frac{d^2 x}{dt^2} \right|_{i+1} - 2 \left. \frac{d^2 x}{dt^2} \right|_i + \left. \frac{d^2 x}{dt^2} \right|_{i-1}}{(\Delta t)^2}$$

$$= \left\{ \left(\frac{x_{i+2} - 2x_{i+1} + x_i}{(\Delta t)^2} \right) - 2 \left(\frac{x_{i+1} - 2x_i + x_{i-1}}{(\Delta t)^2} \right) + \left(\frac{x_i - 2x_{i-1} + x_{i-2}}{(\Delta t)^2} \right) \right\} / (\Delta t)^2$$

$$= \frac{x_{i+2} - 4x_{i+1} + 6x_i - 4x_{i-1} + x_{i-2}}{(\Delta t)^4}$$

11.4

Equation: $\ddot{x} + x = 0$

Central difference solution: $x_{i+1} = (\Delta t)^2 \left\{ \left(\frac{2}{(\Delta t)^2} - 1 \right) x_i - \frac{1}{(\Delta t)^2} x_{i-1} \right\}$

$$= (2 - \Delta t^2) x_i - x_{i-1} \quad \text{--- (E}_1\text{)}$$

For $x_0 = 0$ and $\dot{x}_0 = 1$, Eq. (11.9) gives $x_{-1} = -\Delta t$

$$(i) \text{ With } \Delta t = 1, \quad x_{i+1} = x_i - x_{i-1} \quad \text{--- (E}_2\text{)}$$

Repetitive application of (E₂), with $x_{-1} = -1$ and $x_0 = 0$, gives the results shown in the following table.

$$(ii) \text{ With } \Delta t = 0.5, \quad x_{i+1} = 1.75 x_i - x_{i-1} \quad \text{--- (E}_3\text{)}$$

The results, for $x_{-1} = -0.5$ and $x_0 = 0$, are shown in the table.

comparison of solutions:

Time (t)	value of $x(t)$ obtained with		Exact value of $x(t)$ $x(t) = \sin t$
	$\Delta t = 1$	$\Delta t = 0.5$	
0	0	0	0
0.5	-	0.5	0.4794
1	1	0.8750	0.8415
1.5	-	1.0313	0.9975
2	1	0.9297	0.9093
2.5	-	0.5957	0.5985
3	0	0.1128	0.1411
3.5	-	-0.3983	-0.3508
4	-1	-0.8098	-0.7568
4.5	-	-1.0189	-0.9775
5	-1	-0.9733	-0.9589
5.5	-	-0.6843	-0.7055
6	0	-0.2242	-0.2794

11.5

At t_i , central difference formula gives

$$- \left(\frac{x_i - 2x_{i-1} + x_{i-2}}{(\Delta t)^2} \right) + 0.1 x_i = 0$$

$$\text{i.e. } x_i = \frac{2x_{i-1} - x_{i-2}}{1 - 0.1(\Delta t)^2} = 1.1111 (2x_{i-1} - x_{i-2}) \quad \text{for } \Delta t = 1 \quad \text{--- (E}_1\text{)}$$

$$\text{Since } \dot{x}_0 = \frac{x_0 - x_{-1}}{\Delta t} = 0, \quad x_{-1} = x_0 = 1$$

Eq. (E₁) gives the following results.

i	1	2	3	4	5	6	7	8	9	10
x_{i-1}	1	1.1111	1.3580	1.7832	2.2084	2.6336	3.0588	3.4840	3.9092	4.3344
x_{i-2}	1	1	1.1111	1.3580	1.7832	2.2084	2.6336	3.0588	3.4840	3.9092
x_i from (E_1)	1.1111	1.3580	1.7832	2.2084	2.6336	3.0588	3.4840	3.9092	4.3344	4.7596

11.6

For given data, Eq. (11.7) gives $x_{i+1} = 0.2(7x_i - 3x_{i-1}) \dots (E_1)$
 Eqs. (11.8) and (11.9) give $\ddot{x}_0 = -1$, $x_{-1} = -0.625$

$$\left(\frac{c}{2m}\right)^2 = \left(\frac{1}{2}\right)^2 < \left(\frac{k}{m} = 1\right) \Rightarrow \text{Underdamped case.}$$

Since $\tau_n = \frac{2\pi}{\omega_n} = 2\pi$ sec, we will consider the response for 15 steps.

$i+1$	x_i	x_{i-1}	x_{i+1} from (E_1)
1	0	-0.625	0.375
2	0.375	0	0.525
3	0.525000E+00	0.375000E+00	0.510000E+00
4	0.510000E+00	0.525000E+00	0.399000E+00
5	0.399000E+00	0.510000E+00	0.252600E+00
6	0.252600E+00	0.399000E+00	0.114240E+00
7	0.114240E+00	0.252600E+00	0.837604E-02
8	0.837604E-02	0.114240E+00	-0.568176E-01
9	-0.568176E-01	0.837604E-02	-0.845702E-01
10	-0.845702E-01	-0.568176E-01	-0.843078E-01
11	-0.843078E-01	-0.845702E-01	-0.672887E-01
12	-0.672887E-01	-0.843078E-01	-0.436196E-01
13	-0.436196E-01	-0.672887E-01	-0.206942E-01
14	-0.206942E-01	-0.436196E-01	-0.280008E-02
15	-0.280008E-02	-0.206942E-01	0.849639E-02

11.7

$$\left(\frac{c}{2m}\right)^2 = 1 = \left(\frac{k}{m} = 1\right) \Rightarrow \text{critically damped case.}$$

For the data, Eqs. (11.7) to (11.9) give

$$x_{i+1} = \frac{1}{6}(7x_i - 2x_{i-1}) \dots (E_1)$$

$\ddot{x}_0 = -2$, $x_{-1} = -0.75$. Response is given below.

$i+1$	x_i	x_{i-1}	x_{i+1} from (E_1)
1	0	-0.75	0.25
2	0.25	0	0.2917
3	0.291667E+00	0.250000E+00	0.256944E+00
4	0.256944E+00	0.291667E+00	0.202546E+00
5	0.202546E+00	0.256944E+00	0.150656E+00
6	0.150656E+00	0.202546E+00	0.108250E+00
7	0.108250E+00	0.150656E+00	0.760728E-01
8	0.760728E-01	0.108250E+00	0.526683E-01
9	0.526683E-01	0.760728E-01	0.360888E-01

10	0.360888E-01	0.526683E-01	0.245475E-01
11	0.245475E-01	0.360888E-01	0.166091E-01
12	0.166091E-01	0.245475E-01	0.111948E-01
13	0.111948E-01	0.166091E-01	0.752425E-02
14	0.752425E-02	0.111948E-01	0.504668E-02
15	0.504668E-02	0.752425E-02	0.337971E-02

11.8

For given data, Eq. (11.7) gives $x_{i+1} = 0.875 x_i$

This equation does not involve x_{i-1} and hence the response can't be found since $x_0 = 0$. Hence we change Δt to 0.4. This gives, from Eq. (11.7), $x_{i+1} = 0.0889 (11.5 x_i - 1.25 x_{i-1})$ --- (E₁)

Eqs. (11.8) and (11.9) give $\ddot{x}_0 = -4$, $x_{-1} = -0.72$

Results:

$i+1$	x_{i-1}	x_i	x_{i+1} from (E ₁)
1	-0.720000E+00	0.000000E+00	0.800010E-01
2	0.000000E+00	0.800010E-01	0.817798E-01
3	0.800010E-01	0.817798E-01	0.747091E-01
4	0.817798E-01	0.747091E-01	0.672835E-01
5	0.747091E-01	0.672835E-01	0.604784E-01
6	0.672835E-01	0.604784E-01	0.543471E-01
7	0.604784E-01	0.543471E-01	0.488356E-01
8	0.543471E-01	0.488356E-01	0.438829E-01
9	0.488356E-01	0.438829E-01	0.394323E-01
10	0.438829E-01	0.394323E-01	0.354332E-01
11	0.394323E-01	0.354332E-01	0.318396E-01
12	0.354332E-01	0.318396E-01	0.286105E-01
13	0.318396E-01	0.286105E-01	0.257089E-01

11.9

$$\omega_n = \sqrt{\frac{3000}{4}} = 27.3861, \quad \tau_n = 2\pi/\omega_n = 0.22943; \quad \Delta t = 0.05$$

Eqs. (11.7) to (11.9) give

$$x_{i+1} = \frac{1}{1620} (200 x_i - 1580 x_{i-1} + F_i) \quad \text{where } F(t) = \begin{cases} 200 & ; 0 \leq t \leq 0.2 \\ -500t + 300 & ; 0.2 \leq t \leq 0.6 \end{cases}$$

$$\ddot{x}_0 = 50, \quad x_{-1} = 0.0625 \quad \text{--- (E}_1\text{)}$$

Results:

$i+1$	t_i	$F_i = F(t_i)$	x_i	x_{i-1}	x_{i+1} from (E ₁)
1	0	0.200000E+03	0.000000E+00	0.625000E-01	0.625000E-01
2	0.05	0.200000E+03	0.625000E-01	0.000000E+00	0.131173E+00
3	0.1	0.200000E+03	0.131173E+00	0.625000E-01	0.786942E-01
4	0.15	0.200000E+03	0.786942E-01	0.131173E+00	0.523812E-02
5	0.2	0.200000E+03	0.523812E-02	0.786942E-01	0.473524E-01
6	0.25	0.175000E+03	0.473524E-01	0.523812E-02	0.108762E+00
7	0.3	0.150000E+03	0.108762E+00	0.473524E-01	0.598368E-01
8	0.35	0.125000E+03	0.598368E-01	0.108762E+00	-0.215287E-01
9	0.4	0.100000E+03	-0.215287E-01	0.598368E-01	0.711154E-03
10	0.45	0.750000E+02	0.711154E-03	-0.215287E-01	0.673812E-01
11	0.5	0.500000E+02	0.673812E-01	0.711154E-03	0.384892E-01
12	0.55	0.250000E+02	0.384892E-01	0.673812E-01	-0.455336E-01
13	0.6	-0.305176E-04	-0.455336E-01	0.384892E-01	-0.431603E-01

14	0.65	-0.250001E+02	-0.431603E-01	-0.455336E-01	0.236487E-01
15	0.7	-0.500001E+02	0.236487E-01	-0.431603E-01	0.141500E-01
16	0.75	-0.750001E+02	0.141500E-01	0.236487E-01	-0.676142E-01
17	0.8	-0.100000E+03	-0.676142E-01	0.141500E-01	-0.838765E-01
18	0.85	-0.125000E+03	-0.838765E-01	-0.676142E-01	-0.215709E-01
19	0.9	-0.150000E+03	-0.215709E-01	-0.838765E-01	-0.134502E-01
20	0.95	-0.175000E+03	-0.134502E-01	-0.215709E-01	-0.886470E-01
21	1.00	-0.200000E+03	-0.886470E-01	-0.134502E-01	-0.121283E+00

11.10

Equation: $m\ddot{x} + c\dot{x} + kx = F(t) = t$

Since $\tau_n = 2\pi$, Δt is selected as 0.5.

Eqs. (11.7) to (11.9) give $x_{i+1} = 0.2(7x_i - 3x_{i-1} + t_i) \quad \dots (E_1)$

$$\ddot{x}_0 = 0, \quad x_{-1} = 0$$

Exact solution (from Example 4.6):

For $\delta F = 1$, $k = 1$, $\omega_n = 1$, $\gamma = 0.5$, $\omega_d = 0.86603$, we find

$$x(t) = t - 1 + e^{-0.5t} (\cos 0.86603 t - 0.57735 \sin 0.86603 t) \quad \dots (E_2)$$

Results:

$i+1$	t_i	x_i	x_{i-1}	$x_{i+1}, (E_1)$	$x_{i+1}, (E_2)$
1	0	0.000000E+00	0.000000E+00	0.000000E+00	0.182477E-01
2	0.5	0.000000E+00	0.000000E+00	0.100000E+00	0.126190E+00
3	1	0.100000E+00	0.000000E+00	0.340000E+00	0.364080E+00
4	1.5	0.340000E+00	0.100000E+00	0.716000E+00	0.731292E+00
5	2	0.716000E+00	0.340000E+00	0.119840E+01	0.120253E+01
6	2.5	0.119840E+01	0.716000E+00	0.174816E+01	0.174240E+01
7	3	0.174816E+01	0.119840E+01	0.232838E+01	0.231622E+01
8	3.5	0.232838E+01	0.174816E+01	0.291084E+01	0.289641E+01
9	4	0.291084E+01	0.232838E+01	0.347815E+01	0.346501E+01
10	4.5	0.347815E+01	0.291084E+01	0.402290E+01	0.401335E+01
11	5	0.402290E+01	0.347815E+01	0.454518E+01	0.454011E+01
12	5.5	0.454518E+01	0.402290E+01	0.504950E+01	0.504860E+01
13	6	0.504950E+01	0.454518E+01	0.554220E+01	0.554439E+01
14	6.5	0.554220E+01	0.504950E+01	0.602938E+01	0.603328E+01
15	7	0.602938E+01	0.554220E+01	0.651581E+01	0.652013E+01
16	7.5	0.651581E+01	0.602938E+01	0.700451E+01	0.700828E+01
17	8	0.700451E+01	0.651581E+01	0.749682E+01	0.749949E+01
18	8.5	0.749682E+01	0.700451E+01	0.799285E+01	0.799426E+01
19	9	0.799285E+01	0.749682E+01	0.849190E+01	0.849219E+01
20	9.5	0.849190E+01	0.799285E+01	0.899294E+01	0.899244E+01

11.11

Let $x_1 = x$, $x_2 = \frac{dx_1}{dt} = \frac{dx}{dt}$, $x_3 = \frac{dx_2}{dt} = \frac{d^2x}{dt^2}$, ..., $x_n = \frac{dx_{n-1}}{dt} = \frac{d^{n-1}x}{dt^{n-1}}$

Equation: $\frac{d^n x}{dt^n} = \frac{g(x, t)}{a_n} - \frac{a_{n-1}}{a_n} \frac{d^{n-1}x}{dt^{n-1}} - \dots - \frac{a_1}{a_n} \frac{dx}{dt} \quad \dots (E_1)$

$$= \frac{g(x, t)}{a_n} - \frac{a_{n-1}}{a_n} x_n - \frac{a_{n-2}}{a_n} x_{n-1} - \dots - \frac{a_1}{a_n} x_2 \quad \dots (E_2)$$

(E_1) and (E_2) can be expressed as

$$\frac{d\vec{X}}{dt} = \vec{F}(\vec{X}, t)$$

where $\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$ and $\vec{F} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix} = \begin{Bmatrix} x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \\ \frac{g(x,t)}{a_n} - \frac{a_{n-1}}{a_n} x_n - \dots - \frac{a_1}{a_n} x_2 \end{Bmatrix}$

11.12

The main program for problems (a) and (b) is given below.

```

=====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C =====
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION TIME(40),X(40,1),XX(1),F(1),YI(1),YJ(1),YK(1),YL(1),
2  UU(1)
  XX(1)=1.0
  NEQ=1
  NSTEP=40
  DT=0.1
  T=0.0
  WRITE (58,10)
10  FORMAT (//,3X,5H I ,10H TIME(1),7X,5H X(1),/)
  DO 40 I=1,NSTEP
    CALL RK4 (T,DT,NEQ,XX,F,YI,YJ,YK,YL,UU)
    TIME(I)=T
    DO 20 J=1,NEQ
20  X(I,J)=XX(J)
  WRITE (58,30) I,TIME(I),(X(I,J),J=1,NEQ)
30  FORMAT (2X,15,F10.4,2X,E15.8,2X,E15.8)
40  CONTINUE
  STOP
  END

```

The subroutine FUN and output are given below.

For problem (a)			For problem (b)		
SUBROUTINE FUN (X,F,N,T)			SUBROUTINE FUN (X,F,N,T)		
DIMENSION X(N),F(N)			DIMENSION X(N),F(N)		
F(1)=X(1)-1.5*EXP(-0.5*T)			F(1)=-T*(X(1)**2)		
RETURN			RETURN		
END			END		
I	TIME(I)	X(1)	I	TIME(I)	X(1)
1	0.1000	0.95122939E+00	1	0.1000	0.99502486E+00
2	0.2000	0.90483737E+00	2	0.2000	0.98039210E+00
3	0.3000	0.86070788E+00	3	0.3000	0.95693767E+00
4	0.4000	0.81873059E+00	4	0.4000	0.92592573E+00
5	0.5000	0.77880061E+00	5	0.5000	0.88888866E+00
6	0.6000	0.74081802E+00	6	0.6000	0.84745741E+00
7	0.7000	0.70468783E+00	7	0.7000	0.80321264E+00
8	0.8000	0.67031974E+00	8	0.8000	0.75757557E+00

:	:	:	:		
31	3.1000	0.21224274E+00	31	3.1000	0.17226547E+00
32	3.2000	0.20189069E+00	32	3.2000	0.16339886E+00
33	3.3000	0.19204344E+00	33	3.3000	0.15515921E+00
34	3.4000	0.18267635E+00	34	3.4000	0.14749278E+00
35	3.5000	0.17376597E+00	35	3.5000	0.14035103E+00
36	3.6000	0.16529004E+00	36	3.6000	0.13368998E+00
37	3.7000	0.15722735E+00	37	3.7000	0.12746987E+00
38	3.8000	0.14955774E+00	38	3.8000	0.12165464E+00
39	3.9000	0.14226201E+00	39	3.9000	0.11621163E+00
40	4.0000	0.13532193E+00	40	4.0000	0.11111123E+00

11.13

The program and output are given below.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK2
C
C =====
C THE FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),UU(2)
  XX(1)=0.0
  XX(2)=0.0
  NEQ=2
  NSTEP=40
  DT=0.31416/2
  I=0.0
  WRITE (5/,10)
10  FORMAT (//,3X,5H I ,10H TIME(I),7X,5H X(1),12X,5H X(2),/)
  DO 40 I=1,NSTEP
    CALL RK2 (T,DT,NEQ,XX,F,YI,YJ,UU)
    TIME(I)=T
    DO 20 J=1,NEQ
20   X(I,J)=XX(J)
  WRITE (57,30) I,TIME(I),(X(I,J),J=1,NEQ)
30  FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40  CONTINUE
  STOP
  END
C =====
C SUBROUTINE RK2
C =====
  SUBROUTINE RK2 (T,DT,N,XX,F,XI,XJ,UU)
  DIMENSION XI(N),XJ(N),UU(N),XX(N),F(N)
  DO 10 I=1,N
10   UU(I)=XX(I)
  CALL FUN (XX,F,N,T)
  DO 20 I=1,N
  XI(1)=F(I)*DT
20   XX(1)=UU(I)+XI(I)
  T=T+DT
  CALL FUN (XX,F,N,T)
  DO 30 I=1,N
  XJ(1)=F(I)*DT
30   XX(I)=UU(I)+(XI(I)+XJ(I))/2.0
  RETURN
  END

```



```

C =====
C SUBROUTINE FUN FOR USE IN THE SUBROUTINE RK2
C THIS SUBROUTINE CHANGES FROM PROBLEM TO PROBLEM
C =====
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F0=1.0
      T0=3.1416
      XM=1.0
      XC=0.2
      XK=1.0
      FT=F0*(1.0-SIN(3.1416*T/(2.0*T0)))
      F(2)=(F1-XC*X(2)-XK*X(1))
      RETURN
      END

```

I	TIME(I)	X(1)	X(2)
1	0.1571	0.12337063E-01	0.14845039E+00
2	0.3142	0.46506263E-01	0.27647862E+00
3	0.4712	0.99086702E-01	0.38170516E+00
4	0.6283	0.16633771E+00	0.46245623E+00
5	0.7854	0.24431184E+00	0.51778144E+00
6	0.9425	0.32896915E+00	0.54745305E+00
7	1.0996	0.41628993E+00	0.55194682E+00
8	1.2566	0.50238299E+00	0.53240609E+00
9	1.4137	0.58358723E+00	0.49059010E+00
10	1.5708	0.65656364E+00	0.42880845E+00
:			
36	5.6549	-0.28463572E+00	0.51887971E+00
37	5.8120	-0.19237404E+00	0.65321267E+00
38	5.9690	-0.79548687E-01	0.77926701E+00
39	6.1261	0.52324414E-01	0.89440674E+00
40	6.2832	0.20133464E+00	0.99627036E+00

11.14 The computer program and output are given below.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK3
C
C =====
C THE FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),YK(2),UU(2)
      XX(1)=0.0
      XX(2)=0.0
      NEQ=2
      NSTEP=40
      DT=0.31416/2
      T=0.0
      WRITE (57,10)
10  FORMAT (//,3X,5H I ,10H TIME(1),7X,5H X(1),12X,5H X(2),/)
      DO 40 I=1,NSTEP
      CALL RK3 (I,DT,NEQ,XX,F,YI,YJ,YK,UU)
      TIME(I)=T

```



```

DO 20 J=1,NEQ
20  X(I,J)=XX(J)
WRITE (57,30) I,TIME(I),(X(I,J),J=1,NEQ)
30  FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40  CONTINUE
STOP
END

```

```

C =====
C SUBROUTINE RK3

```

```

C =====
SUBROUTINE RK3 (T,DT,N,XX,F,XI,XJ,XK,UU)
DIMENSION XI(N),XJ(N),XK(N),UU(N),XX(N),F(N)
DO 10 I=1,N
10  UU(I)=XX(I)
CALL FUN (XX,F,N,T)
DO 20 I=1,N
20  XI(I)=F(I)*DT
XX(I)=UU(I)+XI(I)/2.0
T=T+DT/2.0
CALL FUN (XX,F,N,T)
DO 30 I=1,N
30  XJ(I)=F(I)*DT
XX(I)=UU(I)-XI(I)+2.0*XJ(I)
CALL FUN (XX,F,N,T)
DO 40 I=1,N
40  XK(I)=F(I)*DT
XX(I)=UU(I)+(XI(I)+4.0*XJ(I)+XK(I))/6.0
RETURN
END

```

```

C =====
C SUBROUTINE FUN FOR USE IN THE SUBROUTINE RK3
C THIS SUBROUTINE CHANGES FROM PROBLEM TO PROBLEM
C =====

```

```

SUBROUTINE FUN (X,F,N,T)
DIMENSION X(N),F(N)
F(1)=X(2)
F0=1.0
T0=3.1416
XM=1.0
XC=0.2
XK=1.0
FT=F0*(1.0-SIN(3.1416*T/(2.0*T0)))
F(2)=(FT-XC*X(2)-XK*X(1))
RETURN
END

```

I	TIME(I)	X(1)	X(2)
1	0.0785	0.11884968E-01	0.14891791E+00
2	0.1571	0.46078302E-01	0.28356332E+00
3	0.2356	0.10011104E+00	0.40113181E+00
4	0.3142	0.17111324E+00	0.49932912E+00
5	0.3927	0.25589743E+00	0.57641083E+00
6	0.4712	0.35104716E+00	0.63120759E+00
7	0.5498	0.45300832E+00	0.66313553E+00
8	0.6283	0.55818105E+00	0.67219222E+00
9	0.7069	0.66300964E+00	0.65893871E+00
10	0.7854	0.76406908E+00	0.62446821E+00

36	2.8274	-0.56948423E+00	-0.28658101E+00
37	2.9060	-0.60655731E+00	-0.18535951E+00
38	2.9845	-0.62764353E+00	-0.83412744E-01
39	3.0631	-0.63280767E+00	0.17002784E-01
40	3.1416	-0.62246108E+00	0.11373823E+00

11.15

The main program, subroutine FUN and results are given.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK2
C
C =====
C THE FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),UU(2)
  XX(1)=5.0
  XX(2)=0.0
  NEQ=2
  NSTEP=40
  DT=0.01
  T=0.0
  WRITE (61,10)
10  FORMAT (//,3X,5H I ,10H TIME(1),7X,5H X(1),12X,5H X(2),/)
  DO 40 I=1,NSTEP
  CALL RK2 (T,DT,NEQ,XX,F,YI,YJ,UU)
  TIME(I)=T
  DO 20 J=1,NEQ
20  X(I,J)=XX(J)
  WRITE (61,30) I,TIME(I),(X(I,J),J=1,NEQ)
30  FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40  CONTINUE
  STOP
  END

  SUBROUTINE FUN (X,F,N,T)
  DIMENSION X(N),F(N)
  F(1)=X(2)
  F(2)=-1000.0*X(1)
  RETURN
  END

```

I	TIME(I)	X(1)	X(2)
1	0.0100	0.47500000E+01	-0.50000000E+02
2	0.0200	0.40124998E+01	-0.95000000E+02
3	0.0300	0.28618748E+01	-0.13037500E+03
4	0.0400	0.14150311E+01	-0.15247501E+03
5	0.0500	-0.18047059E+00	-0.15900157E+03
6	0.0600	-0.17614628E+01	-0.14924678E+03
7	0.0700	-0.31658576E+01	-0.12416982E+03
8	0.0800	-0.42492628E+01	-0.86302750E+02
9	0.0900	-0.48998270E+01	-0.39494984E+02
10	0.1000	-0.50497856E+01	0.11478039E+02
:			
:			

36	0.3600	0.28330812E+01	0.13901421E+03
37	0.3700	0.40815692E+01	0.10373269E+03
38	0.3800	0.49148178E+01	0.57730366E+02
39	0.3900	0.52463808E+01	0.56956673E+01
40	0.4000	0.50410185E+01	-0.47052921E+02

11.16

The main program, subroutine FUN and results are given.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK3
C
C =====
C THE FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),YK(2),UU(2)
  XX(1)=5.0
  XX(2)=0.0
  NEQ=2
  NSTEP=40
  DT=0.01
  T=0.0
  WRITE (62,10)
10  FORMAT (//,3X,5H I ,10H  TIME(I),7X,5H X(1),12X,5H X(2),/)
  DO 40 I=1,NSTEP
    CALL RK3 (T,DT,NEQ,XX,F,YI,YJ,YK,UU)
    TIME(I)=T
    DO 20 J=1,NEQ
20   X(I,J)=XX(J)
    *WRITE (62,30) I,TIME(I),(X(I,J),J=1,NEQ)
30   FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40   CONTINUE
  STOP
  END

  SUBROUTINE FUN (X,F,N,T)
  DIMENSION X(N),F(N)
  F(1)=X(2)
  F(2)=-1000.0*X(1)
  RETURN
  END

```

I	TIME(I)	X(1)	X(2)
1	0.0050	0.47500000E+01	-0.49166668E+02
2	0.0100	0.40290279E+01	-0.93416672E+02
3	0.0150	0.29089794E+01	-0.12836461E+03
4	0.0200	0.15012785E+01	-0.15055135E+03
5	0.0250	-0.54207087E-01	-0.15778635E+03
6	0.0300	-0.16030624E+01	-0.14936400E+03
7	0.0350	-0.29916553E+01	-0.12613235E+03
8	0.0400	-0.40823741E+01	-0.90407799E+02
9	0.0450	-0.47672653E+01	-0.45744061E+02
10	0.0500	-0.49787188E+01	0.34212494E+01
:			
36	0.1800	0.18843936E+01	0.14399376E+03
37	0.1850	0.32061126E+01	0.11826420E+03
38	0.1900	0.42087383E+01	0.80824219E+02
39	0.1950	0.47930727E+01	0.35397079E+02
40	0.2000	0.49014902E+01	-0.13504654E+02

11.17 The main program, subroutine FUN and output are given.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C =====
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),YK(2),YL(2),
2  UU(2)
  XX(1)=5.0
  XX(2)=0.0
  NEQ=2
  NSTEP=40
  DT=0.01
  T=0.0
  WRITE (59,10)
10  FORMAT (//,3X,5H I ,10H TIME(I),7X,5H X(1),12X,5H X(2),/)
  DO 40 I=1,NSTEP
  CALL RK4 (T,DT,NEQ,XX,F,YI,YJ,YK,YL,UU)
  TIME(I)=T
  DO 20 J=1,NEQ
20  X(I,J)=XX(J)
  WRITE (59,30) I,TIME(I),(X(I,J),J=1,NEQ)
30  FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40  CONTINUE
  STOP
  END
  SUBROUTINE FUN (X,F,N,T)
  DIMENSION X(N),F(N)
  F(1)=X(2)
  F(2)=-1000.0*X(1)
  RETURN
  END

```

I	TIME(I)	X(1)	X(2)
1	0.0100	0.47520833E+01	-0.49166668E+02
2	0.0200	0.40329871E+01	-0.93457642E+02
3	0.0300	0.29140182E+01	-0.12848141E+03
4	0.0400	0.15061308E+01	-0.15076540E+03
5	0.0500	-0.51074505E-01	-0.15810023E+03
6	0.0600	-0.16031944E+01	-0.14975887E+03
7	0.0700	-0.29963315E+01	-0.12656857E+03
8	0.0800	-0.40923543E+01	-0.90828957E+02
9	0.0900	-0.47825933E+01	-0.46083870E+02
10	0.1000	-0.49986143E+01	0.32299538E+01
:			
36	0.3600	0.18898817E+01	0.14634213E+03
37	0.3700	0.32352061E+01	0.12050217E+03
38	0.3800	0.42597318E+01	0.82714409E+02
39	0.3900	0.48618784E+01	0.36725792E+02
40	0.4000	0.49819474E+01	-0.12903667E+02

11.18 The main program, subroutine EXTFUN and results are given.

```

C =====
C
C PROGRAM
C MAIN PROGRAM WHICH CALLS CDIFF
C
C =====
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),XM1(2),F(2),R(2),RR(2),
2     XMK(2),XMI(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
3     S(2),X(25,2),XD(25,2),XDD(25,2)
      DATA N,NSTEP,NSTEP1,DELT/2,24,25,0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/2.0,-2.0,-2.0,2.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL CDIFF (M,C,K,XI,XDI,XDDI,N,NSTEP,DELT,F,R,RR,XM1,XM2,XP1,
2     MC,MK,MCI,XMK,MMC,XMI,ZA,ZB,ZC,LA,LB,S,X,XD,XDD,NSTEP1)
      WRITE (53,10)
10     FORMAT (//,38H SOLUTION BY CENTRAL DIFFERENCE METHOD,/)
      WRITE (53,20) N,NSTEP,DELT
20     FORMAT (12H GIVEN DATA: ,/,3H N=,I5,4X,7H NSTEP=,I5,4X,6H DELT=,
2     E15.8,/)
      WRITE (53,30)
30     FORMAT (10H SOLUTION: ,//,5H STEP,3X,5H TIME,3X,7H X(I,1),3X,
2     8H XD(I,1),2X,9H XDD(I,1),4X,7H X(I,2),3X,8H XD(I,2),2X,
3     9H XDD(I,2),/)
      DO 40 I=1,NSTEP1
      TIME=REAL(I-1)*DELT
40     WRITE (53,50) I,TIME,X(I,1),XD(I,1),XDD(I,1),X(I,2),XD(I,2),
2     XDD(I,2)
50     FORMAT (1X,I4,F8.4,6(1X,E10.4))
      STOP
      END
C =====
C
C SUBROUTINE EXTFUN
C THIS SUBROUTINE IS PROBLEM-DEPENDENT
C
C =====
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=10.0
      RETURN
      END
SOLUTION BY CENTRAL DIFFERENCE METHOD

GIVEN DATA:
N=      2      NSTEP=      24      DELT= 0.24216267E+00

```

XD(I,2) XDD(I,2)

0.0000E+00 0.0000E+00
 0.0000E+00 0.0000E+00
 0.6645E-01 0.5488E+00
 0.2737E+00 0.1162E+01
 0.5084E+00 0.7765E+00
 0.5035E+00 -0.8173E+00
 0.1110E+00 -0.2424E+01
 0.4982E+00 -0.2608E+01
 0.9609E+00 -0.1214E+01
 0.1039E+01 0.5660E+00
 0.7768E+00 0.1602E+01
 0.3511E+00 0.1914E+01
 0.1424E+00 0.2162E+01
 0.6921E+00 0.2378E+01
 0.1192E+01 0.1749E+01
 0.1380E+01 -0.1991E+00
 0.1043E+01 -0.2579E+01
 0.2764E+00 -0.3753E+01
 0.5385E+00 -0.2976E+01
 0.1033E+01 -0.1112E+01
 0.1118E+01 0.4146E+00
 0.9338E+00 0.1106E+01
 0.6152E+00 0.1525E+01
 0.1772E+00 0.2092E+01
 0.3504E+00 0.2265E+01

ogram (which
 iven below.

2,2)

,R(2),RR(2),
 (2),LB(2,2),

XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.5000E+01
0.0000E+00	0.0000E+00	0.1466E+00	0.3077E-07	0.5000E+01
0.2317E+00	0.1913E+01	0.5045E+00	0.1042E+01	0.3604E+01
0.7436E+00	0.2314E+01	0.9859E+00	0.1733E+01	0.2105E+01
0.1207E+01	0.1510E+01	0.1500E+01	0.2056E+01	0.5669E+00
0.1396E+01	0.5531E-01	0.1961E+01	0.2013E+01	-0.9224E+00
0.1224E+01	-0.1478E+01	0.2292E+01	0.1633E+01	-0.2217E+01
0.7260E+00	-0.2635E+01	0.2438E+01	0.9851E+00	-0.3136E+01
0.2610E-01	-0.3145E+01	0.2378E+01	0.1795E+00	-0.3518E+01
-0.7088E+00	-0.2925E+01	0.2127E+01	-0.6433E+00	-0.3277E+01
-0.1312E+01	-0.2061E+01	0.1732E+01	-0.1335E+01	-0.2439E+01
-0.1655E+01	-0.7648E+00	0.1270E+01	-0.1769E+01	-0.1146E+01
-0.1665E+01	0.6808E+00	0.8289E+00	-0.1864E+01	0.3640E+00
-0.1343E+01	0.1979E+01	0.4940E+00	-0.1602E+01	0.1803E+01
-0.7550E+00	0.2874E+01	0.3288E+00	-0.1033E+01	0.2895E+01
-0.2001E-01	0.3196E+01	0.3645E+00	-0.2673E+00	0.3427E+01
0.7167E+00	0.2889E+01	0.5935E+00	0.5466E+00	0.3295E+01
0.1311E+01	0.2019E+01	0.9707E+00	0.1252E+01	0.2527E+01
0.1647E+01	0.7600E+00	0.1422E+01	0.1712E+01	0.1271E+01
0.1662E+01	-0.6399E+00	0.1861E+01	0.1838E+01	-0.2264E+00
0.1353E+01	-0.1908E+01	0.2201E+01	0.1608E+01	-0.1675E+01
0.7832E+00	-0.2800E+01	0.2378E+01	0.1067E+01	-0.2793E+01
0.6342E-01	-0.3145E+01	0.2357E+01	0.3211E+00	-0.3366E+01
-0.6657E+00	-0.2877E+01	0.2143E+01	-0.4839E+00	-0.3283E+01
-0.1263E+01	-0.2053E+01	0.1779E+01	-0.1192E+01	-0.2564E+01

pendent data to be used in the main program (which
 , subroutine EXTFUN and output are given below.

S CONTAIN PROBLEM-DEPENDENT DATA

,K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
 (2,2),XI(2),XDI(2),XDDI(2),XM1(2),F(2),R(2),HR(2),
 2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
),XD(25,2),XDD(25,2)

P, NSTEP1, DELT/2, 24, 25, 0.24216267/

,0.0/

0,0.0/

0.0,0.0,2.0/

0.0,0.0,0.0/

-2.0,-2.0,8.0/

PENDENT DATA

TFUN (F, TIME, N)

)

(5.0*TIME)

DIFFERENCE METHOD

24 DELT= 0.24216267E+00

XD(I,2) XDD(I,2)

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2422	0.5488E+00	0.1133E+01	0.9359E+01	0.0000E+00	0.0000E+00	0.0000E+00
3	0.4843	0.1291E+01	0.2666E+01	0.3301E+01	0.3219E-01	0.6645E-01	0.5488E+00
4	0.7265	0.1307E+01	0.1505E+01	-1.240E+02	0.1325E+00	0.2737E+00	0.1162E+01
5	0.9687	0.2964E+00	-.2054E+01	-.1749E+02	0.2784E+00	0.5084E+00	0.7765E+00
6	1.2108	-.9186E+00	-.4595E+01	-.3493E+01	0.3764E+00	0.5035E+00	-.8173E+00
7	1.4530	-.1279E+01	-.3252E+01	0.1458E+02	0.3322E+00	0.1110E+00	-.2424E+01
8	1.6951	-.6732E+00	0.5066E+00	0.1647E+02	0.1351E+00	-.4982E+00	-.2608E+01
9	1.9373	0.3324E-01	0.2709E+01	0.1722E+01	-.1332E+00	-.9609E+00	-.1214E+01
10	2.1795	0.1288E+00	0.1656E+01	-.1042E+02	-.3683E+00	-.1039E+01	0.5660E+00
11	2.4216	-.1236E+00	-.3238E+00	-.5933E+01	-.5094E+00	-.7768E+00	0.1602E+01
12	2.6638	0.8610E-02	-.2481E+00	0.6558E+01	-.5383E+00	-.3511E+00	0.1914E+01
13	2.9060	0.6165E+00	0.1528E+01	0.8111E+01	-.4404E+00	0.1424E+00	0.2162E+01
14	3.1481	0.9366E+00	0.1916E+01	-.4906E+01	-.2031E+00	0.6921E+00	0.2378E+01
15	3.3903	0.3481E+00	-.5540E+00	-.1549E+02	0.1368E+00	0.1192E+01	0.1749E+01
16	3.6324	-.7189E+00	-.3418E+01	-.8160E+01	0.4650E+00	0.1380E+01	-.1991E+00
17	3.8746	-.1185E+01	-.3166E+01	0.1024E+02	0.6420E+00	0.1043E+01	-.2579E+01
18	4.1168	-.5807E+00	0.2853E+00	0.1826E+02	0.5989E+00	0.2764E+00	-.3753E+01
19	4.3589	0.4129E+00	0.3300E+01	0.6635E+01	0.3812E+00	-.5385E+00	-.2976E+01
20	4.6011	0.8081E+00	0.2867E+01	-.1021E+02	0.9837E-01	-.1033E+01	-.1112E+01
21	4.8433	0.4651E+00	0.1077E+00	-.1259E+02	-.1602E+00	-.1118E+01	0.4146E+00
22	5.0854	0.1098E+00	-.1442E+01	-.2098E+00	-.3539E+00	-.9338E+00	0.1106E+01
23	5.3276	0.2596E+00	-.4244E+00	0.8612E+01	-.4582E+00	-.6152E+00	0.1525E+01
24	5.5697	0.5064E+00	0.8188E+00	0.1655E+01	-.4397E+00	-.1772E+00	0.2092E+01
25	5.8119	0.1089E+00	-.3112E+00	-.1099E+02	-.2885E+00	0.3504E+00	0.2265E+01

11.20 The problem-dependent data to be used in the main program (which calls CDIFF), subroutine EXTFUN and output are given below.

C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA

```

REAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
DIMENSION C(2,2),XI(2),XDI(2),XDD(2),XM1(2),F(2),R(2),RR(2),
2  XMK(2),XM1(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
3  S(2),X(25,2),XD(25,2),XDD(25,2)
DATA N,NSTEP,NSTEP1,DELT/2,24,25,0.25/
DATA X1/0.0,0.0/
DATA XDI/0.0,0.0/
DATA M/2.0,0.0,0.0,1.0/
DATA C/0.0,0.0,0.0,0.0/
DATA K/6.0,-2.0,-2.0,4.0/

```

C END OF PROBLEM-DEPENDENT DATA

```

SUBROUTINE EXTFUN (F,TIME,N)
DIMENSION F(N)
F(1)=5.0
F(2)=20.0*SIN(5.0*TIME)
RETURN
END

```

SOLUTION BY CENTRAL DIFFERENCE METHOD

GIVEN DATA:

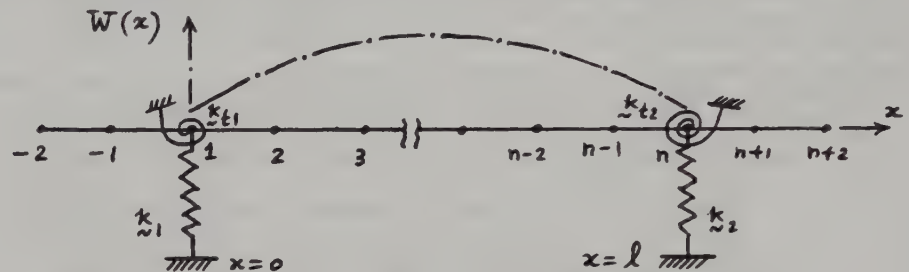
N= 2 NSTEP= 24 DELT= 0.25000000E+00

SOLUTION:

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
------	------	--------	---------	----------	--------	---------	----------

1	0.0000	0.0000E+00	0.0000E+00	0.2500E+01	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2500	0.7813E-01	0.0000E+00	0.2500E+01	0.1186E+01	0.2372E+01	0.1898E+02
3	0.5000	0.3720E+00	0.7440E+00	0.3452E+01	0.2834E+01	0.5668E+01	0.7381E+01
4	0.7500	0.9295E+00	0.1703E+01	0.4218E+01	0.3105E+01	0.3837E+01	-.2202E+02
5	1.0000	0.1663E+01	0.2582E+01	0.2816E+01	0.1517E+01	-.2633E+01	-.2974E+02
6	1.2500	0.2336E+01	0.2813E+01	-.9716E+00	-.2832E+00	-.6776E+01	-.3407E+01
7	1.5000	0.2709E+01	0.2092E+01	-.4790E+01	-.5484E+00	-.4131E+01	0.2456E+02
8	1.7500	0.2697E+01	0.7215E+00	-.6176E+01	0.4430E+00	0.1452E+01	0.2011E+02
9	2.0000	0.2362E+01	-.6938E+00	-.5147E+01	0.9807E+00	0.3058E+01	-.7259E+01
10	2.2500	0.1803E+01	-.1788E+01	-.3606E+01	0.3588E+00	-.1685E+00	-.1855E+02
11	2.5000	0.1084E+01	-.2557E+01	-.2549E+01	-.2105E+00	-.2382E+01	0.8438E+00
12	2.7500	0.3046E+00	-.2996E+01	-.9614E+00	0.5659E+00	0.4142E+00	0.2153E+02
13	3.0000	-.3400E+00	-.2847E+01	0.2152E+01	0.2052E+01	0.4524E+01	0.1135E+02
14	3.2500	-.6363E+00	-.1882E+01	0.5572E+01	0.2337E+01	0.3543E+01	-.1920E+02
15	3.5000	-.5110E+00	-.3421E+00	0.6746E+01	0.7394E+00	-.2625E+01	-.3013E+02
16	3.7500	-.6742E-01	0.1098E+01	0.4772E+01	-.1231E+01	-.7137E+01	-.5967E+01
17	4.0000	0.4318E+00	0.1886E+01	0.1531E+01	-.1764E+01	-.5007E+01	0.2301E+02
18	4.2500	0.9161E+00	0.2007E+01	-.5596E+00	-.9579E+00	0.5471E+00	0.2142E+02
19	4.5000	0.1325E+01	0.1786E+01	-.1206E+01	-.4066E+00	0.2715E+01	-.4080E+01
20	4.7500	0.1616E+01	0.1400E+01	-.1882E+01	-.8160E+00	0.2836E+00	-.1537E+02
21	5.0000	0.1710E+01	0.7696E+00	-.3165E+01	-.9849E+00	-.1157E+01	0.3850E+01
22	5.2500	0.1577E+01	-.7782E-01	-.3614E+01	0.4299E+00	0.2492E+01	0.2534E+02
23	5.5000	0.1332E+01	-.7549E+00	-.1802E+01	0.2808E+01	0.7586E+01	0.1542E+02
24	5.7500	0.1169E+01	-.8162E+00	0.1311E+01	0.4079E+01	0.7298E+01	-.1773E+02
25	6.0000	0.1198E+01	-.2684E+00	0.3071E+01	0.3241E+01	0.8649E+00	-.3374E+02

11.21



At $x=0$ (at node 1):

$$\frac{d}{dx} \left[EI \frac{d^2 W(x)}{dx^2} \right] = EI \frac{d^3 W}{dx^3} = -k_1 W(x)$$

$$EI \frac{d^2 W}{dx^2} = -k_{t1} \frac{dW(x)}{dx}$$

$$\text{i.e.} \quad \frac{EI}{2h^3} (W_3 - 2W_2 + 2W_{-1} - W_{-2}) = -k_{t1} W_1$$

$$\frac{EI}{h^2} (W_2 - 2W_1 + W_{-1}) = -k_{t1} \cdot \frac{1}{2h} (W_2 - W_{-1})$$

At $x=l$ (at node n):

$$\frac{d}{dx} \left[EI \frac{d^2 W(x)}{dx^2} \right] = EI \frac{d^3 W}{dx^3} = k_2 W(x)$$

$$EI \frac{d^2 W}{dx^2} = k_{t2} \frac{dW(x)}{dx}$$

$$\text{i.e.} \quad \frac{EI}{2h^3} (W_{n+2} - 2W_{n+1} + 2W_{n-1} - W_{n-2}) = k_{t2} W_n$$

$$\frac{EI}{h^2} (W_{n+1} - 2W_n + W_{n-1}) = k_{t2} \cdot \frac{1}{2h} (W_{n+1} - W_{n-1})$$

11.22

Given equations can be expressed as

$$\frac{d\vec{X}}{dt} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ \frac{1}{2}(5 - 6x_1 + 2x_3) \\ x_4 \\ 20 \sin 5t + 2x_1 - 4x_3 \end{Bmatrix}$$

where

$$\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} \text{original } x_1 \\ \text{original } \dot{x}_1 \\ \text{original } x_2 \\ \text{original } \dot{x}_2 \end{Bmatrix} \quad \text{and} \quad \vec{X}_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The main program which calls RK4, the subroutine FUN and the results are given.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C =====
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION TIME(25),X(25,4),XX(4),F(4),YI(4),YJ(4),YK(4),YL(4),
2  UU(4)
  XX(1)=0.0
  XX(2)=0.0
  XX(3)=0.0
  XX(4)=0.0
  NEQ=4
  NSTEP=25

```

```

DT=0.25
T=0.0
WRITE (60,10)
10  FORMAT (//,3X,5H I ,10H  TIME(I),7X,5H X(1),12X,5H X(2),
2    12X,5H X(3),12X,5H X(4),/)
DO 40 I=1,NSTEP
CALL RK4 (T,DT,NEQ,XX,F,YI,YJ,YK,YL,UU)
TIME(I)=T
DO 20 J=1,NEQ
20  X(1,J)=XX(J)
WRITE (60,30) I,TIME(I),(X(1,J),J=1,NEQ)
30  FORMAT (2X,I5,F10.4,4(2X,E15.8))
40  CONTINUE
STOP
END

```

```

SUBROUTINE FUN (X,F,N,T)
DIMENSION X(N),F(N)
F(1)=X(2)
F(2)=(5.0-6.0*X(1)+2.0*X(3))/2.0
F(3)=X(4)
F(4)=20.0*SIN(5.0*T)+2.0*X(1)-4.0*X(3)
RETURN
END

```

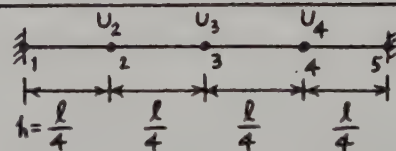
I	TIME(I)	X(1)	X(2)	X(3)	X(4)
1	0.2500	0.76904297E-01	0.62070566E+00	0.24460433E+00	0.26932180E+01
2	0.5000	0.31453162E+00	0.13009652E+01	0.14515579E+01	0.64889636E+01
3	0.7500	0.74014312E+00	0.21077421E+01	0.29793470E+01	0.45641656E+01
4	1.0000	0.13591884E+01	0.27739763E+01	0.32607102E+01	-0.26126151E+01
5	1.2500	0.20711818E+01	0.27800636E+01	0.18916913E+01	-0.72935324E+01
6	1.5000	0.26687763E+01	0.18683126E+01	0.27935052E+00	-0.45067854E+01
7	1.7500	0.29524713E+01	0.35540020E+00	-0.78577667E-01	0.14555850E+01
8	2.0000	0.28473990E+01	-0.11584098E+01	0.60296357E+00	0.29186647E+01
9	2.2500	0.24058905E+01	-0.23112063E+01	0.90024042E+00	-0.96232796E+00
10	2.5000	0.17232550E+01	-0.30908332E+01	0.23876321E+00	-0.35336885E+01
11	2.7500	0.89089936E+00	-0.34831977E+01	-0.34061307E+00	-0.26938438E+00
12	3.0000	0.34346521E-01	-0.32377021E+01	0.25854278E+00	0.46597433E+01
13	3.2500	-0.65400219E+00	-0.21399481E+01	0.14909362E+01	0.39903281E+01
14	3.5000	-0.98161954E+00	-0.43759835E+00	0.17313157E+01	-0.24955246E+01
15	3.7500	-0.87797028E+00	0.11996664E+01	0.39478755E+00	-0.72439308E+01
16	4.0000	-0.43708453E+00	0.22119966E+01	-0.12451535E+01	-0.47495975E+01
17	4.2500	0.16884252E+00	0.25470126E+01	-0.16587874E+01	0.13506827E+01
18	4.5000	0.80284208E+00	0.24732747E+01	-0.92830122E+00	0.34720352E+01
19	4.7500	0.13844182E+01	0.21252785E+01	-0.39376926E+00	0.36745048E+00
20	5.0000	0.18346303E+01	0.14129710E+01	-0.64064902E+00	-0.15653036E+01
21	5.2500	0.20612290E+01	0.37833023E+00	-0.64713440E+00	0.23636723E+01
22	5.5000	0.20297880E+01	-0.56775379E+00	0.71066928E+00	0.81037455E+01
23	5.7500	0.18273093E+01	-0.94346517E+00	0.28840952E+01	0.79655704E+01
24	6.0000	0.16137744E+01	-0.69621664E+00	0.40854921E+01	0.10114455E+01
25	6.2500	0.14960963E+01	-0.25771955E+00	0.34342446E+01	-0.54408941E+01

11.23

Equation of motion

$$\frac{d^2 U}{dx^2} + \alpha^2 U = 0$$

At grid point i , this becomes



$$U_{i+1} - 2U_i + U_{i-1} + \alpha^2 h^2 U_i = 0 \quad \text{or} \quad U_{i+1} - (2-\lambda)U_i + U_{i-1} = 0 \quad \text{--- (E}_1\text{)}$$

Eg. (E₁), when applied to nodes 2, 3 and 4, gives where $\lambda = \alpha^2 h^2 = \rho l^2 \omega^2 / (16E)$.

$$\left. \begin{aligned} U_3 - (2-\lambda)U_2 + U_1 &= 0 \\ U_4 - (2-\lambda)U_3 + U_2 &= 0 \\ U_5 - (2-\lambda)U_4 + U_3 &= 0 \end{aligned} \right\} \quad \text{--- (E}_2\text{)}$$

With boundary conditions $U_1 = U_5 = 0$, (E₂) becomes

$$[A] \vec{U} - \lambda [I] \vec{U} = \vec{0} \quad \text{where} \quad [A] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \vec{U} = \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix}$$

$3 \times 3 \quad 3 \times 1 \quad 3 \times 3 \quad 3 \times 1 \quad 3 \times 1$

The solution of this eigenvalue problem is found using JACOBI. The results are

$$\lambda_1 = 0.585786,$$

$$\lambda_2 = 2.0,$$

$$\lambda_3 = 3.41421$$

$$\omega_1 = 3.06147 \sqrt{\frac{E}{\rho l^2}}, \quad \omega_2 = 5.65685 \sqrt{\frac{E}{\rho l^2}}, \quad \omega_3 = 7.39103 \sqrt{\frac{E}{\rho l^2}}$$

$$\vec{U}^{(1)} = \begin{Bmatrix} 0.5 \\ 0.707 \\ 0.5 \end{Bmatrix}, \quad \vec{U}^{(2)} = \begin{Bmatrix} 0.707 \\ 0 \\ -0.707 \end{Bmatrix}, \quad \vec{U}^{(3)} = \begin{Bmatrix} 0.5 \\ -0.707 \\ 0.5 \end{Bmatrix}$$

11.24 (i) Forced longitudinal vibration:

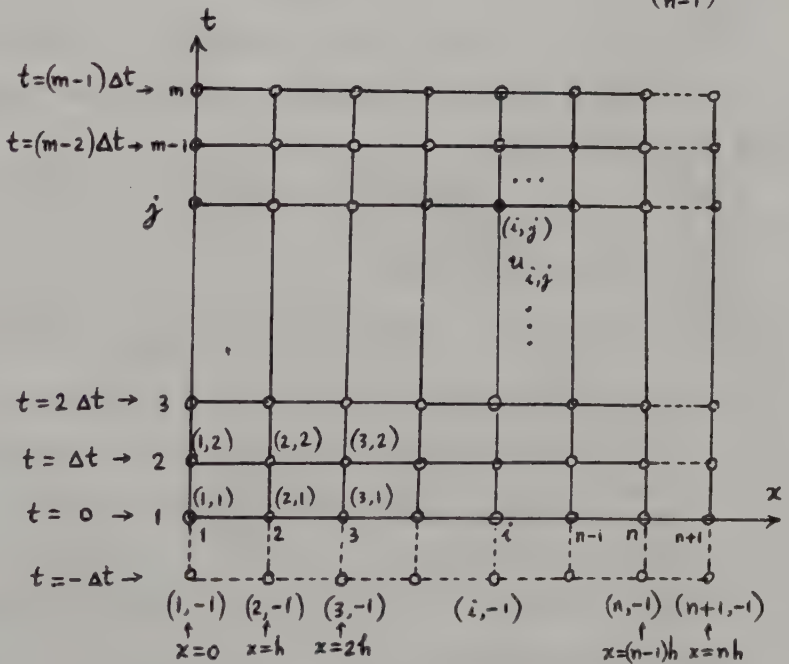
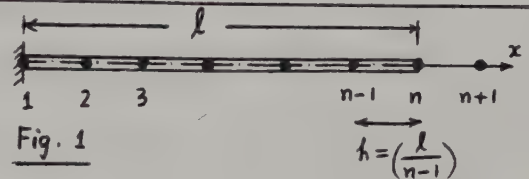
Equation is

$$EA \frac{\partial^2 u}{\partial x^2} + f = \rho A \frac{\partial^2 u}{\partial t^2} \quad \text{--- (E}_1\text{)}$$

where $u = u(x, t)$.

Let the solution be required for $0 \leq t \leq T$.

Set up the finite difference grid along x and t axes as shown in Fig. 2. Note that imaginary grid points are set up at $x = nh$ and $t = -\Delta t$ so that free boundary conditions can be applied at $x = (n-1)h$



$$h = \frac{l}{n-1}, \quad \Delta t = \frac{T}{m-1}$$

Fig. 2

and initial conditions can be applied at $t=0$.

Let $u_{i,j}$ = value of u at the grid point (i,j) .

Eq. (E₁) can be approximated at grid point (i,j) as

$$EA \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \right) + f_{i,j} = \rho A \left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2} \right) \dots (E_2)$$

where $f_{i,j} = f(x=x_i, t=t_j)$.

The procedure to set up the finite difference equations is:

(1). Apply (E₂) at grid points $(2,1), (3,1), \dots, (n,1); (2,2), (3,2), \dots, (n,2); \dots; (2,m-1), (3,m-1), \dots, (n,m-1)$. This gives $(n-1 \times m-1)$ equations in the unknowns associated with all the grid points shown in Fig. 2.

(2). Apply the boundary conditions.

At $x=0$, $u(x,t)=0 \Rightarrow u_{1,1} = u_{1,2} = \dots = u_{1,m-1} = u_{1,m} = 0$

At $x=l$, $\frac{\partial u}{\partial x} = 0 \Rightarrow u_{n+1,j} = u_{n-1,j}$ for $j = 1, 2, \dots, m$

(3). Apply the initial conditions.

At $t=0$, let $u(x,t) = U_0(x)$. Then $u_{i,1} = U_0(x=x_i)$ for $i=1, 2, \dots, n+1$

At $t=0$, let $\frac{\partial u}{\partial t}(x,t) = \dot{U}_0(x)$. Then $\frac{u_{i,2} - u_{i,1}}{2 \Delta t} = \dot{U}_0(x=x_i)$

i.e. $u_{i,-1} = u_{i,2} - 2 \Delta t \dot{U}_0(x_i)$ for $i=1, 2, \dots, n+1$.

(4). Express the resulting equations in matrix form. There will be as many linear equations as there are unknowns.

(ii) Free vibration:

Equation is $\frac{d^2 U}{dx^2} + \alpha^2 U = 0$ where $\alpha^2 = \frac{\rho \omega^2}{E}$ --- (E₃)

At node i , (E₃) becomes (Fig. 1)

$$U_{i+1} - 2U_i + U_{i-1} + \lambda U_i = 0 \text{ where } \lambda = h^2 \alpha^2 \dots (E_4)$$

Applying (E₂) at nodes 2, 3, ..., n gives

$$\left. \begin{aligned} U_3 - (2-\lambda)U_2 + U_1 &= 0 \\ U_4 - (2-\lambda)U_3 + U_2 &= 0 \\ \vdots \\ U_n - (2-\lambda)U_{n-1} + U_{n-2} &= 0 \\ U_{n+1} - (2-\lambda)U_n + U_{n-1} &= 0 \end{aligned} \right\} \dots (E_5)$$

Boundary conditions are $U_1 = 0$ ($U=0$ at $x=0$) and $U_{n+1} = U_{n-1}$ ($\frac{dU}{dx} = 0$ at $x=l$)

(E₅) and (E₆) give

$$\begin{aligned} U_3 - (2-\lambda)U_2 &= 0 \\ U_4 - (2-\lambda)U_3 + U_2 &= 0 \end{aligned}$$

$$\begin{aligned} & \vdots \\ U_n - (2-\lambda)U_{n-1} + U_{n-2} &= 0 \quad \dots (E_7) \\ -(2-\lambda)U_n + 2U_{n-1} &= 0 \end{aligned}$$

For $n=4$, (E_7) can be expressed as

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \dots (E_8)$$

Frequency equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 2 = 0 \quad \dots (E_9)$

Roots are $\lambda_1 = 0.267944$, $\lambda_2 = 2.0$, $\lambda_3 = 3.732050$

$$\therefore \omega_1 = 1.55289 \sqrt{\frac{E}{\rho l^2}}, \quad \omega_2 = 4.24264 \sqrt{\frac{E}{\rho l^2}}, \quad \omega_3 = 5.79555 \sqrt{\frac{E}{\rho l^2}}$$

11.25

Forced torsional vibration:

$$\text{Equation } GJ \frac{\partial^2 \theta}{\partial x^2} + f = J_0 \frac{\partial^2 \theta}{\partial t^2} \quad \dots (E_1)$$

Let the solution be required for $0 \leq t \leq T$.

Set up a finite difference grid along

x (with n points) and along t

(with m points) similar to Fig. 2 of problem 10.23.

Let $\theta_{i,j} = \theta(x=x_i, t=t_j)$ and $f_{i,j} = f(x=x_i, t=t_j)$

Eq. (E_1) at grid point (i,j) gives

$$GJ \left(\frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{h^2} \right) + f_{i,j} = J_0 \left(\frac{\theta_{i,j+1} - 2\theta_{i,j} + \theta_{i,j-1}}{\Delta t^2} \right)$$

The procedure to set up the finite difference grid is: $\dots (E_2)$

(1). Apply Eq. (E_2) at grid points $(2,1), (3,1), \dots, (n-1,1); (2,2), (3,2), \dots, (n-1,2); \dots; (2,m-1), (3,m-1), \dots, (n-1,m-1)$. This gives $(n-2) \times (m-1)$ equations in the $(n) \times (m+1)$ unknowns.

(2). Apply boundary conditions:

At $x=0$, $\theta(x,t)=0$. Hence $\theta_{1,j}=0$ for $j=1,2,\dots,m$

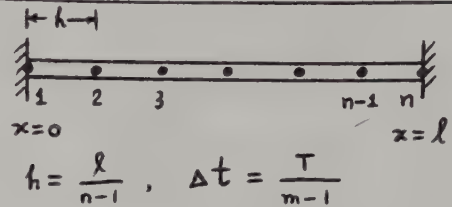
At $x=l$, $\theta(x,t)=0$. Hence $\theta_{n,j}=0$ for $j=1,2,\dots,m$

(3). Apply initial conditions.

At $t=0$, let $\theta(x,t) = \theta_0(x) \Rightarrow \theta_{i,1} = \theta_0(x=x_i)$ for $i=1,2,\dots,n$

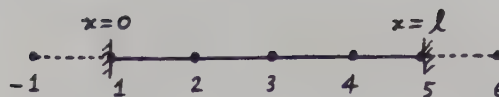
At $t=0$, let $\frac{\partial \theta}{\partial t}(x,t) = \dot{\theta}_0(x) \Rightarrow \theta_{i,-1} = \theta_{i,2} - 2\Delta t \dot{\theta}_0(x_i)$ for $i=1,2,\dots,n$.

(4) Express the resulting equations in matrix form. There will be $(n-2) \times m$ equations in $(n-2) \times m$ unknowns.



11.26

Applying Eq. (10.41) to mesh points 2, 3 and 4 gives



$$\left. \begin{aligned} w_4 - 4w_3 + (6-\lambda)w_2 - 4w_1 + w_{-1} &= 0 \\ w_5 - 4w_4 + (6-\lambda)w_3 - 4w_2 + w_1 &= 0 \\ w_6 - 4w_5 + (6-\lambda)w_4 - 4w_3 + w_2 &= 0 \end{aligned} \right\} \quad (E_1)$$

Boundary conditions are $w = \frac{dw}{dx} = 0$ at $x=0$ and $x=l$

i.e. $w_{-1} = 0, w_1 = w_2; w_5 = 0, w_6 = w_4$

Eq. (E₁) becomes, after applying boundary conditions,

$$\begin{bmatrix} 7 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 7 \end{bmatrix} \begin{Bmatrix} w_2 \\ w_3 \\ w_4 \end{Bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} w_2 \\ w_3 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (E_2)$$

Solution (obtained using the subroutine JACOBI) is

$$\lambda_1 = 1.25544, \quad \lambda_2 = 6.0, \quad \lambda_3 = 12.7446$$

$$\omega_1 = 17.9274 \sqrt{\frac{EI}{PA l^4}}, \quad \omega_2 = 39.1918 \sqrt{\frac{EI}{PA l^4}}, \quad \omega_3 = 57.1193 \sqrt{\frac{EI}{PA l^4}}$$

$$\vec{w}^{(1)} = \begin{Bmatrix} 0.4544 \\ 0.7662 \\ 0.4544 \end{Bmatrix}, \quad \vec{w}^{(2)} = \begin{Bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{Bmatrix}, \quad \vec{w}^{(3)} = \begin{Bmatrix} 0.5418 \\ -0.6426 \\ 0.5418 \end{Bmatrix}$$

11.27

Equation:

$$EI \frac{\partial^4 w}{\partial x^4} + PA \frac{\partial^2 w}{\partial t^2} = f(x, t) \quad \dots (E_1)$$



Let the solution be required for $0 \leq t \leq T$.

$$h = \left(\frac{l}{n-1} \right), \quad \Delta t = \frac{T}{m-1}$$

Set up a finite difference grid along

x (grid points $-1, 1, 2, \dots, n+2$) and along t (grid points $-1, 1, 2, \dots, m$).

Imaginary grid points are located at $x = -h, x = l+h$ and $x = l+2h$, and at $t = -\Delta t$.

Let $w_{i,j} = w(x = x_i, t = t_j)$ and $f_{i,j} = f(x = x_i, t = t_j)$.

Eq. (E₁) at grid point (i, j) gives

$$EI \left(\frac{w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j}}{h^4} \right) + PA \left(\frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{\Delta t^2} \right) = f_{i,j} = f_0 \cos \{ \omega(j-1)\Delta t \} \quad \dots (E_2)$$

The procedure to set up the finite difference equations is:

- (1) Apply (E₂) at grid points $(2,1), (3,1), \dots, (n,1); (2,2), (3,2), \dots, (n,2); \dots; (2,m-1), (3,m-1), \dots, (n,m-1)$. This gives $(n-1) \times (m-1)$ equations in the unknowns associated with all grid points.
- (2). Apply boundary conditions:

At $x=0$, $w = \frac{\partial w}{\partial x} = 0 \Rightarrow w_{1,j} = 0, w_{-1,j} = w_{2,j}$ for $j=1,2,\dots,m$

At $x=l$, $\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0 \Rightarrow w_{n+1,j} - 2w_{n,j} + w_{n-1,j} = 0,$

$$w_{n+2,j} - 2w_{n+1,j} + 2w_{n-1,j} - w_{n-2,j} = 0$$

$$\Rightarrow w_{n+1,j} = 2w_{n,j} - w_{n-1,j}, w_{n+2,j} = 4w_{n,j} - 4w_{n-1,j} + w_{n-2,j}$$

for $j=1,2,\dots,m$

(3). Apply initial conditions:

At $t=0$, let $w(x,t) = W_0(x) \Rightarrow w_{i,1} = W_0(x=x_i)$ for $i=1,2,\dots,n$

At $t=0$, let $\frac{\partial w}{\partial t}(x,t) = \dot{W}_0(x) \Rightarrow w_{i,-1} = w_{i,2} - 2\Delta t \dot{W}_0(x_i)$ for $i=1,2,\dots,n$

(4). Express the resulting equations in matrix form.

11.28

Equation:

$$P\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + f = \rho \frac{\partial^2 w}{\partial t^2} \quad \text{--- (E}_1\text{)}$$

Set up a finite difference grid with n, m and p points along x, y and t axes. Introduce an imaginary grid point at $t = -\Delta t$ to apply the initial conditions.

Let $w_{i,j,k} = w(x=x_i, y=y_j, t=t_k)$

and $f_{i,j,k} = f(x=x_i, y=y_j, t=t_k)$.

(E₁) gives at grid point (i,j,k) :

$$\begin{aligned} P\left(\frac{w_{i+1,j,k} - 2w_{i,j,k} + w_{i-1,j,k}}{h_1^2}\right) + P\left(\frac{w_{i,j+1,k} - 2w_{i,j,k} + w_{i,j-1,k}}{h_2^2}\right) \\ + f_{i,j,k} = \rho \left(\frac{w_{i,j,k+1} - 2w_{i,j,k} + w_{i,j,k-1}}{\Delta t^2}\right) \quad \text{--- (E}_2\text{)} \end{aligned}$$

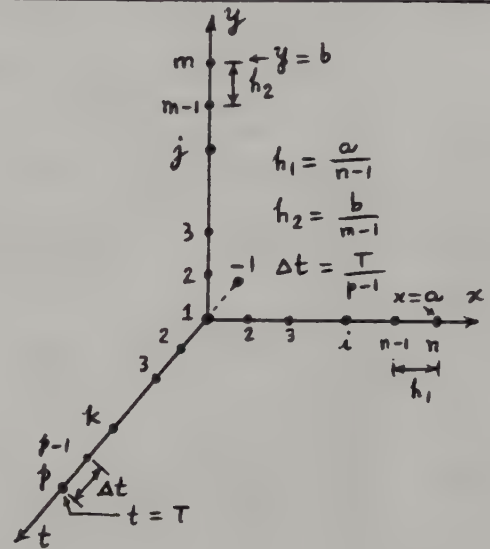
The following procedure is used to derive the final equations:

(1). Apply Eq. (E₂) at grid points (i,j,k) for $i=1,2,\dots,n$; $j=1,2,\dots,m$ and $k=1,2,\dots,p-1$. This gives $n \times m \times (p-1)$ equations in $n \times m \times (p+1)$ unknowns.

(2). Apply boundary conditions:

At $x=0$, $w(x,y,t) = 0 \Rightarrow w_{1,j,k} = 0$ for $j=1,2,\dots,m$; $k=1,2,\dots,p$

At $x=a$, $w(x,y,t) = 0 \Rightarrow w_{n,j,k} = 0$ for $j=1,2,\dots,m$; $k=1,2,\dots,p$



At $y=0$, $w(x,y,t)=0 \Rightarrow w_{i,1,k}=0$ for $i=1,2,\dots,n$; $k=1,2,\dots,p$

At $y=b$, $w(x,y,t)=0 \Rightarrow w_{i,m,k}=0$ for $i=1,2,\dots,n$; $k=1,2,\dots,p$

(3). Apply initial conditions:

At $t=0$, let $w(x,y,t)=W_0(x,y) \Rightarrow w_{i,j,1}=W_0(x_i,y_j)$ for $i=1,2,\dots,n$;
 $j=1,2,\dots,m$

At $t=0$, let $\frac{\partial w}{\partial t}(x,y,t)=\dot{W}_0(x,y) \Rightarrow w_{i,j,-1}=w_{i,j,2}-2\Delta t \dot{W}_0(x_i,y_j)$
 for $i=1,2,\dots,n$; $j=1,2,\dots,m$

(4). Express the resulting equations in matrix form. There will be
 $(n-2) \times (m-2) \times p$ equations in the same number of unknowns.

11.29

The program and output are given below.

```

DIMENSION X(4),XT(4),XTN(4),FT(4),A(4,4),F(2),M(2,2),K(2,2),
2 C(2,2),MI(2,2),MIK(2,2),MIC(2,2),MIF(2),ZK1(4),ZK2(4),
3 ZK3(4),ZK4(4),XD(2),XDD(2),XX(2),X1(2),X2(2)
REAL M,K,MI,MIK,MIC,MIF
DATA N,NSTEP,T,DELT/2,25,0,0,0,0.24216267/
DATA X/0.,0.,0.,0./
DATA M/1.,0.,0.,2./
DATA C/1.0,-1.0,-1.0,1.0/
DATA K/6.,-2.,-2.,8./
N2=N*2
DO 11 I=1,NSTEP
CALL RKM4 (N,N2,T,DELT,X,XT,XTN,FT,A,F,M,K,C,MI,MIK,MIC,
2 MIF,ZK1,ZK2,ZK3,ZK4)
DO 13 II=1,N
XX(II)=X(II)
13 XD(II)=X(II+N)
CALL XMULT (MIC,XD,X1,N)
CALL XMULT (MIK,XX,X2,N)
DO 14 II=1,N
14 XDD(II)=MIF(II)-X1(II)-X2(II)
PRINT 12, I,T,((XX(J),XD(J),XDD(J)),J=1,N)
12 FORMAT (2X,I2,2X,F6.3,1X,3E10.2,1X,3E10.2)
11 CONTINUE
STOP
END

```

C

```

SUBROUTINE RKM4 (N,N2,T,DELT,X,XT,XTN,FT,A,F,M,K,C,MI,
2 MIK,MIC,MIF,ZK1,ZK2,ZK3,ZK4)
REAL M,K,MI,MIK,MIC,MIF
DIMENSION X(N2),XT(N2),XTN(N2),FT(N2),A(N2,N2),F(N),M(N,N),
2 K(N,N),C(N,N),MI(N,N),MIK(N,N),MIC(N,N),MIF(N),
3 ZK1(N2),ZK2(N2),ZK3(N2),ZK4(N2)
DO 11 I=1,N2
11 XT(I)=X(I)
TIME=T
CALL FUN (N,N2,XT,FT,TIME,A,F,M,K,C,MI,MIK,MIC,MIF)
DO 12 I=1,N2
12 ZK1(I)=DELT*FT(I)

```



```

DO 13 I=1,N2
13 XT(I)=X(I)+0.5*ZK1(I)
   TIME=T+0.5*DELT
   CALL FUN (N,N2,XT,FT,TIME,A,F,M,K,C,MI,MIK,MIC,MIF)
   DO 14 I=1,N2
14 ZK2(I)=DELT*FT(I)
   DO 15 I=1,N2
15 XT(I)=X(I)+0.5*ZK2(I)
   CALL FUN (N,N2,XT,FT,TIME,A,F,M,K,C,MI,MIK,MIC,MIF)
   DO 16 I=1,N2
16 ZK3(I)=DELT*FT(I)
   DO 17 I=1,N2
17 XT(I)=X(I)+ZK3(I)
   TIME=T+DELT
   CALL FUN (N,N2,XT,FT,TIME,A,F,M,K,C,MI,MIK,MIC,MIF)
   DO 18 I=1,N2
18 ZK4(I)=DELT*FT(I)
   DO 19 I=1,N2
19 XTN(I)=X(I)+(ZK1(I)+2.0*ZK2(I)+2.0*ZK3(I)+ZK4(I))/6.0
   DO 21 I=1,N2
21 X(I)=XTN(I)
   T=TIME
   RETURN
   END
   SUBROUTINE FUN (N,N2,X,FT,T,A,F,M,K,C,MI,MIK,MIC,MIF)
   REAL M,K,MI,MIK,MIC,MIF
   DIMENSION X(N2),A(N2,N2),FT(N2),M(N,N),K(N,N),C(N,N),MI(N,N),
2 MIK(N,N),MIC(N,N),MIF(N),F(N)
C INVERSE OF M-MATRIX
   DET=M(1,1)*M(2,2)-M(1,2)*M(2,1)
   MI(1,1)=M(2,2)/DET
   MI(1,2)=-M(1,2)/DET
   MI(2,1)=-M(2,1)/DET
   MI(2,2)=M(1,1)/DET
C END OF INVERSE OF M-MATRIX
   CALL MATMUL (MIK,MI,K,N,N,N)
   CALL MATMUL (MIC,MI,C,N,N,N)
C DEFINITION OF F-VECTOR
   CALL FUN2 (X,T,N,N2,F)
C END OF F-VECTOR
   CALL XMULT (MI,F,MIF,N)
   DO 11 I=1,N
   DO 11 J=1,N
   A(I,J)=0.0
   A(I,J+N)=0.0
   A(I+N,J)=-MIK(I,J)
11 A(I,J+N)=-MIC(I,J)
   DO 12 I=1,N
12 A(I,I+N)=1.0
   CALL XMULT (A,X,FT,N2)
   N1=N+1
   DO 13 I=N1,N2
13 FT(I)=FT(I)+MIF(I-N)
   RETURN
   END

```

C

```

SUBROUTINE FUN2 (X, T, N, N2, F)
DIMENSION X(N2), F(N)
F(1)=0. 0
F(2)=10. 0
RETURN
END

```

C

```

SUBROUTINE XMULT (A, B, BB, N)
DIMENSION A(N, N), B(N), BB(N)
DO 10 I=1, N
BB(I)=0. 0
DO 10 J=1, N
10 BB(I)=BB(I)+A(I, J)*B(J)
RETURN
END

```

C

```

SUBROUTINE MATMUL (A, B, C, L, M, N)
DIMENSION A(L, N), B(L, M), C(M, N)
DO 10 I=1, L
DO 10 J=1, N
A(I, J)=0. 0
DO 10 K=1, M
10 A(I, J)=A(I, J)+B(I, K)*C(K, J)
RETURN
END

```

i	t_i	x_{1i}	$\dot{x}_{1i}$	$\ddot{x}_{1i}$	x_{2i}	$\dot{x}_{2i}$	$\ddot{x}_{2i}$
1	0.242	0.14E+00	-0.47E-01	0.66E+00	0.14E+00	0.12E+01	0.40E+01
2	0.484	0.51E+00	-0.35E+00	0.50E+00	0.53E+00	0.22E+01	0.21E+01
3	0.726	0.95E+00	-0.10E+01	0.19E+00	0.11E+01	0.28E+01	-0.17E+00
4	0.969	0.13E+01	-0.20E+01	0.43E+00	0.16E+01	0.30E+01	-0.25E+01
5	1.211	0.14E+01	-0.31E+01	0.17E+01	0.20E+01	0.28E+01	-0.45E+01
6	1.453	0.11E+01	-0.40E+01	0.39E+01	0.22E+01	0.23E+01	-0.56E+01
7	1.695	0.59E+00	-0.42E+01	0.66E+01	0.21E+01	0.16E+01	-0.58E+01
8	1.937	-0.66E-01	-0.36E+01	0.88E+01	0.20E+01	0.92E+00	-0.51E+01
9	2.179	-0.64E+00	-0.22E+01	0.97E+01	0.17E+01	0.25E+00	-0.38E+01
10	2.422	-0.94E+00	-0.18E+00	0.86E+01	0.16E+01	-0.33E+00	-0.22E+01
11	2.664	-0.87E+00	0.19E+01	0.55E+01	0.15E+01	-0.86E+00	-0.66E+00
12	2.906	-0.44E+00	0.37E+01	0.88E+00	0.16E+01	-0.13E+01	0.57E+00
13	3.148	0.21E+00	0.47E+01	-0.42E+01	0.18E+01	-0.18E+01	0.14E+01
14	3.390	0.88E+00	0.48E+01	-0.86E+01	0.19E+01	-0.22E+01	0.19E+01
15	3.632	0.14E+01	0.40E+01	-0.11E+02	0.18E+01	-0.25E+01	0.24E+01
16	3.875	0.16E+01	0.26E+01	-0.11E+02	0.16E+01	-0.26E+01	0.28E+01
17	4.117	0.14E+01	0.11E+01	-0.95E+01	0.12E+01	-0.24E+01	0.33E+01
18	4.359	0.99E+00	-0.17E+00	-0.61E+01	0.74E+00	-0.18E+01	0.39E+01
19	4.601	0.50E+00	-0.10E+01	-0.23E+01	0.32E+00	-0.96E+00	0.42E+01
20	4.843	0.11E+00	-0.13E+01	0.99E+00	0.79E-01	0.15E+00	0.41E+01
21	5.085	-0.39E-01	-0.13E+01	0.30E+01	0.93E-01	0.13E+01	0.33E+01
22	5.328	0.97E-01	-0.12E+01	0.37E+01	0.38E+00	0.23E+01	0.18E+01
23	5.570	0.45E+00	-0.13E+01	0.34E+01	0.88E+00	0.30E+01	-0.21E+00
24	5.812	0.85E+00	-0.17E+01	0.27E+01	0.15E+01	0.32E+01	-0.24E+01
25	6.054	0.11E+01	-0.23E+01	0.25E+01	0.20E+01	0.30E+01	-0.44E+01

11.30

The main program and the subroutine FUN2, along with the output, are given below. Other subroutines are same as those in problem 11.29.

```

DIMENSION X(4), XT(4), XTN(4), FT(4), A(4,4), F(2), M(2,2), K(2,2),
2 C(2,2), MI(2,2), MIK(2,2), MIC(2,2), MIF(2), ZK1(4), ZK2(4),
3 ZK3(4), ZK4(4), XD(2), XDD(2), XX(2), X1(2), X2(2)
REAL M, K, MI, MIK, MIC, MIF
DATA N, NSTEP, T, DELT/2, 25, 0.0, 0.24216267/
DATA X/0., 0., 0., 0./
DATA M/1., 0., 0., 2./
DATA C/0., 0., 0., 0./
DATA K/6., -2., -2., 8./
N2=N*2
DO 11 I=1, NSTEP
CALL RKM4 (N, N2, T, DELT, X, XT, XTN, FT, A, F, M, K, C, MI, MIK, MIC,
2 MIF, ZK1, ZK2, ZK3, ZK4)
DO 13 II=1, N
XX(II)=X(II)
13 XD(II)=X(II+N)
CALL XMULT (MIC, XD, X1, N)
CALL XMULT (MIK, XX, X2, N)
DO 14 II=1, N
14 XDD(II)=MIF(II)-X1(II)-X2(II)
PRINT 12, I, T, ((XX(J), XD(J), XDD(J)), J=1, N)
12 FORMAT (2X, I2, 2X, F6.3, 1X, 3E10.2, 1X, 3E10.2)
11 CONTINUE
STOP
END

```

C

```

SUBROUTINE FUN2 (X, T, N, N2, F)
DIMENSION X(N2), F(N)
F(1)=10.0*SIN(5.0*T)
F(2)=0.0
RETURN
END

```

i	t_i	x_{1i}	$\dot{x}_{1i}$	$\ddot{x}_{1i}$	x_2	$\dot{x}_{2i}$	$\ddot{x}_{2i}$
1	0.242	0.11E+00	0.13E+01	0.87E+01	0.00E+00	0.67E-02	0.11E+00
2	0.484	0.65E+00	0.30E+01	0.27E+01	0.90E-02	0.88E-01	0.62E+00
3	0.726	0.13E+01	0.18E+01	-0.12E+02	0.55E-01	0.30E+00	0.11E+01
4	0.969	0.13E+01	-0.22E+01	-0.17E+02	0.16E+00	0.54E+00	0.64E+00
5	1.211	0.34E+00	-0.50E+01	-0.37E+01	0.29E+00	0.52E+00	-0.84E+00
6	1.453	-0.79E+00	-0.36E+01	0.14E+02	0.38E+00	0.13E+00	-0.73E+01
7	1.695	-0.12E+01	0.39E+00	0.16E+02	0.34E+00	-0.49E+00	-0.75E+01
8	1.937	-0.75E+00	0.28E+01	0.22E+01	0.15E+00	-0.98E+00	-0.14E+01
9	2.179	-0.16E+00	0.16E+01	-0.92E+01	-0.11E+00	-0.11E+01	0.27E+00
10	2.422	-0.27E-01	-0.39E+00	-0.50E+01	-0.36E+00	-0.89E+00	0.14E+01
11	2.664	-0.15E+00	-0.13E+00	0.67E+01	-0.52E+00	-0.48E+00	0.20E+01
12	2.906	0.70E-01	0.19E+01	0.77E+01	-0.58E+00	0.45E-01	0.24E+01
13	3.148	0.66E+00	0.24E+01	-0.53E+01	-0.50E+00	0.66E+00	0.26E+01
14	3.390	0.95E+00	-0.40E+00	-0.16E+02	-0.26E+00	0.12E+01	0.20E+01
15	3.632	0.41E+00	-0.37E+01	-0.87E+01	0.85E-01	0.15E+01	0.75E+01
16	3.875	-0.57E+00	-0.37E+01	0.93E+01	0.43E+00	0.12E+01	-0.20E+01

17	4.117	-0.11E+01	-0.51E-01	0.17E+02	0.65E+00	0.49E+00	-0.36E+01
18	4.359	-0.62E+00	0.32E+01	0.70E+01	0.66E+00	-0.37E+00	-0.30E+01
19	4.601	0.20E+00	0.29E+01	-0.87E+01	0.49E+00	-0.99E+00	-0.13E+01
20	4.843	0.57E+00	0.15E+00	-0.11E+02	0.21E+00	-0.12E+01	-0.20E+00
21	5.085	0.38E+00	-0.13E+01	0.43E+00	-0.81E-01	-0.12E+01	0.71E+00
22	5.328	0.20E+00	0.43E-01	0.81E+01	-0.34E+00	-0.89E+00	0.15E+01
23	5.570	0.41E+00	0.14E+01	0.67E+00	-0.50E+00	-0.41E+00	0.24E+01
24	5.812	0.63E+00	-0.73E-01	-0.12E+02	-0.52E+00	0.23E+00	0.27E+01
25	6.054	0.22E+00	-0.33E+01	-0.11E+02	-0.39E+00	0.80E+00	0.13E+01

11.31

The main program, the subroutine FUN2, and the output are shown below. Other subroutines are given in problem 11.29.

```

      DIMENSION X(4),XT(4),XTN(4),FT(4),A(4,4),F(2),M(2,2),K(2,2),
2     C(2,2),MI(2,2),MIK(2,2),MIC(2,2),MIF(2),ZK1(4),ZK2(4),
3     ZK3(4),ZK4(4),XD(2),XDD(2),XX(2),X1(2),X2(2)
      REAL M,K,MI,MIK,MIC,MIF
      DATA N,NSTEP,T,DELT/2,25,0.0,0.25/
      DATA X/0.,0.,0.,0./
      DATA M/1.,0.,0.,2./
      DATA C/0.,0.,0.,0./
      DATA K/6.0,-2.0,-2.0,4.0/
      N2=N*2
      DO 11 I=1,NSTEP
      CALL RKM4 (N,N2,T,DELT,X,XT,XTN,FT,A,F,M,K,C,MI,MIK,MIC,
2     MIF,ZK1,ZK2,ZK3,ZK4)
      DO 13 II=1,N
      XX(II)=X(II)
13     XD(II)=X(II+N)
      CALL XMULT (MIC,XD,X1,N)
      CALL XMULT (MIK,XX,X2,N)
      DO 14 II=1,N
14     XDD(II)=MIF(II)-X1(II)-X2(II)
      PRINT 12, I,T,((XX(J),XD(J),XDD(J)),J=1,N)
12     FORMAT (2X,I2,2X,F6.3,1X,3E10.2,1X,3E10.2)
11     CONTINUE
      STOP
      END

C
      SUBROUTINE FUN2 (X,T,N,N2,F)
      DIMENSION X(N2),F(N)
      F(1)=5.0
      F(2)=20.0*SIN(5.0*T)
      RETURN
      END

```

i	t_i	x_{1i}	$\dot{x}_{1i}$	$\ddot{x}_{1i}$	x_{2i}	$\dot{x}_{2i}$	$\ddot{x}_{2i}$
1	0.250	0.15E+00	0.12E+01	0.43E+01	0.12E+00	0.14E+01	0.74E+01
2	0.500	0.57E+00	0.21E+01	0.31E+01	0.75E+00	0.35E+01	0.50E+01
3	0.750	0.12E+01	0.27E+01	0.12E+01	0.16E+01	0.31E+01	-0.78E+01
4	1.000	0.19E+01	0.26E+01	-0.19E+01	0.21E+01	0.34E+00	-0.12E+02
5	1.250	0.24E+01	0.17E+01	-0.57E+01	0.19E+01	-0.16E+01	-0.17E+01
6	1.500	0.26E+01	-0.62E-01	-0.75E+01	0.16E+01	-0.49E+00	0.80E+01
7	1.750	0.24E+01	-0.18E+01	-0.58E+01	0.17E+01	0.16E+01	0.51E+01
8	2.000	0.18E+01	-0.27E+01	-0.14E+01	0.22E+01	0.13E+01	-0.80E+01

9	2.250	0.11E+01	-0.25E+01	0.26E+01	0.21E+01	-0.17E+01	-0.13E+02
10	2.500	0.59E+00	-0.16E+01	0.43E+01	0.14E+01	-0.39E+01	-0.77E+01
11	2.750	0.32E+00	-0.56E+00	0.41E+01	0.48E+00	-0.30E+01	0.86E+01
12	3.000	0.29E+00	0.36E+00	0.33E+01	0.24E-01	-0.73E+00	0.67E+01
13	3.250	0.48E+00	0.10E+01	0.20E+01	-0.63E-01	-0.43E+00	-0.46E+01
14	3.500	0.78E+00	0.13E+01	-0.45E+00	-0.39E+00	-0.23E+01	-0.62E+01
15	3.750	0.10E+01	0.75E+00	-0.36E+01	-0.11E+01	-0.33E+01	0.74E+01
16	4.000	0.11E+01	-0.39E+00	-0.51E+01	-0.18E+01	-0.11E+01	0.14E+02
17	4.250	0.85E+00	-0.15E+01	-0.33E+01	-0.16E+01	0.23E+01	0.11E+02
18	4.500	0.41E+00	-0.18E+01	0.91E+00	-0.80E+00	0.34E+01	-0.77E+01
19	4.750	0.33E-01	-0.11E+01	0.45E+01	-0.16E+00	0.15E+01	-0.95E+01
20	5.000	-0.85E-01	0.20E+00	0.55E+01	-0.22E-01	-0.82E-01	-0.14E+01
21	5.250	0.13E+00	0.14E+01	0.43E+01	0.43E-01	0.10E+01	0.70E+01
22	5.500	0.61E+00	0.23E+01	0.26E+01	0.60E+00	0.33E+01	0.64E+01
23	5.750	0.12E+01	0.27E+01	0.53E+00	0.15E+01	0.33E+01	-0.63E+01
24	6.000	0.19E+01	0.25E+01	-0.24E+01	0.20E+01	0.74E+00	-0.12E+02
25	6.250	0.24E+01	0.15E+01	-0.57E+01	0.19E+01	-0.14E+01	-0.10E+01

11.32 The main program, subroutine EXTFUN and results are given.

```

C =====
C
C PROGRAM
C MAIN PROGRAM WHICH CALLS HOBOLT
C
C =====
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
  REAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
  DIMENSION C(2,2),X1(2),XDI(2),XDDI(2),XM1(2),F(2),R(2),RR(2),
2  XMK(2),XMI(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
3  S(2),X(25,2),XD(25,2),XDD(25,2)
  DATA N,NSTEP,NSTEP1,DELT/2,24,25,0.24216267/
  DATA XI/0.0,0.0/
  DATA XDI/0.0,0.0/
  DATA M/1.0,0.0,0.0,2.0/
  DATA C/2.0,-2.0,-2.0,2.0/
  DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
  CALL HOBOLT (M,C,K,XI,XDI,XDDI,N,NSTEP,DELT,r,R,RR,XM1,XM2,XP1,
2  MC,MK,MCI,XMK,MMC,XMI,ZA,ZB,ZC,LA,LB,S,X,XD,XDD,NSTEP1)
  WRITE (56,10)
10  FORMAT (//,27H SOLUTION BY HOUBOLT METHOD,/)
  WRITE (56,20) N,NSTEP,DELT
20  FORMAT (12H GIVEN DATA: ,/,3H N=,I5,4X,7H NSTEP=,I5,4X,6H DELT=,
2  E15.8,/)
  WRITE (56,30)
30  FORMAT (10H SOLUTION: ,//,5H STEP,3X,5H TIME,3X,7H X(1,1),3X,
2  8H XD(1,1),2X,9H XDD(1,1),4X,7H X(1,2),3X,8H XD(1,2),2X,
3  9H XDD(1,2),/)
  DO 40 I=1,NSTEP1
    TIME=REAL(I-1)*DELT
40  WRITE (56,50) 1,TIME,X(1,1),XD(1,1),XDD(1,1),X(1,2),XD(1,2),
2  XDD(1,2)
50  FORMAT (1X,I4,F8.4,6(1X,E10.4))
  STOP
END

```

```

C =====
C
C SUBROUTINE EXTFUN
C
C =====

```

```

      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=10.0
      RETURN
      END

```

SOLUTION BY HOUBOLT METHOD

GIVEN DATA:

N= 2 NSTEP= 24 DELT= 0.24216267E+00

SOLUTION:

STEP	TIME	X(1,1)	XD(1,1)	XDD(1,1)	X(1,2)	XD(1,2)	XDD(1,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.5000E+01
2	0.2422	0.0000E+00	0.0000E+00	0.0000E+00	0.1466E+00	0.3077E-07	0.5000E+01
3	0.4843	0.1122E+00	0.2317E+00	0.1913E+01	0.5045E+00	0.1042E+01	0.3604E+01
4	0.7265	0.3347E+00	0.1144E+01	0.1847E+01	0.9817E+00	0.2090E+01	0.4620E+00
5	0.9687	0.6302E+00	0.1320E+01	0.6120E+00	0.1488E+01	0.2029E+01	-0.1031E+01
6	1.2108	0.9375E+00	0.1209E+01	-0.8452E+00	0.1943E+01	0.1656E+01	-0.2280E+01
7	1.4530	0.1188E+01	0.8260E+00	-0.2123E+01	0.2278E+01	0.1051E+01	-0.3150E+01
8	1.6951	0.1325E+01	0.2515E+00	-0.2923E+01	0.2451E+01	0.3152E+00	-0.3543E+01
9	1.9373	0.1315E+01	-0.3944E+00	-0.3090E+01	0.2442E+01	-0.4369E+00	-0.3412E+01
10	2.1795	0.1154E+01	-0.9807E+00	-0.2621E+01	0.2261E+01	-0.1091E+01	-0.2781E+01
11	2.4216	0.8691E+00	-0.1393E+01	-0.1644E+01	0.1943E+01	-0.1550E+01	-0.1745E+01
12	2.6638	0.5118E+00	-0.1556E+01	-0.3726E+00	0.1542E+01	-0.1749E+01	-0.4632E+00
13	2.9060	0.1456E+00	-0.1442E+01	0.9391E+00	0.1125E+01	-0.1661E+01	0.8628E+00
14	3.1481	-0.1649E+00	-0.1078E+01	0.2049E+01	0.7604E+00	-0.1309E+01	0.2024E+01
15	3.3903	-0.3664E+00	-0.5339E+00	0.2766E+01	0.5042E+00	-0.7540E+00	0.2837E+01
16	3.6324	-0.4260E+00	0.9206E-01	0.2981E+01	0.3955E+00	-0.9085E-01	0.3175E+01
17	3.8746	-0.3362E+00	0.6895E+00	0.2676E+01	0.4483E+00	0.5704E+00	0.2990E+01
18	4.1168	-0.1154E+00	0.1157E+01	0.1922E+01	0.6498E+00	0.1122E+01	0.2320E+01
19	4.3589	0.1962E+00	0.1419E+01	0.8631E+00	0.9635E+00	0.1476E+01	0.1286E+01
20	4.6011	0.5442E+00	0.1437E+01	-0.3110E+00	0.1335E+01	0.1579E+01	0.6234E-01
21	4.8433	0.8692E+00	0.1214E+01	-0.1400E+01	0.1702E+01	0.1419E+01	-0.1144E+01
22	5.0854	0.1118E+01	0.7933E+00	-0.2225E+01	0.2004E+01	0.1030E+01	-0.2134E+01
23	5.3276	0.1250E+01	0.2515E+00	-0.2656E+01	0.2193E+01	0.4803E+00	-0.2750E+01
24	5.5697	0.1247E+01	-0.3186E+00	-0.2633E+01	0.2240E+01	-0.1354E+00	-0.2897E+01
25	5.8119	0.1112E+01	-0.8215E+00	-0.2174E+01	0.2142E+01	-0.7137E+00	-0.2562E+01

11.33

Problem - dependent data: (in the main program which calls HOUBOLT)

REAL M(2,2), K(2,2), MC(2,2), MK(2,2), MCI(2,2), MMC(2,2)

DIMENSION C(2,2), XI(2), XDI(2), XDDI(2), XM1(2), F(2), R(2), RR(2),

XMK(2), XMI(2), XM2(2), XP1(2), ZA(2), ZB(2), ZC(2), LA(2), LB(2,2),

S(2), X(25,2), XD(25,2), XDD(25,2)

DATA N, NSTEP, NSTEP1, DELT/2, 24, 25, 0.24216267/

DATA XI/0.0, 0.0/

DATA XDI/0.0, 0.0/

```
DATA M/1.0, 0.0, 0.0, 2.0/
DATA C/0.0, 0.0, 0.0, 0.0/
DATA K/6.0, -2.0, -2.0, 8.0/
```

The subroutine EXTFUN is:

```
SUBROUTINE EXTFUN (F, TIME, N)
  F(1) = 10.0 * SIN (5.0 * TIME)
  F(2) = 0.0
  RETURN
END
```

output is:

SOLUTION BY HOBOLT METHOD

GIVEN DATA:

= 2 NSTEP= 24 DELT= 0.24216267E+00

SOLUTION:

STEP	TIME	X(1,1)	XD(1,1)	XDD(1,1)	X(1,2)	XD(1,2)	XDD(1,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2422	0.5188E+00	0.1133E+01	0.9359E+01	0.0000E+00	0.0000E+00	0.0000E+00
3	0.4843	0.1291E+01	0.2666E+01	0.3301E+01	0.3219E-01	0.6645E-01	0.5488E+00
4	0.7265	0.1709E+01	0.2732E+00	-0.1468E+02	0.1166E+00	0.4842E+00	0.1233E+01
5	0.9687	0.1416E+01	-0.3036E+01	-0.1793E+02	0.2405E+00	0.5754E+00	0.4540E+00
6	1.2108	0.6297E+00	-0.4023E+01	-0.5329E+01	0.3603E+00	0.4264E+00	-0.8116E+00
7	1.4530	-0.1184E+00	-0.2266E+01	0.9874E+01	0.4248E+00	0.8146E-01	-0.1818E+01
8	1.6951	-0.4980E+00	-0.3518E+00	0.1192E+02	0.4001E+00	-0.3328E+00	-0.2098E+01
9	1.9373	-0.6401E+00	-0.2775E+00	0.1812E+01	0.2793E+00	-0.7070E+00	-0.1757E+01
10	2.1795	-0.8086E+00	-0.1113E+01	-0.4945E+01	0.7756E-01	-0.9789E+00	-0.1119E+01
11	2.4216	-0.9610E+00	-0.5377E+00	0.9972E+00	-0.1725E+00	-0.1088E+01	-0.2709E+00
12	2.6638	-0.7905E+00	0.1793E+01	0.1074E+02	-0.4206E+00	-0.9512E+00	0.8920E+00
13	2.9060	-0.1896E+00	0.3518E+01	0.9172E+01	-0.6026E+00	-0.5268E+00	0.2221E+01
14	3.1481	0.4916E+00	0.2498E+01	-0.4595E+01	-0.6598E+00	0.1026E+00	0.3131E+01
15	3.3903	0.7673E+00	-0.3680E+00	-0.1520E+02	-0.5657E+00	0.7373E+00	0.3030E+01
16	3.6324	0.5394E+00	-0.2116E+01	-0.1026E+02	-0.3402E+00	0.1175E+01	0.1900E+01
17	3.8746	0.1733E+00	-0.1293E+01	0.3880E+01	-0.3937E-01	0.1321E+01	0.3308E+00
18	4.1168	0.3681E-01	0.4163E+00	0.1018E+02	0.2687E+00	0.1193E+01	-0.1038E+01
19	4.3589	0.8754E-01	0.5378E+00	0.2471E+01	0.5218E+00	0.8458E+00	-0.2000E+01
20	4.6011	0.1885E-01	-0.9523E+00	-0.7265E+01	0.6694E+00	0.3224E+00	-0.2659E+01
21	4.8433	-0.2572E+00	-0.1689E+01	-0.5037E+01	0.6773E+00	-0.3030E+00	-0.2967E+01
22	5.0854	-0.4425E+00	-0.1669E+00	0.6634E+01	0.5392E+00	-0.8809E+00	-0.2599E+01
23	5.3276	-0.2320E+00	0.2106E+01	0.1195E+02	0.2874E+00	-0.1229E+01	-0.1382E+01
24	5.5697	0.2522E+00	0.2397E+01	0.2591E+01	-0.1222E-01	-0.1246E+01	0.3011E+00
25	5.8119	0.5522E+00	0.2275E+00	-0.1095E+02	-0.2861E+00	-0.9763E+00	0.1697E+01

11.34

The data to be used in the main program which calls HOBOLT, subroutine EXTFUN and the output are given below.

C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA

```
REAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),M*MC(2,2)
DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),XM1(2),F(2),R(2),RR(2),
2 XMK(2),XMI(2),XM2(2),XPI(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
3 S(2),X(25,2),XD(25,2),XDD(25,2)
DATA N,NSTEP,NSTEP1,DELT/2,24,25,0.25/
```



```

DATA XI/0.0,0.0/
DATA XDI/0.0,0.0/
DATA M/2.0,0.0,0.0,1.0/
DATA C/0.0,0.0,0.0,0.0/
DATA K/6.0,-2.0,-2.0,4.0/
C END OF PROBLEM-DEPENDENT DATA

SUBROUTINE EXTFUN (F,TIME,N)
DIMENSION F(N)
F(1)=5.0
F(2)=20.0*SIN(5.0*TIME)
RETURN
END

```

SOLUTION BY HOUBOLT METHOD

GIVEN DATA:

N= 2 NSIEP= 24 DELT= 0.25000000E+00

SOLUTION:

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.2500E+01	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2500	0.7813E-01	0.0000E+00	0.2500E+01	0.1186E+01	0.2372E+01	0.1898E+02
3	0.5000	0.3720E+00	0.7440E+00	0.3452E+01	0.2834E+01	0.5668E+01	0.7341E+01
4	0.7500	0.8908E+00	0.2538E+01	0.3748E+01	0.3920E+01	0.1861E+01	-0.2533E+02
5	1.0000	0.1570E+01	0.2954E+01	0.1544E+01	0.3756E+01	-0.4081E+01	-0.3106E+02
6	1.2500	0.2281E+01	0.2728E+01	-0.1597E+01	0.2744E+01	-0.5200E+01	-0.7080E+01
7	1.5000	0.2872E+01	0.1932E+01	-0.4272E+01	0.1845E+01	-0.2097E+01	0.1713E+02
8	1.7500	0.3225E+01	0.7715E+00	-0.5759E+01	0.1416E+01	-0.2943E+00	0.1328E+02
9	2.0000	0.3261E+01	-0.5946E+00	-0.6317E+01	0.3659E+00	-0.2500E+01	-0.8222E+01
10	2.2500	0.2943E+01	-0.2030E+01	-0.6253E+01	0.7463E-01	-0.5007E+01	-0.1377E+02
11	2.5000	0.2285E+01	-0.3287E+01	-0.5188E+01	-0.8318E+00	-0.3088E+01	0.6572E+01
12	2.7500	0.1378E+01	-0.4014E+01	-0.2594E+01	-0.9609E+00	0.2095E+01	0.2512E+02
13	3.0000	0.3796E+00	-0.3959E+01	0.1120E+01	-0.2410E+00	0.4673E+01	0.1473E+02
14	3.2500	-0.5214E+00	-0.3160E+01	0.4552E+01	0.4875E+00	0.1810E+01	-0.1331E+02
15	3.5000	-0.1171E+01	-0.1890E+01	0.6491E+01	0.4775E+00	-0.2512E+01	-0.2376E+02
16	3.7500	-0.1480E+01	-0.4332E+00	0.6889E+01	-0.4997E-01	-0.2851E+01	-0.4747E+01
17	4.0000	-0.1417E+01	0.1036E+01	0.6435E+01	-0.3148E+00	0.5060E+00	0.1668E+02
18	4.2500	-0.9987E+00	0.2360E+01	0.5417E+01	-0.7897E-01	0.7262E+01	0.1182E+02
19	4.5000	-0.2950E+00	0.3294E+01	0.3460E+01	0.7496E-01	-0.3250E+00	-0.1063E+02
20	4.7500	0.5658E+00	0.3586E+01	0.4555E+00	-0.3473E+00	-0.3500E+01	-0.1713E+02
21	5.0000	0.1421E+01	0.3191E+01	-0.2697E+01	-0.3347E+00	-0.2132E+01	0.3933E+01
22	5.2500	0.2126E+01	0.2332E+01	-0.4688E+01	-0.8089E+00	0.3101E+01	0.2547E+02
23	5.5000	0.2599E+01	0.1315E+01	-0.5055E+01	0.2427E+00	0.6341E+01	0.1821E+02
24	5.7500	0.2816E+01	0.3215E+00	-0.4484E+01	0.1464E+01	0.4216E+01	-0.9383E+01
25	6.0000	0.2786E+01	-0.6007E+00	-0.3789E+01	0.2068E+01	0.1335E+00	-0.2246E+02

11.35 The problem-dependent data to be used in the main program (given in Problem 10.34 and which calls WILSON), subroutine EXTFUN and output are given.

THE FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA

```

REAL M(2,2), MI(2,2), K(2,2)
DIMENSION C(2,2), X1(2), XD1(2), XDD1(2), X(25,2), XD(25,2), XDD(25,2),
2 X1(2), F(2), F1(2), F2(2), F3(2), K(2), GA(2), LB(2,2), S(2), ZA(2),
3 FK(2,2), XN1(2), XN2(2), XN3(2), XN4(2)

```



```

DATA N,NSTEP,NSTEP1,TH,DELTA/2,24,25,1.4,0.24216267/
DATA X1/0.0,0.0/
DATA XD1/0.0,0.0/
DATA M/1.0,0.0,0.0,2.0/
DATA C/2.0,-2.0,-2.0,2.0/
DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA

```

```

SUBROUTINE EXIFUN (F,TIME,N)
DIMENSION F(N)
F(1)=0.0
F(2)=10.0
RETURN
END

```

SOLUTION BY WILSON METHOD

IVEN DATA:

= 2 -- NSTEP= 24 -- TH= 0.14000000E+01 -- DELTA= 0.24216267E+00

SOLUTION:

TEP	TIME	X(1,1)	XD(1,1)	XDD(1,1)	X(1,2)	XD(1,2)	XDD(1,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.5000E+01
2	0.2422	0.1551E-01	0.1922E+00	0.1587E+01	0.1330E+00	0.1042E+01	0.3610E+01
3	0.4843	0.1136E+00	0.6390E+00	0.2103E+01	0.4767E+00	0.1735E+01	0.2112E+01
4	0.7265	0.3250E+00	0.1087E+01	0.1593E+01	0.9440E+00	0.2063E+01	0.5986E+00
5	0.9687	0.6233E+00	0.1328E+01	0.4052E+00	0.1447E+01	0.2036E+01	-0.8224E+00
6	1.2108	0.9431E+00	0.1256E+01	-0.9998E+00	0.1905E+01	0.1691E+01	-0.2029E+01
7	1.4530	0.1206E+01	0.8697E+00	-0.2195E+01	0.2246E+01	0.1095E+01	-0.2892E+01
8	1.6951	0.1346E+01	0.2556E+00	-0.2877E+01	0.2423E+01	0.3457E+00	-0.3297E+01
9	1.9373	0.1323E+01	-0.4452E+00	-0.2910E+01	0.2411E+01	-0.4384E+00	-0.3179E+01
10	2.1795	0.1136E+01	-0.1078E+01	-0.2318E+01	0.2218E+01	-0.1131E+01	-0.2545E+01
11	2.4216	0.8171E+00	-0.1511E+01	-0.1252E+01	0.1879E+01	-0.1620E+01	-0.1489E+01
12	2.6638	0.4274E+00	-0.1656E+01	0.5148E-01	0.1456E+01	-0.1822E+01	-0.1836E+00
13	2.9060	0.4036E-01	-0.1489E+01	0.1330E+01	0.1023E+01	-0.1705E+01	0.1149E+01
14	3.1481	-0.2713E+00	-0.1044E+01	0.2341E+01	0.6543E+00	-0.1291E+01	0.2276E+01
15	3.3903	-0.4500E+00	-0.4093E+00	0.2903E+01	0.4155E+00	-0.6520E+00	0.2998E+01
16	3.6324	-0.4638E+00	0.2960E+00	0.2923E+01	0.3474E+00	0.9684E-01	0.3186E+01
17	3.8746	-0.3114E+00	0.9418E+00	0.2410E+01	0.4606E+00	0.8226E+00	0.2808E+01
18	4.1168	-0.2189E-01	0.1411E+01	0.1468E+01	0.7335E+00	0.1396E+01	0.1931E+01
19	4.3589	0.3512E+00	0.1622E+01	0.2714E+00	0.1116E+01	0.1717E+01	0.7159E+00
20	4.6011	0.7399E+00	0.1539E+01	-0.9597E+00	0.1540E+01	0.1729E+01	-0.6189E+00
21	4.8433	0.1074E+01	0.1180E+01	-0.2005E+01	0.1929E+01	0.1432E+01	-0.1834E+01
22	5.0854	0.1294E+01	0.6124E+00	-0.2640E+01	0.2213E+01	0.8814E+00	-0.2711E+01
23	5.3276	0.1362E+01	-0.5939E-01	-0.2868E+01	0.2343E+01	0.1781E+00	-0.3097E+01
24	5.5697	0.1267E+01	-0.7143E+00	-0.2541E+01	0.2297E+01	-0.5512E+00	-0.2926E+01
25	5.8119	0.1027E+01	-0.1235E+01	-0.1763E+01	0.2085E+01	-0.1176E+01	-0.2233E+01

11.36 The problem-dependent data to be used in the main program given in Problem 10.34 (which calls WILSON), subroutine EXIFUN and the results are given.

C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA

```

REAL M(2,2),X1(2,2),K(2,2)
DIMENSION C(2,2),X1(2),XD1(2),XDD1(2),X(25,2),XD(25,2),XDD(25,2),
2 X1(2),F(2),F1(2),F2(2),FT(2),K(2),LA(2),LB(2,2),S(2),ZA(2),
3 TH(2,2),XN1(2),XN2(2),XN3(2),XN4(2)

```

```

DATA N,NSTEP,NSTEP1,TH,DELTA/2,24,25,1.4,0.24216267/
DATA X1/0.0,0.0/
DATA XD1/0.0,0.0/
DATA M/1.0,0.0,0.0,2.0/
DATA C/0.0,0.0,0.0,0.0/
DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
SUBROUTINE EXTFUN (F,TIME,N)
DIMENSION F(N)
F(1)=10.0*SIN(5.0*TIME)
F(2)=0.0
RETURN
END

```

SOLUTION BY WILSON METHOD

GIVEN DATA:

N= 2 NSTEP= 24 TH= 0.14000000E+01 DELTA= 0.24216267E+00

SOLUTION:

STEP	TIME	X(1,1)	XD(1,1)	XDD(1,1)	X(1,2)	XD(1,2)	XDD(1,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2422	0.8209E-01	0.1017E+01	0.8399E+01	0.1461E-02	0.1810E-01	0.1495E+00
3	0.4843	0.5226E+00	0.2406E+01	0.3071E+01	0.1381E-01	0.9867E-01	0.5160E+00
4	0.7265	0.1062E+01	0.1503E+01	-1.053E+02	0.5495E-01	0.2498E+00	0.7325E+00
5	0.9687	0.1074E+01	-1.1584E+01	-1.496E+02	0.1330E+00	0.3781E+00	0.3265E+00
6	1.2108	0.3655E+00	-3.797E+01	-3.306E+01	0.2246E+00	0.3401E+00	-6.398E+00
7	1.4530	-5.063E+00	-2.807E+01	0.1148E+02	0.2799E+00	0.8172E-01	-1.494E+01
8	1.6951	-8.380E+00	0.1135E+00	0.1263E+02	0.2548E+00	-2.930E+00	-1.600E+01
9	1.9373	-5.584E+00	0.1706E+01	0.5215E+00	0.1431E+00	-6.043E+00	-9.714E+00
10	2.1795	-2.170E+00	0.7537E+00	-8.390E+01	-2.359E-01	-7.389E+00	-1.402E+00
11	2.4216	-2.285E+00	-6.334E+00	-3.066E+01	-2.004E+00	-6.951E+00	0.5020E+00
12	2.6638	-3.632E+00	-3.125E-01	0.8040E+01	-3.488E+00	-5.089E+00	0.1036E+01
13	2.9060	-1.304E+00	0.1973E+01	0.8514E+01	-4.357E+00	-1.847E+00	0.1641E+01
14	3.1481	0.4742E+00	0.2513E+01	-4.054E+01	-4.280E+00	0.2662E+00	0.2083E+01
15	3.3903	0.8633E+00	0.2840E+00	-1.436E+02	-3.050E+00	0.7389E+00	0.1822E+01
16	3.6324	0.5643E+00	-2.534E+01	-8.916E+01	-8.396E-01	0.1040E+01	0.6659E+00
17	3.8746	-1.616E+00	-2.845E+01	0.6349E+01	0.1726E+00	0.1017E+01	-8.543E+00
18	4.1168	-5.959E+00	-4.607E+00	0.1334E+02	0.3836E+00	0.6825E+00	-1.911E+01
19	4.3589	-4.018E+00	0.1711E+01	0.4600E+01	0.4907E+00	0.1932E+00	-2.130E+01
20	4.6011	0.2833E-01	0.1349E+01	-7.598E+01	0.4783E+00	-2.815E+00	-1.791E+01
21	4.8433	0.1332E+00	-4.775E+00	-7.484E+01	0.3623E+00	-6.571E+00	-1.311E+01
22	5.0854	-9.143E-01	-9.221E+00	0.3812E+01	0.1706E+00	-9.025E+00	-7.150E+00
23	5.3276	-1.416E+00	0.7616E+00	0.1009E+02	-5.983E-01	-9.628E+00	0.2168E+00
24	5.5697	0.2564E+00	0.2184E+01	0.1656E+01	-2.753E+00	-7.698E+00	0.1377E+01
25	5.8119	0.7047E+00	0.9850E+00	-1.156E+02	-4.136E+00	-3.408E+00	0.2166E+01

11.37

The problem-dependent data to be used in the main program given in Problem 10.34 (which calls WILSON), subroutine EXTFUN and the output are given.

C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA

```

REAL M(2,2),MI(2,2),K(2,2)
DIMENSION C(2,2),X1(2),XD1(2),XDD1(2),X(25,2),XD(25,2),XDD(25,2),
2 X1(2),F(2),F1(2),F2(2),FT(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
3 TR(2,2),XN1(2),XN2(2),XN3(2),XN4(2)

```

```

DATA N,NSTEP,NSTEP1,TH,DELTA/2,24,25,1.4,0.25/
DATA X1/0.0,0.0/
DATA XDI/0.0,0.0/
DATA M/2.0,0.0,0.0,1.0/
DATA C/0.0,0.0,0.0,0.0/
DATA K/6.0,-2.0,-2.0,4.0/
C END OF PROBLEM-DEPENDENT DATA

```

```

SUBROUTINE EXTIFUN (F,TIME,N)
DIMENSION F(N)
F(1)=5.0
F(2)=20.0*SIN(5.0*TIME)
RETURN
END

```

SOLUTION BY WILSON METHOD

GIVEN DATA:

N= 2 NSTEP= 24 TH= 0.14000000E+01 DELTA= 0.25000000E+00

SOLUTION:

STEP	TIME	X(1,1)	XD(1,1)	XDD(1,1)	X(1,2)	XD(1,2)	XDD(1,2)
1	0.0000	0.0000E+00	0.0000E+00	0.2500E+01	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2500	0.7846E-01	0.6290E+00	0.2532E+01	0.1849E+00	0.2219E+01	0.1775E+02
3	0.5000	0.3173E+00	0.1292E+01	0.2772E+01	0.1185E+01	0.5347E+01	0.7278E+01
4	0.7500	0.7246E+00	0.1956E+01	0.2539E+01	0.2473E+01	0.3847E+01	-.1928E+02
5	1.0000	0.1277E+01	0.2398E+01	0.9958E+00	0.2760E+01	-.1840E+01	-.2622E+02
6	1.2500	0.1880E+01	0.2322E+01	-.1598E+01	0.1722E+01	-.5495E+01	-.3023E+01
7	1.5000	0.2387E+01	0.1638E+01	-.3881E+01	0.5026E+00	-.3266E+01	0.2085E+02
8	1.7500	0.2666E+01	0.5579E+00	-.4757E+01	0.2826E+00	0.1285E+01	0.1556E+02
9	2.0000	0.2661E+01	-.5871E+00	-.4402E+01	0.8331E+00	0.2090E+01	-.9119E+01
10	2.2500	0.2384E+01	-.1595E+01	-.3661E+01	0.9796E+00	-.1282E+01	-.1786E+02
11	2.5000	0.1980E+01	-.2397E+01	-.2755E+01	0.3022E+00	-.3333E+01	0.1457E+01
12	2.7500	0.1212E+01	-.2882E+01	-.1122E+01	-.2795E+00	-.4970E+00	0.2123E+02
13	3.0000	0.4833E+00	-.2841E+01	0.1444E+01	0.1675E+00	0.3704E+01	0.1238E+02
14	3.2500	-.1555E+00	-.2164E+01	0.3975E+01	0.1194E+01	0.3364E+01	-.1510E+02
15	3.5000	-.5604E+00	-.1028E+01	0.5115E+01	0.1459E+01	-.1662E+01	-.2510E+02
16	3.7500	-.6636E+00	0.1787E+00	0.4536E+01	0.4757E+00	-.5336E+01	-.4289E+01
17	4.0000	-.4914E+00	0.1142E+01	0.3169E+01	-.7410E+00	-.3392E+01	0.1984E+02
18	4.2500	-.1194E+00	0.1784E+01	0.1965E+01	-.1005E+01	0.1134E+01	0.1637E+02
19	4.5000	0.3705E+00	0.2138E+01	0.8735E+00	-.4563E+00	0.2272E+01	-.7272E+01
20	4.7500	0.9225E+00	0.2166E+01	-.6563E+00	-.2094E+00	-.6720E+00	-.1628E+02
21	5.0000	0.1425E+01	0.1778E+01	-.2441E+01	-.6840E+00	-.2317E+01	0.3120E+01
22	5.2500	0.1783E+01	0.1046E+01	-.3417E+01	-.9452E+00	0.1110E+01	0.2429E+02
23	5.5000	0.1943E+01	0.2598E+00	-.2874E+01	0.1076E-01	0.6216E+01	0.1655E+02
24	5.7500	0.1934E+01	-.2762E+00	-.1414E+01	0.1780E+01	0.6725E+01	-.1248E+02
25	6.0000	0.1832E+01	-.4927E+00	-.3190E+00	0.2923E+01	0.1835E+01	-.2663E+02

11.38

```

C =====
C
C MAIN PROGRAM WHICH CALLS WILSON
C
C =====

```



```

C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
  REAL M(2,2),MI(2,2),K(2,2)
  DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
2 XI(2),F(2),F1(2),F2(2),FT(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
  DATA N,NSTEP,NSTEP1,TH,DELTA/2,24,25,1.4,0.24216267/
  DATA XI/0.0,0.0/
  DATA XDI/0.0,0.0/
  DATA M/1.0,0.0,0.0,2.0/
  DATA C/0.0,0.0,0.0,0.0/
  DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
  CALL WILSON (M,C,K,MI,XI,XDI,XDDI,X,XD,XDD,XT,F,F1,F2,FT,R,LA,
2 LB,S,ZA,TK,XN1,XN2,XN3,XN4,NSTEP1,DELTA,TH,N)
  PRINT 10
10  FORMAT (//,26H SOLUTION BY WILSON METHOD,/)
  PRINT 20, N,NSTEP,TH,DELTA
20  FORMAT (12H GIVEN DATA:,,3H N=,15,4X,7H NSTEP=,15,4X,4H TH=,
2 E15.8,4X,7H DELTA=,E15.8,/)
  PRINT 30
30  FORMAT (10H SOLUTION:,,//,5H STEP,3X,5H TIME,3X,7H X(1,1),3X,
2 8H XD(1,1),2X,9H XDD(1,1),4X,7H X(1,2),3X,8H XD(1,2),2X,
3 9H XDD(1,2),/)
  DO 40 I=1,NSTEP1
    TIME=REAL(I-1)*DELTA
40  PRINT 50, I,TIME,X(1,1),XD(1,1),XDD(1,1),X(1,2),XD(1,2),XDD(1,2)
50  FORMAT (1X,I4,F8.4,6(1X,E10.4))
  STOP
  END

C =====
C
C SUBROUTINE WILSON
C
C =====
  SUBROUTINE WILSON (M,C,K,MI,XI,XDI,XDDI,X,XD,XDD,XI,F,F1,F2,FT,
2 R,LA,LB,S,ZA,TK,XN1,XN2,XN3,XN4,NSTEP1,DELTA,TH,N)
  REAL M(N,N),MI(N,N),K(N,N)
  DIMENSION C(N,N),XI(N),XDI(N),XDDI(N),X(NSTEP1,N),XD(NSTEP1,N),
2 XDD(NSTEP1,N),XT(N),F(N),F1(N),F2(N),FT(N),R(N),LA(N),LB(N,2),
3 S(N),ZA(N),TK(N,N),XN1(N),XN2(N),XN3(N),XN4(N)
  DO 5 I=1,N
    DO 5 J=1,N
5    MI(I,J)=M(I,J)
    CALL SIMUL (MI,ZA,N,0,LA,LB,S)
    CALL EXTFUN (F,0.0,N)
    DO 20 I=1,N
      R(I)=F(I)
    DO 10 J=1,N
10    R(I)=R(I)-C(I,J)*XDI(J)-K(I,J)*XI(J)
20  CONTINUE
    CALL XMULT (MI,R,XDDI,N)
    DO 25 J=1,N
      X(1,J)=XI(J)
      XD(1,J)=XDI(J)
25    XDD(1,J)=XDDI(J)
      A1=6.0/((TH*DELTA)**2)

```



```

A2=3.0/(TH*DELTA)
A3=2.0*A2
A4=TH*DELTA/2.0
A5=A1/TH
A6=-A3/TH
A7=1.0-(3.0/TH)
A8=DELTA/2.0
A9=(DELTA**2)/6.0
DO 30 I=1,N
DO 30 J=1,N
30 TK(I,J)=K(I,J)+A1*M(I,J)+A2*C(I,J)
CALL SIMUL (TK,ZA,N,0,LA,LB,S)
DO 100 II=2,NSTEP1
DO 40 I=1,N
XN1(I)=A1*X(II-1,I)+A3*XD(II-1,I)+2.0*XDD(II-1,I)
40 XN2(I)=A2*X(II-1,I)+2.0*XD(II-1,I)+A4*XDD(II-1,I)
CALL XMULT (M,XN1,XN3,N)
CALL XMULT (C,XN2,XN4,N)
TIME1=REAL(II-2)*DELTA
TIME2=REAL(II-1)*DELTA
CALL EXTFUN (F1,TIME1,N)
CALL EXTFUN (F2,TIME2,N)
DO 45 J=1,N
45 FT(J)=F1(J)+TH*(F2(J)-F1(J))+XN3(J)+XN4(J)
CALL XMULT (TK,FT,XT,N)
DO 50 J=1,N
XDD(II,J)=A5*(XT(J)-X(II-1,J))+A6*XD(II-1,J)+A7*XDD(II-1,J)
XD(II,J)=XD(II-1,J)+A8*(XDD(II,J)+XDD(II-1,J))
50 X(II,J)=X(II-1,J)+DELTA*XD(II-1,J)+A9*(XDD(II,J)+2.0*XDD(II-1,J))
100 CONTINUE
RETURN
END

```

```

C =====
C
C SUBROUTINE EXTFUN
C
C =====
SUBROUTINE EXTFUN (F,TIME,N)
DIMENSION F(N)
F(1)=0.0
F(2)=10.0
RETURN
END

```

```

C =====
C
C SUBROUTINE XMULT
C
C =====
SUBROUTINE XMULT (A,B,BB,N)
DIMENSION A(N,N),B(N),BB(N)
DO 10 I=1,N
BB(I)=0.0
DO 10 J=1,N
10 BB(I)=BB(I)+A(I,J)*B(J)
RETURN
END

```

SOLUTION BY WILSON METHOD

GIVEN DATA:

N= 2 NSTEP= 24 TH= 0.14000000E+01 DELTA= 0.24216267E+00

SOLUTION:

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.3000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.5000E+01
2	0.2422	0.3344E-02	0.4143E-01	0.3422E+00	0.1392E+00	0.1119E+01	0.4244E+01
3	0.4843	0.2893E-01	0.1926E+00	0.9065E+00	0.5201E+00	0.1967E+01	0.2756E+01
4	0.7265	0.1072E+00	0.4742E+00	0.1419E+01	0.1058E+01	0.2395E+01	0.7809E+00
5	0.9687	0.2649E+00	0.8337E+00	0.1550E+01	0.1641E+01	0.2336E+01	-.1267E+01
6	1.2108	0.5076E+00	0.1152E+01	0.1076E+01	0.2153E+01	0.1825E+01	-.2958E+01
7	1.4530	0.8074E+00	0.1281E+01	-.4223E-02	0.2498E+01	0.9863E+00	-.3964E+01
8	1.6951	0.1104E+01	0.1106E+01	-.1442E+01	0.2619E+01	0.6096E-02	-.4131E+01
9	1.9373	0.1316E+01	0.5916E+00	-.2807E+01	0.2506E+01	-.9176E+00	-.3497E+01
10	2.1795	0.1369E+01	-.1866E+00	-.3621E+01	0.2193E+01	-.1615E+01	-.2266E+01
11	2.4216	0.1218E+01	-.1052E+01	-.3526E+01	0.1750E+01	-.1979E+01	-.7362E+00
12	2.6638	0.8710E+00	-.1772E+01	-.2420E+01	0.1264E+01	-.1974E+01	0.7743E+00
13	2.9060	0.3897E+00	-.2127E+01	-.5128E+00	0.8208E+00	-.1638E+01	0.2002E+01
14	3.1481	-.1186E+00	-.1981E+01	0.1717E+01	0.4905E+00	-.1059E+01	0.2781E+01
15	3.3903	-.5290E+00	-.1330E+01	0.3658E+01	0.3183E+00	-.3521E+00	0.3056E+01
16	3.6324	-.7333E+00	-.3125E+00	0.4747E+01	0.3207E+00	0.3643E+00	0.2860E+01
17	3.8746	-.6708E+00	0.8244E+00	0.4643E+01	0.4872E+00	0.9876E+00	0.2287E+01
18	4.1168	-.3477E+00	0.1791E+01	0.3340E+01	0.7852E+00	0.1440E+01	0.1451E+01
19	4.3589	0.1627E+00	0.2337E+01	0.1170E+01	0.1167E+01	0.1672E+01	0.4655E+00
20	4.6011	0.7389E+00	0.2323E+01	-.1288E+01	0.1575E+01	0.1660E+01	-.5639E+00
21	4.8433	0.1243E+01	0.1757E+01	-.3380E+01	0.1951E+01	0.1406E+01	-.1535E+01
22	5.0854	0.1558E+01	0.7952E+00	-.4566E+01	0.2239E+01	0.9371E+00	-.2339E+01
23	5.3276	0.1617E+01	-.3112E+00	-.4571E+01	0.2392E+01	0.3075E+00	-.2860E+01
24	5.5697	0.1418E+01	-.1283E+01	-.3457E+01	0.2382E+01	-.4004E+00	-.2987E+01
25	5.8119	0.1024E+01	-.1894E+01	-.1588E+01	0.2201E+01	-.1082E+01	-.2639E+01

11.39

The problem-dependent data to be used in the main program given in Problem 10.38 (which calls NUMARK), subroutine EXTFUN and output are given.

```

C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
  REAL M(2,2),MI(2,2),K(2,2)
  DIMENSION C(2,2),X1(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
  2 XT(2),F(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
  3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
  DATA N,NSTEP,NSTEP1,ALPHA,BETA,DELTA/2,24,25,0.16666667,0.5,
  2 0.24216267/
  DATA X1/0.0,0.0/
  DATA XDI/0.0,0.0/
  DATA M/1.0,0.0,0.0,2.0/
  DATA C/2.0,-2.0,-2.0,2.0/
  DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
  SUBROUTINE EXTFUN (F,TIME,N)
  DIMENSION F(N)
  F(1)=0.0
  F(2)=10.0
  RETURN
  END

```

SOLUTION BY NEWMARK METHOD

GIVEN DATA:

N= 2 NSIEP= 24 ALPHA= 0.16666667E+00 BETA= 0.50000000E+00
DELTA= 0.24216267E+00

SOLUTION:

STEP	TIME	X(1,1)	XD(1,1)	XDD(1,1)	X(1,2)	XD(1,2)	XDD(1,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.5000E+01
2	0.2422	0.1776E-01	0.2200E+00	0.1817E+01	0.1335E+00	0.1048E+01	0.3656E+01
3	0.4843	0.1287E+00	0.7143E+00	0.2266E+01	0.4799E+00	0.1753E+01	0.2170E+01
4	0.7265	0.3613E+00	0.1178E+01	0.1568E+01	0.9532E+00	0.2093E+01	0.6342E+00
5	0.9687	0.6793E+00	0.1393E+01	0.2007E+00	0.1464E+01	0.2067E+01	-.8511E+00
6	1.2108	0.1008E+01	0.1259E+01	-.1303E+01	0.1927E+01	0.1704E+01	-.2145E+01
7	1.4530	0.1263E+01	0.7991E+00	-.2497E+01	0.2268E+01	0.1071E+01	-.3080E+01
8	1.6951	0.1377E+01	0.1221E+00	-.3094E+01	0.2433E+01	0.2741E+00	-.3505E+01
9	1.9373	0.1317E+01	-.6142E+00	-.2987E+01	0.2398E+01	-.5540E+00	-.3334E+01
10	2.1795	0.1088E+01	-.1246E+01	-.2231E+01	0.2173E+01	-.1270E+01	-.2581E+01
11	2.4216	0.7330E+00	-.1639E+01	-.1013E+01	0.1802E+01	-.1748E+01	-.1365E+01
12	2.6638	0.3203E+00	-.1712E+01	0.4055E+00	0.1353E+01	-.1901E+01	0.9807E-01
13	2.9060	-.6943E-01	-.1453E+01	0.1736E+01	0.9094E+00	-.1703E+01	0.1543E+01
14	3.1481	-.3607E+00	-.9133E+00	0.2721E+01	0.5536E+00	-.1189E+01	0.2700E+01
15	3.3903	-.4977E+00	-.1994E+00	0.3176E+01	0.3513E+00	-.4559E+00	0.3354E+01
16	3.6324	-.4544E+00	0.5510E+00	0.3022E+01	0.3395E+00	0.3593E+00	0.3379E+01
17	3.8746	-.2394E+00	0.1195E+01	0.2295E+01	0.5196E+00	0.1104E+01	0.2773E+01
18	4.1168	0.1059E+00	0.1610E+01	0.1138E+01	0.8573E+00	0.1639E+01	0.1647E+01
19	4.3589	0.5158E+00	0.1720E+01	-.2284E+00	0.1289E+01	0.1865E+01	0.2165E+00
20	4.6011	0.9129E+00	0.1506E+01	-.1545E+01	0.1732E+01	0.1740E+01	-.1251E+01
21	4.8433	0.1222E+01	0.1008E+01	-.2564E+01	0.2105E+01	0.1288E+01	-.2478E+01
22	5.0854	0.1386E+01	0.3231E+00	-.3096E+01	0.2337E+01	0.5967E+00	-.3235E+01
23	5.3276	0.1374E+01	-.4202E+00	-.3043E+01	0.2385E+01	-.2046E+00	-.3382E+01
24	5.5697	0.1189E+01	-.1081E+01	-.2419E+01	0.2241E+01	-.9643E+00	-.2893E+01
25	5.8119	0.8670E+00	-.1537E+01	-.1342E+01	0.1933E+01	-.1540E+01	-.1862E+01

11.40

The problem-dependent data to be used in the main program given in Problem 10.38 (which calls NUMARK), subroutine EXTFUN and output are given.

C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA

```

REAL M(2,2),MI(2,2),K(2,2)
DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
2 XI(2),F(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
3 IK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
DATA N,NSIEP,NSTEP1,ALPHA,BETA,DELTA/2,24,25,0.16666667,0.5,
2 0.24216267/
DATA X1/0.0,0.0/
DATA XD1/0.0,0.0/
DATA M/1.0,0.0,0.0,2.0/
DATA C/0.0,0.0,0.0,0.0/
DATA K/6.0,-2.0,-2.0,8.0/

```

C END OF PROBLEM-DEPENDENT DATA

```

SUBROUTINE EXTFUN (F,TIME,N)
DIMENSION F(N)
F(1)=10.0*SIN(5.0*TIME)
F(2)=0.0
RETURN
END

```


SOLUTION BY NEWMARK METHOD

GIVEN DATA:

N= 2 NSTEP= 24 ALPHA= 0.16666667E+00 BETA= 0.50000000E+00
DELTA= 0.24216267E+00

SOLUTION:

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2422	0.8642E-01	0.1071E+01	0.8842E+01	0.8129E-03	0.1007E-01	0.8317E-01
3	0.4843	0.5509E+00	0.2542E+01	0.3308E+01	0.9875E-02	0.8206E-01	0.5114E+00
4	0.7265	0.1120E+01	0.1570E+01	-0.1134E+02	0.4878E-01	0.2560E+00	0.9251E+00
5	0.9687	0.1119E+01	-0.1784E+01	-0.1636E+02	0.1345E+00	0.4383E+00	0.5807E+00
6	1.2108	0.3303E+00	-0.4220E+01	-0.3762E+01	0.2457E+00	0.4296E+00	-0.6523E+00
7	1.4530	-0.6401E+00	-0.3127E+01	0.1279E+02	0.3182E+00	0.1190E+00	-0.1913E+01
8	1.6951	-0.1003E+01	0.2052E+00	0.1473E+02	0.2886E+00	-0.3739E+00	-0.2158E+01
9	1.9373	-0.6501E+00	0.2182E+01	0.1601E+01	0.1439E+00	-0.7835E+00	-0.1226E+01
10	2.1795	-0.1786E+00	0.1284E+01	-0.9018E+01	-0.6887E-01	-0.9202E+00	0.9693E-01
11	2.4216	-0.8717E-01	-0.3477E+00	-0.4458E+01	-0.2797E+00	-0.7836E+00	0.1031E+01
12	2.6638	-0.1894E+00	-0.2747E-01	0.7103E+01	-0.4342E+00	-0.4713E+00	0.1547E+01
13	2.9060	0.2207E-01	0.1815E+01	0.8110E+01	-0.4984E+00	-0.3990E-01	0.2016E+01
14	3.1481	0.5744E+00	0.2232E+01	-0.4664E+01	-0.4456E+00	0.4895E+00	0.2357E+01
15	3.3903	0.8748E+00	-0.1784E+00	-0.1524E+02	-0.2622E+00	0.1008E+01	0.1924E+01
16	3.6324	0.4459E+00	-0.3111E+01	-0.8975E+01	0.2292E-01	0.1284E+01	0.3542E+00
17	3.8746	-0.4039E+00	-0.3220E+01	0.8070E+01	0.3241E+00	0.1121E+01	-0.1700E+01
18	4.1168	-0.8682E+00	-0.2884E+00	0.1614E+02	0.5329E+00	0.5516E+00	-0.3000E+01
19	4.3589	-0.5593E+00	0.2449E+01	0.6467E+01	0.5797E+00	-0.1601E+00	-0.2878E+01
20	4.6011	0.8159E-01	0.2258E+01	-0.8046E+01	0.4672E+00	-0.7250E+00	-0.1787E+01
21	4.8433	0.3764E+00	0.1102E+00	-0.9692E+01	0.2506E+00	-0.1017E+01	-0.6259E+00
22	5.0854	0.2234E+00	-0.8793E+00	0.1519E+01	-0.5534E-02	-0.1063E+01	0.2506E+00
23	5.3276	0.1303E+00	0.3584E+00	0.8703E+01	-0.2470E+00	-0.8969E+00	0.1118E+01
24	5.5697	0.3961E+00	0.1522E+01	0.9086E+00	-0.4220E+00	-0.5091E+00	0.2084E+01
25	5.8119	0.6650E+00	0.1770E+00	-0.1202E+02	-0.4793E+00	0.5587E-01	0.2582E+01

11.41

The problem-dependent data to be used in the main program given in Problem 10.38 (which calls NUMARK), subroutine EXTFUN and results are given.

C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA

```

REAL M(2,2),MI(2,2),K(2,2)
DIMENSION C(2,2),XI(2),XD1(2),XDD1(2),X(25,2),XD(25,2),XDD(25,2),
2 XT(2),F(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
DATA N,NSTEP,NSTEP1,ALPHA,BETA,DELTA/
2 2,24,25,0.16666667,0.5,0.25/
DATA XI/0.0,0.0/
DATA XD1/0.0,0.0/
DATA M/2.0,0.0,0.0,1.0/
DATA C/0.0,0.0,0.0,0.0/
DATA K/6.0,-2.0,-2.0,4.0/

```

C END OF PROBLEM-DEPENDENT DATA

```

SUBROUTINE EXTFUN (F,TIME,N)
DIMENSION F(N)
F(1)=5.0
F(2)=20.0*SIN(5.0*TIME)
RETURN
END

```


SOLUTION BY NEWMARK METHOD

GIVEN DATA:

N= 2 NSTEP= 24 ALPHA= 0.16666667E+00 BETA= 0.50000000E+00

SOLUTION:

DELTA= 0.24216267E+00

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.2500E+01	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2500	0.7769E-01	0.6198E+00	0.2458E+01	0.1914E+00	0.2296E+01	0.1837E+02
3	0.5000	0.3129E+00	0.1276E+01	0.2789E+01	0.1228E+01	0.5553E+01	0.7683E+01
4	0.7500	0.7202E+00	0.1988E+01	0.2905E+01	0.2565E+01	0.3982E+01	-.2025E+02
5	1.0000	0.1293E+01	0.2534E+01	0.1469E+01	0.2847E+01	-.2048E+01	-.2798E+02
6	1.2500	0.1940E+01	0.2517E+01	-.1606E+01	0.1715E+01	-.6001E+01	-.3641E+01
7	1.5000	0.2488E+01	0.1742E+01	-.4594E+01	0.3703E+00	-.3674E+01	0.2226E+02
8	1.7500	0.2769E+01	0.4543E+00	-.5706E+01	0.9920E-01	0.1312E+01	0.1763E+02
9	2.0000	0.2712E+01	-.8750E+00	-.4927E+01	0.7083E+00	0.2480E+01	-.8290E+01
10	2.2500	0.2353E+01	-.1940E+01	-.3596E+01	0.9630E+00	-.8686E+00	-.1850E+02
11	2.5000	0.1768E+01	-.2694E+01	-.2435E+01	0.3681E+00	-.3089E+01	0.7367E+00
12	2.7500	0.1035E+01	-.3095E+01	-.7734E+00	-.1674E+00	-.3396E+00	0.2126E+02
13	3.0000	0.2664E+00	-.2940E+01	0.2019E+01	0.3183E+00	0.3851E+01	0.1227E+02
14	3.2500	-.3745E+00	-.2064E+01	0.4988E+01	0.1364E+01	0.3319E+01	-.1652E+02
15	3.5000	-.7216E+00	-.6614E+00	0.6231E+01	0.1566E+01	-.2150E+01	-.2722E+02
16	3.7500	-.7048E+00	0.7455E+00	0.5024E+01	0.4094E+00	-.6182E+01	-.5035E+01
17	4.0000	-.3862E+00	0.1704E+01	0.2642E+01	-.1016E+01	-.4117E+01	0.2155E+02
18	4.2500	0.1030E+00	0.2133E+01	0.7952E+00	-.1396E+01	0.9885E+00	0.1929E+02
19	4.5000	0.6503E+00	0.2201E+01	-.2522E+00	-.8012E+00	0.2745E+01	-.5238E+01
20	4.7500	0.1181E+01	0.1991E+01	-.1429E+01	-.3879E+00	0.1236E+00	-.1573E+02
21	5.0000	0.1617E+01	0.1437E+01	-.3003E+01	-.6516E+00	-.1444E+01	0.3194E+01
22	5.2500	0.1874E+01	0.5849E+00	-.3813E+01	-.6909E+00	0.2016E+01	0.2449E+02
23	5.5000	0.1912E+01	-.2353E+00	-.2748E+01	0.4886E+00	0.7060E+01	0.1586E+02
24	5.7500	0.1792E+01	-.6351E+00	-.4500E+00	0.2425E+01	0.7132E+01	-.1527E+02
25	6.0000	0.1636E+01	-.5461E+00	0.1163E+01	0.3569E+01	0.1377E+01	-.3077E+02

11.42

```

C =====
C
C MAIN PROGRAM WHICH CALLS NUMARK
C
C =====
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
  REAL M(2,2),MI(2,2),K(2,2)
  DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
2 XT(2),F(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
  DATA N,NSTEP,NSTEP1,ALPHA,BETA,DELTA/2,24,25,0.16666667,0.5,
2 0.24216267/
  DATA XI/0.0,0.0/
  DATA XDI/0.0,0.0/
  DATA M/1.0,0.0,0.0,2.0/
  DATA C/0.0,0.0,0.0,0.0/
  DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA

```

```

      CALL NUMARK (M,C,K,MI,XI,XDI,XDDI,X,XD,XDD,XT,F,R,LA,LB,S,ZA,
2 TK,XN1,XN2,XN3,XN4,NSTEP1,DELTA,ALPHA,BETA,N)
      PRINT 10
10  FORMAT (//,27H SOLUTION BY NEWMARK METHOD,/)
      PRINT 20, N,NSTEP,ALPHA,BETA,DELTA
20  FORMAT (12H GIVEN DATA:/,3H N=,15,4X,7H NSTEP=,15,4X,7H ALPHA=,
2 E15.8,2X,6H BETA=,E15.8,2X,7H DELTA=,E15.8,/)
      PRINT 30
30  FORMAT (10H SOLUTION:,,5H STEP,3X,5H TIME,3X,7H X(1,1),3X,
2 8H XD(1,1),2X,9H XDD(1,1),4X,7H X(1,2),3X,8H XD(1,2),2X,
3 9H XDD(1,2),/)
      DO 40 I=1,NSTEP1
      TIME=REAL(I-1)*DELTA
40  PRINT 50, I,TIME,X(1,1),XD(1,1),XDD(1,1),X(1,2),XD(1,2),XDD(1,2)
50  FORMAT (1X,14,F8.4,6(1X,E10.4))
      STOP
      END

C =====
C
C SUBROUTINE NUMARK
C
C =====
      SUBROUTINE NUMARK (M,C,K,MI,XI,XDI,XDDI,X,XD,XDD,XT,F,R,LA,LB,S,
2 ZA,TK,XN1,XN2,XN3,XN4,NSTEP1,DELTA,ALPHA,BETA,N)
      REAL M(N,N),MI(N,N),K(N,N)
      DIMENSION C(N,N),XI(N),XDI(N),XDDI(N),X(NSTEP1,N),XD(NSTEP1,N),
2 XDD(NSTEP1,N),XT(N),F(N),R(N),LA(N),LB(N,2),
3 S(N),ZA(N),TK(N,N),XN1(N),XN2(N),XN3(N),XN4(N)
      DO 5 I=1,N
      DO 5 J=1,N
5  MI(I,J)=M(I,J)
      CALL SIMUL (MI,ZA,N,0,LA,LB,S)
      CALL EXTFUN (F,0.0,N)
      DO 20 I=1,N
      R(I)=F(I)
      DO 10 J=1,N
10  R(I)=R(I)-C(I,J)*XDI(J)-K(I,J)*XI(J)
20  CONTINUE
      CALL XMULT (MI,R,XDDI,N)
      DO 25 J=1,N
      X(1,J)=XI(J)
      XD(1,J)=XDI(J)
25  XDD(1,J)=XDDI(J)
      A1=1.0/(ALPHA*(DELTA**2))
      A2=1.0/(ALPHA*DELTA)
      A3=(1.0/(2.0*ALPHA))-1.0
      A4=(1.0-BETA)*DELTA
      A5=8BETA*DELTA
      A6=BETA/(ALPHA*DELTA)
      A7=(BETA/ALPHA)-1.0
      A8=(A7-1.0)*DELTA/2.0
      DO 30 I=1,N
      DO 30 J=1,N
30  TK(I,J)=A1*M(I,J)+A6*C(I,J)+K(I,J)
      CALL SIMUL (TK,ZA,N,0,LA,LB,S)
      DO 100 II=2,NSTEP1

```

```

TIME=REAL(II-1)*DELTA
CALL EXTFUN (F,TIME,N)
DO 40 J=1,N
  XN1(J)=A1*X(II-1,J)+A2*XD(II-1,J)+A3*XDD(II-1,J)
40  XN2(J)=A6*X(II-1,J)+A7*XD(II-1,J)+A8*XDD(II-1,J)
  CALL XMULT (M,XN1,XN3,N)
  CALL XMULT (C,XN2,XN4,N)
  DO 45 J=1,N
45  XN1(J)=F(J)+XN3(J)+XN4(J)
  CALL XMULT (TK,XN1,XT,N)
  DO 50 J=1,N
  X(II,J)=XT(J)
  XDD(II,J)=A1*(X(II,J)-X(II-1,J))-A2*XD(II-1,J)-A3*XDD(II-1,J)
50  XD(II,J)=XD(II-1,J)+A4*XDD(II-1,J)+A5*XDD(II,J)
100 CONTINUE
  RETURN
  END

```

```

C =====
C
C SUBROUTINE EXTFUN
C
C =====

```

```

SUBROUTINE EXTFUN (F,TIME,N)
DIMENSION F(N)
F(1)=0.0
F(2)=10.0
RETURN
END

```

```

C =====
C
C SUBROUTINE XMULT
C
C =====

```

```

SUBROUTINE XMULT (A,B,BB,N)
DIMENSION A(N,N),B(N),BB(N)
DO 10 I=1,N
  BB(I)=0.0
  DO 10 J=1,N
10  BB(I)=BB(I)+A(I,J)*B(J)
RETURN
END

```

SOLUTION BY NEWMARK METHOD

GIVEN DATA:

N= 2 NSTEP= 24 ALPHA= 0.16666667E+00 BETA= 0.50000000E+00

SOLUTION:

DELTA= 0.24216267E+00

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.5000E+01
2	0.2422	0.2606E-02	0.3228E-01	0.2666E+00	0.1411E+00	0.1143E+01	0.4438E+01
3	0.4843	0.2461E-01	0.1757E+00	0.9181E+00	0.5329E+00	0.2030E+01	0.2893E+01
4	0.7265	0.1005E+00	0.4775E+00	0.1574E+01	0.1088E+01	0.2471E+01	0.7467E+00
5	0.9687	0.2644E+00	0.8845E+00	0.1788E+01	0.1687E+01	0.2382E+01	-.1483E+01

6	1.2108	0.5257E+00	0.1252E+01	0.1251E+01	0.2203E+01	0.1805E+01	-.3285E+01
7	1.4530	0.8530E+00	0.1398E+01	-.5063E-01	0.2534E+01	0.8884E+00	-.4281E+01
8	1.6951	0.1173E+01	0.1175E+01	-.1792E+01	0.2623E+01	-.1529E+00	-.4318E+01
9	1.9373	0.1389E+01	0.5460E+00	-.3400E+01	0.2467E+01	-.1097E+01	-.3481E+01
10	2.1795	0.1413E+01	-.3806E+00	-.4253E+01	0.2114E+01	-.1766E+01	-.2042E+01
11	2.4216	0.1200E+01	-.1369E+01	-.3914E+01	0.1643E+01	-.2058E+01	-.3706E+00
12	2.6638	0.7690E+00	-.2124E+01	-.2317E+01	0.1148E+01	-.1960E+01	0.1175E+01
13	2.9060	0.2111E+00	-.2384E+01	0.1724E+00	0.7195E+00	-.1536E+01	0.2333E+01
14	3.1481	-.3348E+00	-.2017E+01	0.2854E+01	0.4223E+00	-.8929E+00	0.2976E+01
15	3.3903	-.7196E+00	-.1078E+01	0.4907E+01	0.2946E+00	-.1570E+00	0.3102E+01
16	3.6324	-.8292E+00	0.2025E+00	0.5664E+01	0.3445E+00	0.5568E+00	0.2793E+01
17	3.8746	-.6221E+00	0.1475E+01	0.4843E+01	0.5550E+00	0.1156E+01	0.2158E+01
18	4.1168	-.1445E+00	0.2382E+01	0.2647E+01	0.8899E+00	0.1574E+01	0.1296E+01
19	4.3589	0.4811E+00	0.2667E+01	-.2884E+00	0.1299E+01	0.1766E+01	0.2842E+00
20	4.6011	0.1091E+01	0.2257E+01	-.3098E+01	0.1724E+01	0.1702E+01	-.8069E+00
21	4.8433	0.1529E+01	0.1281E+01	-.4966E+01	0.2103E+01	0.1377E+01	-.1882E+01
22	5.0854	0.1689E+01	0.2673E-01	-.5390E+01	0.2372E+01	0.8102E+00	-.2799E+01
23	5.3276	0.1548E+01	-.1150E+01	-.4326E+01	0.2480E+01	0.6281E-01	-.3374E+01
24	5.5697	0.1163E+01	-.1938E+01	-.2188E+01	0.2396E+01	-.7600E+00	-.3421E+01
25	5.8119	0.6543E+00	-.2166E+01	0.3099E+00	0.2118E+01	-.1515E+01	-.2817E+01

Chapter 12

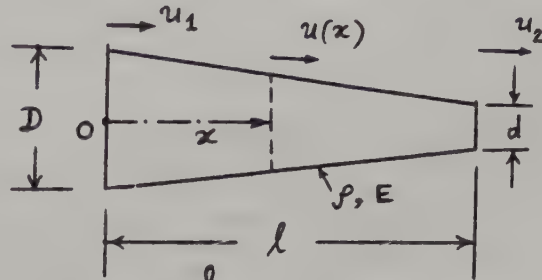
Finite Element Method

12.1

As diameter $h(x) = D$ at $x=0$
and $= d$ at $x=l$, we have

$$h(x) = D + \left(\frac{d-D}{l}\right)x$$

(i) stiffness matrix:



$$V(t) = \text{strain energy of element} = \frac{1}{2} \int_0^l EA \left(\frac{\partial u}{\partial x}\right)^2 dx$$

with

$$u(x,t) = \left(1 - \frac{x}{l}\right) u_1(t) + \left(\frac{x}{l}\right) u_2(t)$$

and

$$A(x) = \frac{\pi h^2}{4} = \frac{\pi}{4} \left[D^2 + \left(\frac{d-D}{l}\right)^2 x^2 + 2D\left(\frac{d-D}{l}\right)x \right]$$

Thus strain energy expression becomes

$$V = \frac{\pi E}{24 l} (D^2 + d^2 + dD) (u_1^2 + u_2^2 - 2 u_1 u_2)$$

$$= \frac{1}{2} \vec{u}^T [k] \vec{u} \equiv \frac{1}{2} (u_1 \quad u_2) \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

This gives the element stiffness matrix as

$$[k] = \frac{\pi E}{12 l} (D^2 + d^2 + dD) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(ii) consistent mass matrix:

$$T(t) = \text{kinetic energy of element} = \frac{1}{2} \int_0^l \rho A \left(\frac{\partial u}{\partial t}\right)^2 dx$$

with

$$\dot{u}(x,t) \equiv \frac{\partial u}{\partial t}(x,t) = \left(1 - \frac{x}{l}\right) \dot{u}_1(t) + \left(\frac{x}{l}\right) \dot{u}_2(t)$$

Substituting for $A(x)$ and $\dot{u} = \frac{\partial u}{\partial t}$, kinetic energy expression can be derived as

$$\begin{aligned}
 T &= \frac{1}{2} \int_0^l \frac{\pi}{4} \left\{ D^2 + \left(\frac{d-D}{l} \right)^2 x^2 + 2D \left(\frac{d-D}{l} \right) x \right\} \left\{ \dot{u}_1^2 + x \left(-\frac{2}{l} \dot{u}_1 + \frac{2}{l} \dot{u}_1 \dot{u}_2 \right) \right. \\
 &\quad \left. + x^2 \left(\frac{\dot{u}_1^2}{l^2} + \frac{\dot{u}_2^2}{l^2} - \frac{2 \dot{u}_1 \dot{u}_2}{l^2} \right) \right\} dx \\
 &= \frac{\pi \rho l}{8} \left\{ \dot{u}_1^2 \left(\frac{D^2}{5} + \frac{d^2}{30} + \frac{Dd}{10} \right) + \dot{u}_2^2 \left(\frac{D^2}{30} + \frac{d^2}{5} + \frac{Dd}{10} \right) \right. \\
 &\quad \left. + 2 \dot{u}_1 \dot{u}_2 \left(\frac{D^2}{20} + \frac{d^2}{20} + \frac{Dd}{15} \right) \right\} \\
 &= \frac{1}{2} \dot{\mathbf{u}}^T [\mathbf{m}_c] \dot{\mathbf{u}} \equiv \frac{1}{2} (\dot{u}_1 \quad \dot{u}_2) \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}
 \end{aligned}$$

This gives the consistent mass matrix as

$$[\mathbf{m}_c] = \frac{\pi \rho l}{4} \begin{bmatrix} \left(\frac{D^2}{5} + \frac{d^2}{30} + \frac{Dd}{10} \right) & \left(\frac{D^2}{20} + \frac{d^2}{20} + \frac{Dd}{15} \right) \\ \left(\frac{D^2}{20} + \frac{d^2}{20} + \frac{Dd}{15} \right) & \left(\frac{D^2}{30} + \frac{d^2}{5} + \frac{Dd}{10} \right) \end{bmatrix}$$

(iii) Lumped mass matrix:

$$\begin{aligned}
 \text{Total mass of element} &= \frac{\pi \rho}{4} \int_0^l h^2(x) \cdot dx \\
 &= \frac{\pi \rho}{4} \int_0^l \left\{ D^2 + \left(\frac{d-D}{l} \right)^2 x^2 + 2D \left(\frac{d-D}{l} \right) x \right\} \cdot dx \\
 &= \frac{\pi \rho l}{12} (D^2 + d^2 + Dd)
 \end{aligned}$$

Distributing the mass at the two nodes,

$$[\mathbf{m}_l] = \frac{\pi \rho l}{24} \begin{bmatrix} (D^2 + d^2 + Dd) & 0 \\ 0 & (D^2 + d^2 + Dd) \end{bmatrix}$$

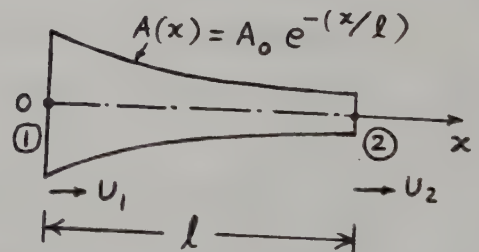
12.2 Assume linear displacement model

$$u(x) = \alpha_1 + \alpha_2 x = U_1 + \left(\frac{U_2 - U_1}{l} \right) x$$

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{U_2 - U_1}{l}$$

$$\sigma_x = E \epsilon_x = E \left(\frac{U_2 - U_1}{l} \right)$$

$$\text{strain energy} = \iiint_{V(e)} \frac{1}{2} \sigma_x \epsilon_x dV = \frac{1}{2} \int_{x=0}^l E \left(\frac{U_2 - U_1}{l} \right)^2 A(x) dx$$



$$= \frac{E}{2l^2} (u_2 - u_1)^2 \int_{x=0}^l A_0 \cdot e^{-\left(\frac{x}{l}\right)} \cdot dx = \frac{E}{2l^2} (u_2 - u_1)^2 A_0 \left[-l \cdot e^{-\frac{x}{l}} \right]_0^l$$

$$= \frac{E}{2l^2} (u_2 - u_1)^2 l A_0 \left(1 - \frac{1}{e} \right) = \frac{1}{2} \frac{EA_0}{l} (0.6321) (u_2 - u_1)^2$$

$$\equiv \frac{1}{2} \vec{U}^T [k] \vec{U} = \frac{1}{2} (u_1 \ u_2) [k] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore [k] = \frac{EA_0}{l} (0.6321) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

12.3

```

=====
C
C      PROBLEM 12.3   PROGRAM FOR STRESS ANALYSIS OF PLANAR TRUSSES
C
C=====
C      DATA FOR PROBLEM 12.10 (TEST EXAMPLE)
C      DIMENSION A(4),EL(4),GS(8,3),PP(8,1),P(8,1),GSS(4,4)
10 2 ,STRS(4,1),X(4),Y(4),LOC(4,2),IFIX(4)
C      DOUBLE PRECISION DIFF(2)
C      DATA LOC/1,3,3,2,3,2,4,4/
C      M=1
C      DATA NN,NE,ND,NB,NFIX,E/4,4,8,4,4,30.0E+6/
C      DATA IFIX/1,2,3,4/
C      DATA X/C.,100.,50.,200./
C      DATA Y/D.,7.,25.,100./
C      DO 10 I=1,8
10  P(I,1)=0.0
C      P(8,1)=-1000.0
C      DATA A/2.,2.,1.,1./
C      END OF PROBLEM-DEPENDENT DATA
C      CALL TRUSS (NN,NE,ND,NB,M,LOC,X,Y,E,A,EL,NFIX,IFIX,P,GS,DIFF,
2  GSS,ND2)
C      PRINT 21
21  FORMAT (2X,20H NODAL DISPLACEMENTS,/)
C      DO 11 I=1,ND
C      PP(I,1)=P(I,1)
11  PRINT 12, I, P(I,1)
12  FORMAT (4X,I5,2X,E15.6)
C      DO 45 I=1,NFIX
C      DO 45 J=1,M
C      II=IFIX(I)
C      PP(II,J)=0.0
45  CONTINUE
C      PRINT 22
22  FORMAT (//,2X, 21H STRESSES IN ELEMENTS,/)
C      DO 80 K=1,NE
C      DO 70 KK=1,M
C      I=LOC(K,1)
C      J=LOC(K,2)
C      I1=2*I-2+1
C      I2=I1+1

```

```

      I3=2*J-2+1
      I4=I3+1
      XL=(X(J)-X(I))/EL(K)
      XM=(Y(J)-Y(I))/EL(K)
      STRS(K, KK)=(E/EL(K))*(XL*(PP(I3, KK)-PP(I1, KK))+XM*(PP(I4, KK)-
2 PP(I2, KK)))
70  CONTINUE
80  CONTINUE
    DO 120 I=1, NE
120 PRINT 130, I, LOC(I,1), LOC(I,2), STRS(I,1)
130  FORMAT (2X, 3I4, 2X, 2E15.8)
      STOP
      END

C =====
C
C SUBROUTINE TRUSS
C
C =====
C      NN = NUMBER OF NODES,  NE = NUMBER OF ELEMENTS
C      ND = NUMBER OF DEGREES OF FREEDOM,  NB = SEMI-BANDWIDTH
C      M = NUMBER OF LOAD CONDITIONS,  LOC = NODE CONNECTIVITY MATRIX
C      X,Y = X - AND Y- COORDINATES OF NODES
C      E = YOUNGS MODULUS,  A = AREAS OF CROSS SECTION OF ELEMENTS
C      EL = LENGTHS OF ELEMENTS,  NFIX = NUMBER OF D.O.F. WHICH ARE FIXED
C      IFIX = FIXED DEGREES OF FREEDOM NUMBERS,  P = LOAD VECTOR
C      SUBROUTINE TRUSS (NN, NE, ND, NB, M, LOC, X, Y, E, A, EL, NFIX, IFIX, P, GS,
2 DIFF, GSS, ND2)
      DIMENSION LOC(NE,2), X(NN), Y(NN), A(NE), EL(NE), IFIX(NFIX),
2 P(ND, M), GS(ND, NB), GSS(ND2, ND2)
      DIMENSION B(4,4), N(4)
      DOUBLE PRECISION DIFF(M)
      DO 5 I=1, ND
      DO 5 J=1, NB
5      GS(I, J)=0.0
      DO 6 I=1, ND2
      DO 6 J=1, ND2
6      GSS(I, J)=0.0
      DO 200 K=1, NE
      I=LOC(K,1)
      J=LOC(K,2)
      EL(K)=SQRT((X(J)-X(I))**2+(Y(J)-Y(I))**2)
      CON=A(K)*E/EL(K)
      XL=(X(J)-X(I))/EL(K)
      XM=(Y(J)-Y(I))/EL(K)
      B(1,1)=XL**2
      B(1,2)=XL*XM
      B(1,3)=-B(1,1)
      B(1,4)=-B(1,2)
      B(2,2)=XM**2
      B(2,3)=-XL*XM
      B(2,4)=-XM**2
      B(3,3)=XL**2
      B(3,4)=XL*XM
      B(4,4)=XM**2
      DO 10 II=1, 4
      DO 10 JJ=1, II

```



```

10  B(II,JJ)=B(JJ,II)
    DO 20 II=1,4
    DO 20 JJ=1,4
20  B(II,JJ)=B(II,JJ)*CON
    DO 90 II=1,2
    N(II)=2*I-2+II
90  N(II+2)=2*J-2+II
    DO 100 II=1,4
    DO 100 JJ=1,4
    IK=N(II)
    JK=N(JJ)
    IF (IK .GT. ND2 .OR. JK .GT. ND2) GO TO 91
    GSS(IK,JK)=GSS(IK,JK)+B(II,JJ)
91  CONTINUE
    IN=JK-IK+1
    IF (IN .LE. 0) GO TO 100
    GS(IK,IN)=GS(IK,IN)+B(II,JJ)
100 CONTINUE
200 CONTINUE
    DO 110 II=1,NFIX
    IX=IFIX(II)
110  GS(IX,1)=GS(IX,1)+1.0E6
    CALL DECOMP (ND,NB,GS)
    CALL SOLVE (ND,NB,M,GS,P,DIFF)
    RETURN
    END

```

NODAL DISPLACEMENTS

1	-0.116462e-14
2	0.232924e-08
3	-0.128973e-14
4	0.170199e-07
5	0.116462e-02
6	0.232925e-02
7	0.514656e-01
8	-0.703219e-01

STRESSES IN ELEMENTS

1	1	3	0.11180378e+04
2	3	2	0.38109762e-02
3	3	4	0.22360752e+04
4	2	4	-0.28284324e+04

(a) Derivation of
element stiffness
matrix:

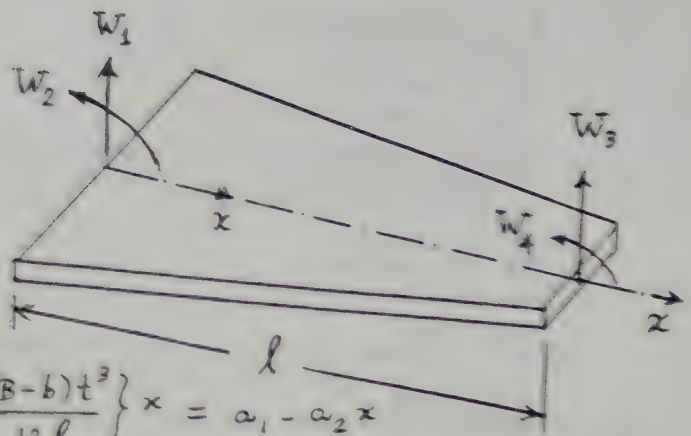
At x ,

thickness = t

$$\text{width} = w(x) = B - \left(\frac{B-b}{l} \right) x$$

$$I(x) = \frac{1}{12} w(x) t^3 = \frac{Bt^3}{12} - \left\{ \frac{(B-b)t^3}{12l} \right\} x = a_1 - a_2 x$$

$$\text{with } a_1 = \frac{Bt^3}{12} \text{ and } a_2 = \frac{(B-b)t^3}{12l}$$



Deflection of beam

$$w(x) = \sum_{i=1}^4 N_i(x) \cdot W_i \quad (E_1)$$

where $N_i(x)$ are defined by Eqs. (12.33) - (12.36).

Strain energy of element is given by

$$V = \frac{1}{2} \int_0^l EI(x) \left\{ \frac{d^2 w(x)}{dx^2} \right\}^2 dx \quad (E_2)$$

$$\text{with } \frac{d^2 N_1}{dx^2} = -\frac{6}{l^2} + \frac{12}{l^3} x, \quad \frac{d^2 N_2}{dx^2} = -\frac{4}{l} + \frac{6}{l^2} x,$$

$$\frac{d^2 N_3}{dx^2} = \frac{6}{l^2} - \frac{12}{l^3} x, \quad \frac{d^2 N_4}{dx^2} = -\frac{2}{l} + \frac{6}{l^2} x$$

$$\left\{ \frac{d^2 w(x)}{dx^2} \right\}^2 = c_1^2 + c_2^2 x^2 + 2c_1 c_2 x = (c_1 + c_2 x)^2 \quad (E_3)$$

$$\text{where } c_1 = -\frac{6}{l^2} W_1 - \frac{4}{l} W_2 + \frac{6}{l^2} W_3 - \frac{2}{l} W_4$$

$$\text{and } c_2 = \frac{12}{l^3} W_1 + \frac{6}{l^2} W_2 - \frac{12}{l^3} W_3 + \frac{6}{l^2} W_4$$

Integration in Eq. (E₂) gives

$$\begin{aligned} V = \frac{1}{2} E \bigg\{ & W_1^2 \left[\frac{36}{l^4} (a_1 l - a_2 l^2/2) - \frac{72}{l^5} (a_1 l^2 - 2a_2 l^3/3) \right. \\ & + \left. \frac{144}{l^6} (a_1 l^3/3 - a_2 l^4/4) \right] + W_2^2 \left[\frac{16}{l^2} (a_1 l - a_2 l^2/2) \right. \\ & - \left. \frac{24}{l^3} (a_1 l^2 - 2a_2 l^3/3) + \frac{36}{l^4} (a_1 l^3/3 - a_2 l^4/4) \right] \\ & + W_3^2 \left[\frac{36}{l^4} (a_1 l - a_2 l^2/2) - \frac{72}{l^5} (a_1 l^2 - 2a_2 l^3/3) \right. \\ & + \left. \frac{144}{l^6} (a_1 l^3/3 - a_2 l^4/4) \right] + W_4^2 \left[\frac{4}{l^2} (a_1 l - a_2 l^2/2) \right. \\ & - \left. \frac{12}{l^3} (a_1 l^2 - 2a_2 l^3/3) + \frac{36}{l^4} (a_1 l^3/3 - a_2 l^4/4) \right] \end{aligned}$$

$$\begin{aligned}
& + 2W_1W_2 \left[\frac{24}{l^3} (a_1 l - a_2 l^2/2) - \frac{42}{l^4} (a_1 l^2 - 2a_2 l^3/3) \right. \\
& + \left. \frac{72}{l^5} (a_1 l^3/3 - a_2 l^4/4) \right] + 2W_1W_3 \left[-\frac{36}{l^4} (a_1 l - a_2 l^2/2) \right. \\
& + \left. \frac{72}{l^5} (a_1 l^2 - 2a_2 l^3/3) - \frac{144}{l^6} (a_1 l^3/3 - a_2 l^4/4) \right] \\
& + 2W_1W_4 \left[\frac{12}{l^3} (a_1 l - a_2 l^2/2) - \frac{30}{l^4} (a_1 l^2 - 2a_2 l^3/3) \right. \\
& + \left. \frac{72}{l^5} (a_1 l^3/3 - a_2 l^4/4) \right] + 2W_2W_3 \left[-\frac{24}{l^3} (a_1 l - a_2 l^2/2) \right. \\
& + \left. \frac{42}{l^4} (a_1 l^2 - 2a_2 l^3/3) - \frac{72}{l^5} (a_1 l^3/3 - a_2 l^4/4) \right] \\
& + 2W_2W_4 \left[\frac{8}{l^2} (a_1 l - a_2 l^2/2) - \frac{18}{l^3} (a_1 l^2 - 2a_2 l^3/3) \right. \\
& + \left. \frac{36}{l^4} (a_1 l^3/3 - a_2 l^4/4) \right] + 2W_3W_4 \left[-\frac{12}{l^3} (a_1 l - a_2 l^2/2) \right. \\
& + \left. \frac{30}{l^4} (a_1 l^2 - 2a_2 l^3/3) - \frac{72}{l^5} (a_1 l^3/3 - a_2 l^4/4) \right] \} \quad (E_4)
\end{aligned}$$

By writing $V = \frac{1}{2} \vec{W}^T [k] \vec{W}$

with $\vec{W} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix}$, the stiffness matrix can be identified.

Defining $\alpha_1 = \frac{Bt^3}{12}$, $d_1 = \frac{(B-b)t^3}{12}$, $a_2 = \frac{d_1}{l}$, $a_2 l = d_1$, the elements of $[k]$ can be expressed as:

$$\begin{aligned}
k_{11} &= E \left\{ \alpha_1 \left(\frac{12}{l^3} \right) - d_1 \left(\frac{6}{l^3} \right) \right\}, \quad k_{22} = E \left\{ \alpha_1 \left(\frac{4}{l} \right) - d_1 \left(\frac{1}{l} \right) \right\}, \\
k_{33} &= E \left\{ \alpha_1 \left(\frac{12}{l^3} \right) - d_1 \left(\frac{6}{l^3} \right) \right\}, \quad k_{44} = E \left\{ \alpha_1 \left(\frac{4}{l} \right) - d_1 \left(\frac{3}{l} \right) \right\}, \\
k_{12} &= E \left\{ \alpha_1 \left(\frac{6}{l^2} \right) - d_1 \left(\frac{2}{l^2} \right) \right\}, \quad k_{13} = E \left\{ \alpha_1 \left(-\frac{12}{l^3} \right) + d_1 \left(\frac{6}{l^3} \right) \right\}, \\
k_{14} &= E \left\{ \alpha_1 \left(\frac{6}{l^2} \right) - d_1 \left(\frac{4}{l^2} \right) \right\}, \quad k_{23} = E \left\{ \alpha_1 \left(-\frac{6}{l^2} \right) + d_1 \left(\frac{2}{l^2} \right) \right\}, \\
k_{24} &= E \left\{ \alpha_1 \left(\frac{2}{l} \right) - d_1 \left(\frac{1}{l} \right) \right\}, \quad k_{34} = E \left\{ \alpha_1 \left(-\frac{6}{l^2} \right) + d_1 \left(\frac{4}{l^2} \right) \right\}
\end{aligned}$$

(b) Stresses induced in the beam:

$$\begin{aligned}
B &= 0.25 \text{ m}, \quad b = 0.10 \text{ m}, \quad t = 0.025 \text{ m}, \quad l = 2 \text{ m}, \quad E = 2.07 \times 10^{11} \text{ N/m}^2, \\
P &= 1000 \text{ N}, \quad \alpha_1 = 32552.0833 \times 10^{-11}, \quad d_1 = 19531.25 \times 10^{-11}.
\end{aligned}$$

stiffness matrix can be computed as:

$$[K] = \begin{matrix} & \begin{matrix} w_1 & w_2 & w_3 & w_4 \end{matrix} \\ \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix} & \begin{bmatrix} 70750 & 80860 & -70750 & 60640 \\ 80860 & 114600 & -80860 & 47170 \\ -70750 & -80860 & 70750 & -60640 \\ 60640 & 47170 & -60640 & 74120 \end{bmatrix} \end{matrix}$$

Equilibrium equations:

$$[K] \vec{W} = \vec{F}$$

i.e. $\begin{bmatrix} 70750 & -60640 \\ -60640 & 74120 \end{bmatrix} \begin{Bmatrix} w_3 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1000 \end{Bmatrix}$

The solution of these equations is

$$w_3 = 0.03871 \text{ m}, \quad w_4 = 0.04516 \text{ rad}$$

stress at root:

$$\sigma_{\max} \Big|_{x=0} = \frac{M y_{\max}}{I} \Big|_{x=0} = \frac{EI(x) \frac{d^2 w(x)}{dx^2} y_{\max}}{I(x)} \Big|_{x=0}$$

$$= E \frac{d^2 w(x)}{dx^2} \frac{t}{2} \Big|_{x=0}$$

$$= \frac{Et}{2} \left[-\frac{6}{l^2} w_1^0 - \frac{4}{l} w_2^0 + \frac{6}{l^2} w_3 - \frac{2}{l} w_4 \right]$$

$$= \frac{Et}{2} \left[\frac{6}{l^2} w_3 - \frac{2}{l} w_4 \right]$$

$$= \frac{(2.07 \times 10^{11})(0.025)}{2} \left\{ \frac{6}{4} \times 0.03871 - \frac{2}{2} \times 0.04516 \right\}$$

$$= 3.3392 \times 10^7 \text{ N/m}^2$$

12.5

$$A^{(i)} = 2 \text{ in}^2; \quad i = 1, 2, 3, 4$$

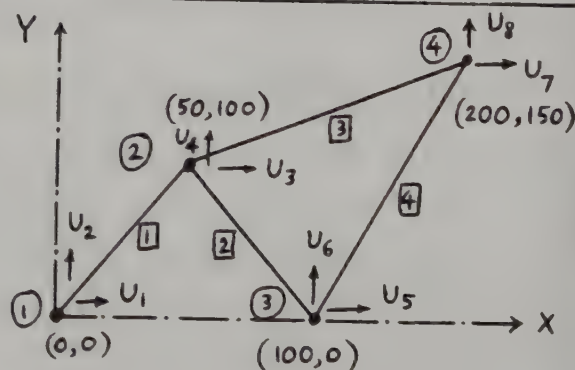
$$E = 30 \times 10^6 \text{ psi}$$

$$l^{(1)} = \sqrt{50^2 + 100^2} = 111.8034 \text{ in}$$

$$l^{(2)} = \sqrt{50^2 + 100^2} = 111.8034 \text{ in}$$

$$l^{(3)} = \sqrt{150^2 + 50^2} = 158.1139 \text{ in}$$

$$l^{(4)} = \sqrt{100^2 + 150^2} = 180.2776 \text{ in}$$



$$[k^i] = \frac{E^{(i)} A^{(i)}}{l^{(i)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

From Eq. (12.52), the global stiffness matrix of element i is

$$[\bar{k}^{(i)}] = [\lambda^{(i)}]^T [k^{(i)}] [\lambda^{(i)}] \quad \text{--- (E}_1\text{)}$$

where $[\lambda^{(i)}] = \begin{bmatrix} \cos \theta_i & \sin \theta_i & 0 & 0 \\ 0 & 0 & \cos \theta_i & \sin \theta_i \end{bmatrix}$ --- (E₂)

and θ_i is the angle made by the element with respect to X-axis.

Thus (E₁) gives

$$[\bar{k}^{(i)}] = \frac{E^{(i)} A^{(i)}}{l^{(i)}} \begin{bmatrix} \cos^2 \theta_i & \cos \theta_i \sin \theta_i & -\cos^2 \theta_i & -\cos \theta_i \sin \theta_i \\ \cos \theta_i \sin \theta_i & \sin^2 \theta_i & -\cos \theta_i \sin \theta_i & -\sin^2 \theta_i \\ -\cos^2 \theta_i & -\cos \theta_i \sin \theta_i & \cos^2 \theta_i & \cos \theta_i \sin \theta_i \\ -\cos \theta_i \sin \theta_i & -\sin^2 \theta_i & \cos \theta_i \sin \theta_i & \sin^2 \theta_i \end{bmatrix}$$

Here $\theta_1 = 63.4349^\circ$ (line ① ②), $\theta_2 = 116.5651^\circ$ (line ③ ②),

$\theta_3 = 18.4350^\circ$ (line ② ④), $\theta_4 = 56.3099^\circ$ (line ③ ④)

$$\frac{E^{(1)} A^{(1)}}{l^{(1)}} = \frac{(30 \times 10^6)(2)}{111.8034} = 0.53666 \times 10^6 \text{ lbf/in}, \cos \theta_1 = 0.4472, \sin \theta_1 = 0.8944$$

$$[\bar{k}^{(1)}] = \begin{bmatrix} 107332 & 214664 & -107332 & -214664 \\ 214664 & 429328 & -214664 & -429328 \\ -107332 & -214664 & 107332 & 214664 \\ -214664 & -429328 & 214664 & 429328 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix}$$

$$\frac{E^{(2)} A^{(2)}}{l^{(2)}} = \frac{(30 \times 10^6)(2)}{111.8034} = 0.53666 \times 10^6 \text{ lbf/in}, \cos \theta_2 = -0.4472, \sin \theta_2 = 0.8944$$

$$[\bar{k}^{(2)}] = \begin{bmatrix} 107332 & -214664 & -107332 & 214664 \\ -214664 & 429328 & 214664 & -429328 \\ -107332 & 214664 & 107332 & -214664 \\ 214664 & -429328 & -214664 & 429328 \end{bmatrix} \begin{matrix} U_5 \\ U_6 \\ U_3 \\ U_4 \end{matrix}$$

$$\frac{E^{(3)} A^{(3)}}{l^{(3)}} = \frac{(30 \times 10^6)(2)}{158.1139} = 0.37947 \times 10^6 \text{ lbf/in}, \cos \theta_3 = 0.9487, \sin \theta_3 = 0.3162$$

$$[\bar{k}^{(3)}] = \begin{matrix} & \begin{matrix} U_3 & U_4 & U_7 & U_8 \end{matrix} \\ \begin{matrix} U_3 \\ U_4 \\ U_7 \\ U_8 \end{matrix} & \begin{bmatrix} 341526 & 113842 & -341526 & -113842 \\ 113842 & 37947 & -113842 & -37947 \\ -341526 & -113842 & 341526 & 113842 \\ -113842 & -37947 & 113842 & 37947 \end{bmatrix} \end{matrix}$$

$$\frac{E^{(4)} A^{(4)}}{l^{(4)}} = \frac{(30 \times 10^6)(2)}{180.2776} = 0.33282 \times 10^6 \text{ lbf/in}, \cos \theta_4 = 0.5547, \sin \theta_4 = 0.8321$$

$$[\bar{k}^{(4)}] = \begin{matrix} & \begin{matrix} U_5 & U_6 & U_7 & U_8 \end{matrix} \\ \begin{matrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{matrix} & \begin{bmatrix} 102405 & 153810 & -102405 & -153810 \\ 153810 & 230415 & -153810 & -230415 \\ -102405 & -153810 & 102405 & 153810 \\ -153810 & -230415 & 153810 & 230415 \end{bmatrix} \end{matrix}$$

$$\textcircled{12.6} \quad \vec{U}^{(1)} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_8 \end{Bmatrix} \equiv [A^{(1)}] \vec{U}$$

$$\vec{U}^{(2)} = \begin{Bmatrix} U_5 \\ U_6 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_8 \end{Bmatrix} \equiv [A^{(2)}] \vec{U}$$

$$\vec{U}^{(3)} = \begin{Bmatrix} U_3 \\ U_4 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_8 \end{Bmatrix} \equiv [A^{(3)}] \vec{U}$$

$$\vec{U}^{(4)} = \begin{Bmatrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_8 \end{Bmatrix} \equiv [A^{(4)}] \vec{U}$$

Assembled stiffness matrix of the truss is given by Eq. (12.63):

$$[\bar{k}] = \sum_{e=1}^4 [A^{(e)}]^T [\bar{k}^{(e)}] [A^{(e)}]$$

	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	
	107332	214664	-107332	-214664	0	0	0	0	U_1
	214664	429328	-214664	-429328	0	0	0	0	U_2
	-107332	-214664	107332 + 107332 + 341526	214664 - 214664 + 113842	-107332	214664	-341526	-113842	U_3
	-214664	-429328	214664 - 214664 + 113842	429328 + 429328 + 37947	214664	-429328	-113842	-37947	U_4
=	0	0	-107332	214664	107332 + 102405	-214664 + 153810	-102405	-153810	U_5
	0	0	214664	-429328	-214664 + 153810	429328 + 230415	-153810	-230415	U_6
	0	0	-341526	-113842	-102405	-153810	341526 + 102405	113842 + 153810	U_7
	0	0	-113842	-37947	-153810	-230415	113842 + 153810	37947 + 230415	U_8

Since nodes ① and ③ are fixed, $U_1 = U_2 = U_5 = U_6 = 0$ and the final equilibrium equations can be expressed as

$$[K] \vec{U} = \vec{F} \quad \text{where}$$

$$[K] = \begin{bmatrix} 556190 & 113842 & -341526 & -113842 \\ 113842 & 896603 & -113842 & -37947 \\ -341526 & -113842 & 443931 & 267652 \\ -113842 & -37947 & 267652 & 268362 \end{bmatrix} \text{ lbf/in, } \vec{U} = \begin{Bmatrix} U_3 \\ U_4 \\ U_7 \\ U_8 \end{Bmatrix} \text{ in,}$$

$$\text{and } \vec{F} = \begin{Bmatrix} F_3 \\ F_4 \\ F_7 \\ F_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1000 \end{Bmatrix} \text{ lbf.}$$

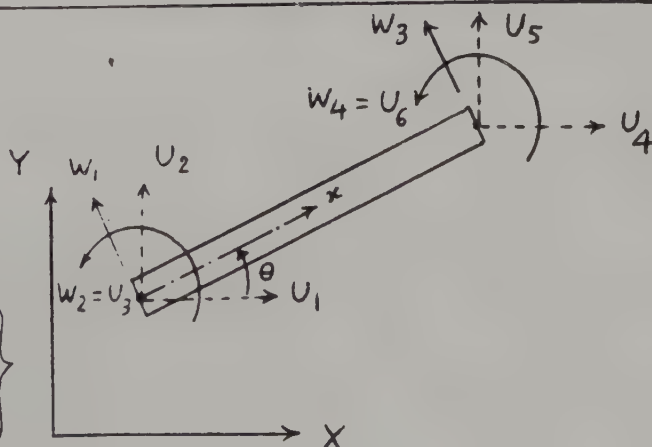
12.7

$$w_1 = U_1 \cos(90^\circ + \theta) + U_2 \cos \theta + U_3(0)$$

$$w_2 = U_1(0) + U_2(0) + U_3(1)$$

i.e.,

$$\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \begin{bmatrix} -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$



$$\vec{W} = [\lambda] \vec{U}$$

where $\vec{W} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{Bmatrix}$, $\vec{U} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$, and

$$[\lambda]_{4 \times 6} = \begin{bmatrix} -\sin \theta & \cos \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

For a beam element with degrees of freedom $\vec{W}$,

$$[k^{(e)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

and

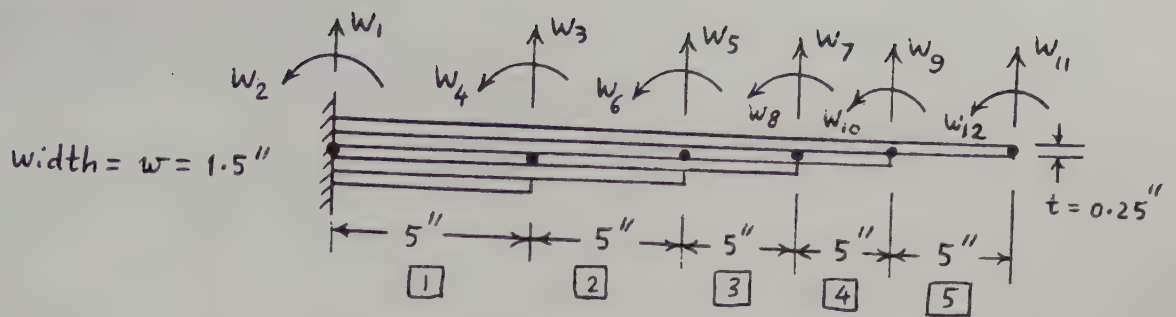
$$[m^{(e)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

Matrices in global system (X, Y system) are given by

$$[\bar{k}^{(e)}] = [\lambda]^T [k^{(e)}] [\lambda]$$

and

$$[\bar{m}^{(e)}] = [\lambda]^T [m^{(e)}] [\lambda]$$



$$[K^{(e)}] = \frac{E^{(e)} I^{(e)}}{l^{(e)3}} \begin{bmatrix} 12 & 6l^{(e)} & -12 & 6l^{(e)} \\ 6l^{(e)} & 4l^{(e)2} & -6l^{(e)} & 2l^{(e)2} \\ -12 & -6l^{(e)} & 12 & -6l^{(e)} \\ 6l^{(e)} & 2l^{(e)2} & -6l^{(e)} & 4l^{(e)2} \end{bmatrix}$$

$$E^{(e)} = E = 30 \times 10^6 \text{ psi}; \quad e = 1 \text{ to } 5$$

$$l^{(e)} = l = 5''; \quad e = 1 \text{ to } 5$$

$$I^{(1)} = \frac{1}{12} (1.5) (1.25)^3 = 0.24414 \text{ in}^4 = 125 I_5$$

$$I^{(2)} = \frac{1}{12} (1.5) (1.00)^3 = 0.125 \text{ in}^4 = 64 I_5$$

$$I^{(3)} = \frac{1}{12} (1.5) (0.75)^3 = 0.05273 \text{ in}^4 = 27 I_5$$

$$I^{(4)} = \frac{1}{12} (1.5) (0.5)^3 = 0.01563 \text{ in}^4 = 8 I_5$$

$$I^{(5)} = \frac{1}{12} (1.5) (0.25)^3 = 1.953125 \times 10^{-3} \text{ in}^4 = I_5$$

Assembled stiffness matrix, after applying the boundary conditions, is given by Eq. (E₁).

Load vector:

$$\vec{F} = \begin{Bmatrix} F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2000 \\ 0 \end{Bmatrix} \quad \text{lb} \quad (E_2)$$

$$[K] = \frac{EI_5}{l^3} \begin{bmatrix} w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 & w_{10} & w_{11} & w_{12} \\ 2268 & -366l & -768 & 384l & 0 & 0 & 0 & 0 & 0 & 0 \\ -366l & 756l^2 & -384l & 128l^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -768 & -384l & 1092 & -222l & -324 & 162l & 0 & 0 & 0 & 0 \\ 384l & 128l^2 & -222l & 364l^2 & -162l & 54l^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -324 & -162l & 420 & -114l & -96 & 48l & 0 & 0 \\ 0 & 0 & 162l & 54l^2 & -114l & 140l^2 & -48l & 16l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -96 & -48l & 108 & -42l & -12 & 6l \\ 0 & 0 & 0 & 0 & 48l & 16l^2 & -42l & 36l^2 & -6l & 2l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 0 & 0 & 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_{10} \\ w_{11} \\ w_{12} \end{matrix} \quad (E_1)$$

with $E = 30 \times 10^6$ psi, $I_5 = 1.953125 \times 10^{-3}$ in⁴ and $l = 5$ in.

Solution of equilibrium equations, $[K] \vec{w} = \vec{F}$, gives

$$\begin{aligned} \vec{w}^T &= \{w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}\}^T \\ &= \{-0.07964, -0.03072, -0.3554, -0.07738, \\ &\quad -0.9531, -0.1564, -2.179, -0.3164, -5.184, -0.7431\}^T \text{ inch.} \end{aligned}$$

12.9

Assembled stiffness matrix is given in the solution of Problem 12.8.

For the assembled mass matrix, we note

$$[m^{(e)}] = \frac{\rho^{(e)} A^{(e)} l^{(e)}}{420} \begin{bmatrix} 156 & 22l^{(e)} & 54 & -13l^{(e)} \\ 22l^{(e)} & 4l^{(e)2} & 13l^{(e)} & -3l^{(e)2} \\ 54 & 13l^{(e)} & 156 & -22l^{(e)} \\ -13l^{(e)} & -3l^{(e)2} & -22l^{(e)} & 4l^{(e)2} \end{bmatrix}$$

with

$$\rho^{(e)} = \frac{0.283}{386.4} = 7.324 \times 10^{-4} \text{ lb-sec}^2/\text{in}^4; e = 1 \text{ to } 5$$

$$l^{(e)} = l = 5''; e = 1 \text{ to } 5$$

$$A^{(1)} = 1.5 \times 1.25 = 1.875 \text{ in}^2 = 5A_5; \quad A^{(2)} = 1.5 \times 1.0 = 1.5 \text{ in}^2 = 4A_5$$

$$A^{(3)} = 1.5 \times 0.75 = 1.125 \text{ in}^2 = 3A_5; \quad A^{(4)} = 1.5 \times 0.5 = 0.75 \text{ in}^2 = 2A_5$$

$$A^{(5)} = 1.5 \times 0.25 = 0.375 \text{ in}^2 = A_5$$

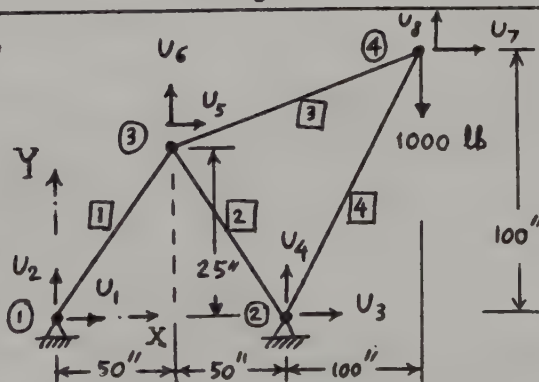
Assembled mass matrix, after applying boundary conditions, is:

$$[M] = \frac{\rho l A_5}{420} \begin{matrix} & \begin{matrix} w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 & w_{10} & w_{11} & w_{12} \end{matrix} \\ \begin{matrix} 1404 & -22l & 216 & -52l & 0 & 0 & 0 & 0 & 0 & 0 \\ -22l & 36l^2 & 52l & -12l^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 216 & 52l & 1092 & -22l & 162 & -39l & 0 & 0 & 0 & 0 \\ -52l & -12l^2 & -22l & 28l^2 & 39l & -9l^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 162 & 39l & 780 & -22l & 108 & -26l & 0 & 0 \\ 0 & 0 & -39l & -9l^2 & -22l & 20l^2 & 26l & -6l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 108 & 26l & 468 & -22l & 54 & -13l \\ 0 & 0 & 0 & 0 & -22l & -6l^2 & -22l & 12l^2 & 13l & -3l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 54 & 13l & 156 & -22l \\ 0 & 0 & 0 & 0 & 0 & 0 & -13l & -3l^2 & -22l & 4l^2 \end{matrix} \end{matrix}$$

with $\rho = 7.324 \times 10^{-4} \text{ lb-sec}^2/\text{in}^4$, $l = 5''$ and $A_5 = 0.375 \text{ in}^2$.

12.10

element e	area of cs, $A^{(e)}$	length $l^{(e)}$	Young's modulus, $E^{(e)}$
1	2 in ²	55.9017"	30 x 10 ⁶ psi
2	2 in ²	55.9017"	30 x 10 ⁶ psi
3	1 in ²	167.7051"	30 x 10 ⁶ psi
4	1 in ²	141.4214"	30 x 10 ⁶ psi



element e	global node correspond- ing to local node		coordinates of local nodes (in)				direction cosines	
	1	2					l_{ij}	m_{ij}
			x_i	y_i	x_j	y_j		
1	1	3	0	0	50	25	0.8944	0.4472
2	3	2	50	25	100	0	0.8944	-0.4472
3	3	4	50	25	200	100	0.8944	0.4472
4	2	4	100	0	200	100	0.7071	0.7071

$$[K^{(e)}] = \frac{A^{(e)} E^{(e)}}{l^{(e)}} \begin{bmatrix} l_{ij}^2 & l_{ij} m_{ij} & -l_{ij}^2 & -l_{ij} m_{ij} \\ l_{ij} m_{ij} & m_{ij}^2 & -l_{ij} m_{ij} & -m_{ij}^2 \\ -l_{ij}^2 & -l_{ij} m_{ij} & l_{ij}^2 & l_{ij} m_{ij} \\ -l_{ij} m_{ij} & -m_{ij}^2 & l_{ij} m_{ij} & m_{ij}^2 \end{bmatrix}$$

This gives

$$[K^{(1)}] = \frac{2(30 \times 10^6)}{55.9017} \begin{bmatrix} U_1 & U_2 & U_3 & U_4 \\ 0.8 & 0.4 & -0.8 & -0.4 \\ 0.4 & 0.2 & -0.4 & -0.2 \\ -0.8 & -0.4 & 0.8 & 0.4 \\ -0.4 & -0.2 & 0.4 & 0.2 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix}$$

$$[K^{(2)}] = \frac{2(30 \times 10^6)}{55.9017} \begin{bmatrix} U_5 & U_6 & U_7 & U_8 \\ 0.8 & -0.4 & -0.8 & 0.4 \\ -0.4 & 0.2 & 0.4 & -0.2 \\ -0.8 & 0.4 & 0.8 & -0.4 \\ 0.4 & -0.2 & -0.4 & 0.2 \end{bmatrix} \begin{matrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{matrix}$$

$$[K^{(3)}] = \frac{1(30 \times 10^6)}{167.7051} \begin{bmatrix} U_5 & U_6 & U_7 & U_8 \\ 0.8 & 0.4 & -0.8 & -0.4 \\ 0.4 & 0.2 & -0.4 & -0.2 \\ -0.8 & -0.4 & 0.8 & 0.4 \\ -0.4 & -0.2 & 0.4 & 0.2 \end{bmatrix} \begin{matrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{matrix}$$

$$[K^{(4)}] = \frac{1(30 \times 10^6)}{141.4214} \begin{bmatrix} U_3 & U_4 & U_7 & U_8 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{matrix} U_3 \\ U_4 \\ U_7 \\ U_8 \end{matrix}$$

Assembled stiffness matrix, after deleting the rows and columns corresponding to degrees of freedom U_1, U_2, U_3 and U_4 , is:

$$[K] = 10^6 \begin{bmatrix} 1.8603 & 0.0716 & -0.1431 & -0.0716 \\ 0.0716 & 0.4652 & -0.0716 & -0.0358 \\ -0.1431 & -0.0716 & 0.2492 & 0.1777 \\ -0.0716 & -0.0358 & 0.1777 & 0.1419 \end{bmatrix} \begin{matrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{matrix}$$

Load vector: $\vec{F} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1000 \end{Bmatrix} \text{ lb}, \quad \vec{U} = \begin{Bmatrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix}$

Equilibrium equations:

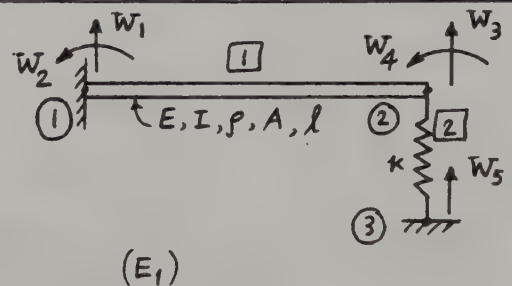
$$[K] \vec{U} = \vec{F}$$

Solution of these equations is

$$\vec{U} = \begin{Bmatrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{Bmatrix} 0.001168 \\ 0.002341 \\ 0.051650 \\ -0.070540 \end{Bmatrix} \text{ in.}$$

12.11 (a) ONE ELEMENT IDEALIZATION

$$[K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$



where $I = \frac{1}{12} b h^3 = \frac{1}{12} \left(\frac{50}{100} \right) \left(\frac{25}{1000} \right)^3 = 6.5104 \times 10^{-8} \text{ m}^4$.

$$[K^{(2)}] = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} w_5 \\ w_3 \end{matrix} \quad (E_2)$$

To incorporate the boundary conditions $w_1 = w_2 = w_5 = 0$, we delete the first two rows and columns in (E_1) and first row and column in (E_2) . The assembled matrix becomes

$$\begin{aligned}
 [K] &= \frac{EI}{l^3} \begin{bmatrix} \left(12 + \frac{l^3}{EI}\right) & -6l \\ -6l & 4l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix} \\
 &= \frac{(2.07 \times 10^{11})(6.5104 \times 10^{-8})}{(0.25)^3} \begin{bmatrix} 12 + \frac{(10^5)(0.25)^3}{(2.07 \times 10^{11})(6.5104 \times 10^{-8})} & -6(0.25) \\ -6(0.25) & 4(0.25)^2 \end{bmatrix} \\
 &= 8.6250 \times 10^5 \begin{bmatrix} 12.1159 & -1.5 \\ -1.5 & 0.25 \end{bmatrix}
 \end{aligned}$$

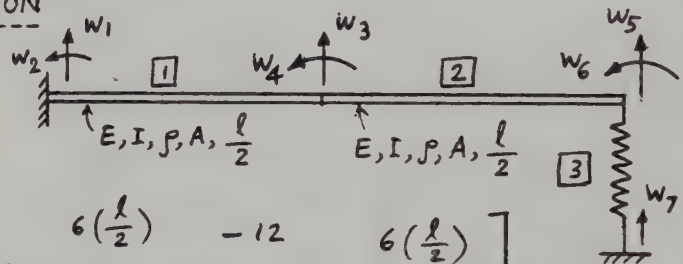
Load vector is $\vec{P} = \begin{Bmatrix} P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} 500 \\ 0 \end{Bmatrix}$

Equilibrium equations become $[K] \vec{W} = \vec{P}$

i.e., $\begin{bmatrix} 12.1159 & -1.5 \\ -1.5 & 0.25 \end{bmatrix} \begin{Bmatrix} w_3 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 57.971 \times 10^{-5} \\ 0 \end{Bmatrix}$

The solution is given by $w_3 = 1.8605 \times 10^{-4} \text{ m}; w_4 = 11.1629 \times 10^{-4} \text{ rad.}$

(b) TWO ELEMENT IDEALIZATION



$$[K^{(1)}] = [K^{(2)}] = \frac{EI}{\left(\frac{l}{2}\right)^3} \begin{bmatrix} 12 & 6\left(\frac{l}{2}\right) & -12 & 6\left(\frac{l}{2}\right) \\ 6\left(\frac{l}{2}\right) & 4\left(\frac{l}{2}\right)^2 & -6\left(\frac{l}{2}\right) & 2\left(\frac{l}{2}\right)^2 \\ -12 & -6\left(\frac{l}{2}\right) & 12 & -6\left(\frac{l}{2}\right) \\ 6\left(\frac{l}{2}\right) & 2\left(\frac{l}{2}\right)^2 & -6\left(\frac{l}{2}\right) & 4\left(\frac{l}{2}\right)^2 \end{bmatrix} \quad (E_1)$$

Substituting $E = 2.07 \times 10^{11}$, $I = 6.5104 \times 10^{-8}$ and $l = 0.25$, Eq. (E₁) becomes

$$[K^{(i)}] = 69 \times 10^5 \begin{bmatrix} 12 & 0.75 & -12 & 0.75 \\ 0.75 & 0.0625 & -0.75 & 0.03125 \\ -12 & -0.75 & 12 & -0.75 \\ 0.75 & 0.03125 & -0.75 & 0.0625 \end{bmatrix} \begin{matrix} w_1 & w_2 & w_3 & w_4 \\ w_2 & w_4 & w_3 & w_6 \end{matrix}$$

$$[K^{(3)}] = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} w_7 \\ w_5 \end{matrix}$$

Assembled stiffness matrix is

$$[K] = 69 \times 10^5 \begin{bmatrix} (12+12) & (-0.75+0.75) & -12 & 0.75 \\ (-0.75+0.75) & (0.0625+0.0625) & -0.75 & 0.03125 \\ -12 & -0.75 & (12+\frac{10^5}{69 \times 10^5}) & -0.75 \\ 0.75 & 0.03125 & -0.75 & 0.0625 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \\ w_5 \\ w_6 \end{matrix}$$

Load vector is

$$\vec{P} = \begin{Bmatrix} P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 500 \\ 0 \end{Bmatrix}$$

Equilibrium equations are $[K] \vec{w} = \vec{P}$

i.e.,

$$69 \times 10^5 \begin{bmatrix} 24 & 0 & -12 & 0.75 \\ 0 & 0.125 & -0.75 & 0.03125 \\ -12 & -0.75 & 12.0145 & -0.75 \\ 0.75 & 0.03125 & -0.75 & 0.0625 \end{bmatrix} \begin{Bmatrix} w_3 \\ w_4 \\ w_5 \\ w_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 500 \\ 0 \end{Bmatrix}$$

The solution is given by :

$$w_3 = 0.5814 \times 10^{-4} \text{ m}, \quad w_4 = 8.372 \times 10^{-4} \text{ rad}, \quad w_5 = 1.86 \times 10^{-4} \text{ m},$$

$$w_6 = 11.16 \times 10^{-4} \text{ rad}.$$

12.12

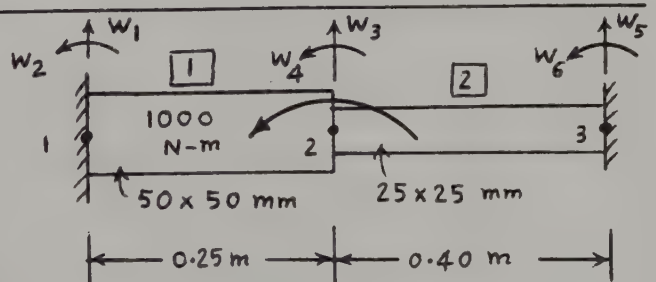
Element 1:

$$I = \frac{1}{12} \left(\frac{50}{1000} \right) \left(\frac{50}{1000} \right)^3$$

$$= 0.5208 \times 10^{-6} \text{ m}^4$$

$$\frac{EI}{l^3} = \frac{(2.1 \times 10^{11}) (0.5208 \times 10^{-6})}{(0.25)^3}$$

$$= 0.7 \times 10^7$$



$$\begin{aligned}
 [K^{(1)}] &= 0.7 \times 10^7 \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ 12 & 1.5 & -12 & 1.5 \\ 1.5 & 0.25 & -1.5 & 0.125 \\ -12 & -1.5 & 12 & -1.5 \\ 1.5 & 0.125 & -1.5 & 0.25 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix} \\
 &= 0.7 \times 10^7 \begin{bmatrix} w_3 & w_4 \\ 12 & -1.5 \\ -1.5 & 0.25 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix} \quad \text{after applying boundary conditions}
 \end{aligned}$$

Element 2:

$$I = \frac{1}{12} \left(\frac{25}{1000} \right) \left(\frac{25}{1000} \right)^3 = 0.3255 \times 10^{-7} \text{ m}^4$$

$$\frac{EI}{l^3} = \frac{(2.1 \times 10^{11}) (0.3255 \times 10^{-7})}{(0.4)^3} = 10.6805 \times 10^4$$

$$[K^{(2)}] = 10.6805 \times 10^4 \begin{bmatrix} w_3 & w_4 & w_5 & w_6 \\ 12 & 2.4 & -12 & 2.4 \\ 2.4 & 0.64 & -2.4 & 0.32 \\ -12 & -2.4 & 12 & -2.4 \\ 2.4 & 0.32 & -2.4 & 0.64 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \\ w_5 \\ w_6 \end{matrix}$$

$$= 10.6805 \times 10^4 \begin{bmatrix} w_3 & w_4 \\ 12 & 2.4 \\ 2.4 & 0.64 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix} \quad \text{after applying boundary conditions}$$

Assembled stiffness matrix:

$$[K] = \begin{bmatrix} (8.4 \times 10^7 + 0.1282 \times 10^7) & (-1.05 \times 10^7 + 0.0256 \times 10^7) \\ (-1.05 \times 10^7 + 0.0256 \times 10^7) & (0.175 \times 10^7 + 0.0068 \times 10^7) \end{bmatrix}$$

Equilibrium equations:

$$10^7 \begin{bmatrix} 8.5282 & -1.0244 \\ -1.0244 & 0.1818 \end{bmatrix} \begin{Bmatrix} w_3 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10^3 \end{Bmatrix}$$

Solution is given by

$$w_3 = 2.0446 \times 10^{-4} \text{ m}, \quad w_4 = 1.7021 \times 10^{-3} \text{ rad}$$

Stresses in elements:

$$\text{Bending moment} = M = EI \frac{d^2 w(x)}{dx^2}$$

$$\text{with } w(x) = \sum_{i=1}^4 W_i N_i(x)$$

$$\sigma_{\max} = \frac{M \cdot c}{I} = \frac{M h}{2I} = \frac{E h}{2} \frac{d^2 w}{dx^2}$$

$$= \frac{E h}{2} \left[\frac{W_1}{l^3} (12x - 6l) + \frac{W_2}{l^2} (6x - 4l) + \frac{W_3}{l^3} (6l - 12x) + \frac{W_4}{l^2} (6x - 2l) \right] \quad \dots (E_1)$$

For element 1,

$$W_1 = 0, W_2 = 0, W_3 = 2.0446 \times 10^{-4}, W_4 = 1.7021 \times 10^{-3},$$

$$l = 0.25, h = 0.05, E = 2.1 \times 10^{11}. \text{ Hence Eq. (E}_1\text{) gives}$$

$$\sigma_{\max}|_{\text{fixed end}} = \sigma_{\max}(x=0) = 3.1560 \times 10^7 \text{ N/m}^2$$

$$\sigma_{\max}|_{\text{loaded end}} = \sigma_{\max}(x=0.25) = 3.9929 \times 10^7 \text{ N/m}^2$$

For element 2,

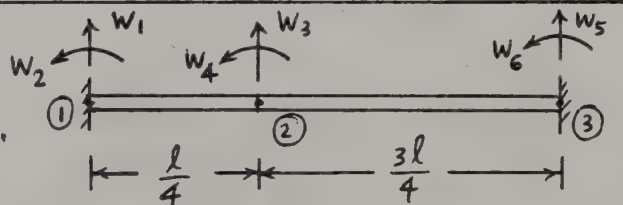
$$W_1 = 2.0446 \times 10^{-4}, W_2 = 1.7021 \times 10^{-3}, W_3 = 0, W_4 = 0,$$

$$l = 0.4, h = 0.025, E = 2.1 \times 10^{11}. \text{ Hence Eq. (E}_1\text{) gives}$$

$$\sigma_{\max}|_{\text{loaded end}} = \sigma_{\max}(x=0) = -6.4807 \times 10^7 \text{ N/m}^2$$

$$\sigma_{\max}|_{\text{fixed end}} = \sigma_{\max}(x=0.4) = 4.2467 \times 10^7 \text{ N/m}^2$$

12.13



For element 1:

$$[K^{(1)}] = \frac{EI}{(l/4)^3} \begin{bmatrix} 12 & \frac{3l}{2} & -12 & \frac{3l}{2} \\ \frac{3l}{2} & \frac{l^2}{4} & -\frac{3l}{2} & \frac{l^2}{8} \\ -12 & -\frac{3l}{2} & 12 & -\frac{3l}{2} \\ \frac{3l}{2} & \frac{l^2}{8} & -\frac{3l}{2} & \frac{l^2}{4} \end{bmatrix} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{matrix}$$

For element 2:

$$[K^{(2)}] = \frac{EI}{(3l/4)^3} \begin{bmatrix} w_3 & w_4 & w_5 & w_6 \\ 12 & \frac{9l}{2} & -12 & \frac{9l}{2} \\ \frac{9l}{2} & \frac{9l^2}{4} & -\frac{9l}{2} & \frac{9l^2}{8} \\ -12 & -\frac{9l}{2} & 12 & -\frac{9l}{2} \\ \frac{9l}{2} & \frac{9l^2}{8} & -\frac{9l}{2} & \frac{9l^2}{4} \end{bmatrix} \begin{matrix} w_3 \\ w_4 \\ w_5 \\ w_6 \end{matrix}$$

Assembled stiffness matrix:

$$[K] = \frac{64EI}{l^3} \begin{bmatrix} (12 + \frac{12}{27}) & (-\frac{3l}{2} + \frac{9l}{54}) \\ (-\frac{3l}{2} + \frac{9l}{54}) & (\frac{l^2}{4} + \frac{9l^2}{108}) \end{bmatrix} = \frac{64EI}{3l^3} \begin{bmatrix} \frac{112}{3} & -4l \\ -4l & l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

Equilibrium equations:

$$[K] \vec{w} = \vec{P}$$

i.e.,

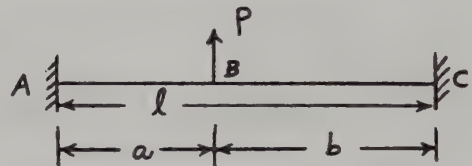
$$\frac{64EI}{3l^3} \begin{bmatrix} 112/3 & -4l \\ -4l & l^2 \end{bmatrix} \begin{Bmatrix} w_3 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Solution is given by

$$w_3 = \frac{9Pl^3}{4096EI} \quad \text{and} \quad w_4 = \frac{9Pl^2}{1024EI}$$

Simple beam deflection formula:

$$y_{AB} = -\frac{Pb^2x^2}{6EI l^3} [x(3a+b) - 3al]$$



at $x = a$,

$$y_B = -\frac{Pb^2a^2}{6EI l^3} [3a^2 + ab - 3al]$$

when $a = l/4$ and $b = 3l/4$,

$$y_B = \frac{9}{4096} \frac{Pl^3}{EI}$$

$$\text{slope} = \frac{dy_{AB}}{dx} = -\frac{Pb^2}{6EI l^2} [3x^2(3a+b) - 6xal]$$

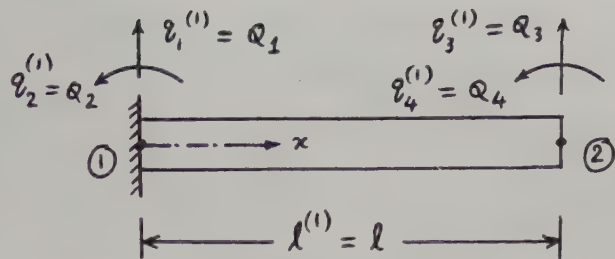
$$\text{at } x = a, \quad \left. \frac{dy}{dx} \right|_B = -\frac{Pb^2}{6EI l^3} [9a^3 + 3a^2b - 6a^2l]$$

$$\text{When } a = l/4 \text{ and } b = 3l/4, \quad \left. \frac{dy}{dx} \right|_B = \frac{9Pl^2}{1024EI}$$

$\therefore$ Both results are same. Reason: shape functions = static deflection relations.

12.14

$$[\tilde{K}] = [K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$



$$[\tilde{M}] = [M^{(1)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

Since node ① is fixed, $Q_1 = Q_2 = 0$. By deleting the first two rows and columns in $[K]$ and $[M]$, the eigenvalue problem becomes

$$([K] - \omega^2 [M]) \begin{Bmatrix} Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\left| \frac{EI}{l^3} \begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix} - \frac{\rho A l \omega^2}{420} \begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix} \right| = 0$$

Let $\lambda = \frac{\rho A l^4 \omega^2}{420 EI}$. Then $\begin{vmatrix} 12 - 156\lambda & -6l + 22l\lambda \\ -6l + 22l\lambda & 4l^2 - 4l^2\lambda \end{vmatrix} = 0$

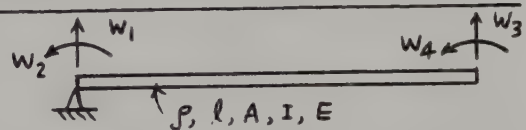
i.e. $140l^2\lambda^2 - 408l^2\lambda + 12l^2 = 0$

i.e. $\lambda = 0.029715, 2.884571$

i.e. $\omega_1^2 = 12.4803 \frac{EI}{\rho A l^4}$, $\omega_2^2 = 1211.5198 \frac{EI}{\rho A l^4}$

$\therefore \omega_1 = 3.5327 \sqrt{\frac{EI}{\rho A l^4}}$, $\omega_2 = 34.8069 \sqrt{\frac{EI}{\rho A l^4}}$

12.15



$$[\tilde{K}] = [K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \quad (E_1)$$

$$[\tilde{M}] = [M^{(1)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \quad (E_2)$$

Applying the boundary condition $w_1 = 0$, the eigenvalue problem can be expressed as

$$-\omega^2 [M] \vec{w} + [K] \vec{w} = \vec{0}$$

i.e.,

$$\left[-\omega^2 \frac{\rho A l}{420} \begin{bmatrix} 4l^2 & 13l & -3l^2 \\ 13l & 156 & -22l \\ -3l^2 & -22l & 4l^2 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 4l^2 & -6l & 2l^2 \\ -6l & 12 & -6l \\ 2l^2 & -6l & 4l^2 \end{bmatrix} \right] \begin{Bmatrix} w_2 \\ w_3 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

i.e.,

$$\begin{bmatrix} (4l^2 - \lambda l^2) & (-6l - 13l\lambda) & (2l^2 + 3l^2\lambda) \\ (-6l - 13l\lambda) & (12 - 156\lambda) & (-6l + 22l\lambda) \\ (2l^2 + 3l^2\lambda) & (-6l + 22l\lambda) & (4l^2 - 4l^2\lambda) \end{bmatrix} \begin{Bmatrix} w_2 \\ w_3 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (E_3)$$

Where $\lambda = \frac{\omega^2 \rho A l^4}{420 EI}$

Eq. (E₃) gives the frequency equation

$$\begin{vmatrix} 4l^2(1-\lambda) & -l(6+13\lambda) & l^2(2+3\lambda) \\ -l(6+13\lambda) & 12(1-13\lambda) & l(-6+22\lambda) \\ l^2(2+3\lambda) & l(-6+22\lambda) & 4l^2(1-\lambda) \end{vmatrix} = 0$$

which reduces to

$$-\lambda (196\lambda^2 - 2436\lambda + 1680) = 0$$

Roots are:

$$\lambda_1 = 0; \quad \lambda_{2,3} = \frac{2436 \pm \sqrt{(2436)^2 - 4(196)(1680)}}{2(196)}$$

$$= 0.7329; \quad 11.6957$$

Thus

$$\omega_1 = 0$$

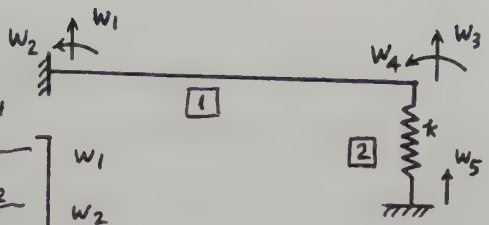
$$\omega_2 = 17.5447 \sqrt{\frac{EI}{\rho A l^4}}$$

$$\omega_3 = 70.0870 \sqrt{\frac{EI}{\rho A l^4}}$$

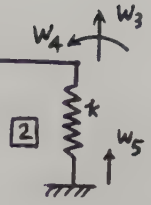
These values can be compared with the exact values (see

Fig. 8.15): $\omega_1 = 0$, $\omega_2 = 15.4182 \sqrt{\frac{EI}{\rho A l^4}}$, $\omega_3 = 49.9649 \sqrt{\frac{EI}{\rho A l^4}}$.

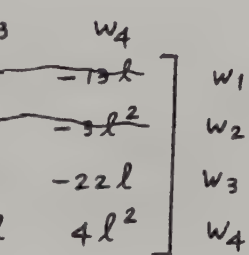
12.16 Element matrices:



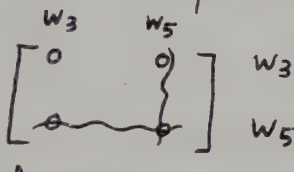
$$[K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$



$$[K^{(2)}] = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} w_3 \\ w_5 \end{matrix}$$

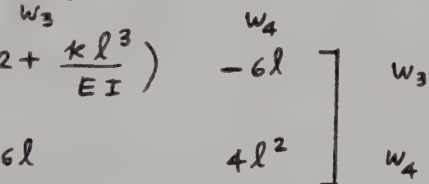


$$[M^{(1)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

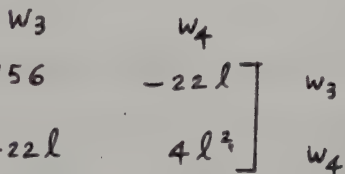


$$[M^{(2)}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} w_3 \\ w_5 \end{matrix} \quad (\text{mass of spring is assumed to be zero})$$

Assembled matrices:



$$[K] = \frac{EI}{l^3} \begin{bmatrix} \left(12 + \frac{k l^3}{EI}\right) & -6l \\ -6l & 4l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix}$$



$$[M] = \frac{\rho A l}{420} \begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

Eigenvalue equation:

$$\left| -\frac{\rho A l \omega^2}{420} \begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} \left(12 + \frac{k l^3}{EI}\right) & -6l \\ -6l & 4l^2 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} 12 + \frac{k l^3}{EI} - 156 \lambda & -6l + 22l \lambda \\ -6l + 22l \lambda & 4l^2 - 4l^2 \lambda \end{vmatrix} = 0$$

where $\lambda = \left(\frac{\rho A l^4 \omega^2}{420 EI} \right)$

Upon expansion, the frequency equation reduces to

$$35 \lambda^2 - \left(102 + \frac{\kappa l^3}{EI} \right) \lambda + \left(3 + \frac{\kappa l^3}{EI} \right) = 0$$

which implies that

$$\lambda_{1,2} = \left\{ \frac{\left(102 + \frac{\kappa l^3}{EI} \right) \pm \left(9984 + 64 \frac{\kappa l^3}{EI} + \frac{\kappa^2 l^6}{E^2 I^2} \right)^{\frac{1}{2}}}{70} \right\}$$

12.17

Element matrices:

$$[K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

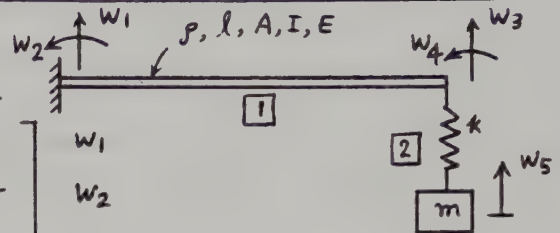
$$[M^{(1)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

$$[K^{(2)}] = \kappa \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} w_3 \\ w_5 \end{matrix}, \quad [M^{(2)}] = \begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix} \begin{matrix} w_3 \\ w_5 \end{matrix}$$

Assembled matrices (after applying boundary conditions):

$$[K] = \frac{EI}{l^3} \begin{bmatrix} 12 + \frac{\kappa l^3}{EI} & -6l & -\frac{\kappa l^3}{EI} \\ -6l & 4l^2 & 0 \\ -\frac{\kappa l^3}{EI} & 0 & \frac{\kappa l^3}{EI} \end{bmatrix} \begin{matrix} w_3 \\ w_4 \\ w_5 \end{matrix}$$

$$= \frac{EI}{l^3} \begin{bmatrix} 13 & -6l & -1 \\ -6l & 4l^2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$



$$[M] = \frac{\rho A l}{420} \begin{bmatrix} 156 + 0 & -22l & 0 \\ -22l & 4l^2 & 0 \\ 0 & 0 & m \left(\frac{420}{\rho A l} \right) \end{bmatrix} \begin{matrix} w_3 \\ w_4 \\ w_5 \end{matrix}$$

$$= \frac{\rho A l}{420} \begin{bmatrix} 156 & -22l & 0 \\ -22l & 4l^2 & 0 \\ 0 & 0 & 420 \end{bmatrix}$$

Frequency equation:

$$| -\omega^2 [M] + [K] | = 0$$

i.e.,

$$\left| -\frac{\rho A l \omega^2}{420} \begin{bmatrix} 156 & -22l & 0 \\ -22l & 4l^2 & 0 \\ 0 & 0 & 420 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 13 & -6l & -1 \\ -6l & 4l^2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} 13 - 156\lambda & -6l + 22l\lambda & -1 \\ -6l + 22l\lambda & 4l^2 - 4l^2\lambda & 0 \\ -1 & 0 & 1 - 420\lambda \end{vmatrix} = 0 \quad (E_1)$$

where

$$\lambda = \left(\frac{\rho A l^4 \omega^2}{420 EI} \right)$$

Eg. (E₁) can be simplified to obtain

$$-58800 \lambda^3 + 173180 \lambda^2 - 7128 \lambda + 12 = 0 \quad (E_2)$$

Roots of Eg. (E₂) given by Program # 7 are:

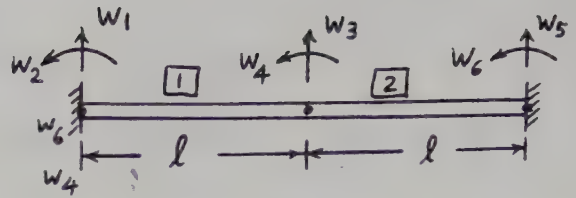
$$\lambda_1 = 0.0017555 \Rightarrow \omega_1 = 0.8587 \sqrt{EI/(\rho A l^4)}$$

$$\lambda_2 = 0.039951 \Rightarrow \omega_2 = 4.0965 \sqrt{EI/(\rho A l^4)}$$

$$\lambda_3 = 2.90351 \Rightarrow \omega_3 = 34.9210 \sqrt{EI/(\rho A l^4)}$$

12.18

Element matrices:



$$[K^{(e)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} w_1 & w_3 \\ w_2 & w_4 \\ w_3 & w_5 \\ w_4 & w_6 \end{matrix}$$

$$[M^{(e)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \begin{matrix} w_1 & w_3 \\ w_2 & w_4 \\ w_3 & w_5 \\ w_4 & w_6 \end{matrix}$$

Assembled matrices, after incorporating boundary conditions $w_1 = w_2 = w_5 = w_6 = 0$:

$$[K] = \frac{EI}{l^3} \begin{bmatrix} 12+12 & -6l+6l \\ -6l+6l & 4l^2+4l^2 \end{bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

$$[M] = \frac{\rho A l}{420} \begin{bmatrix} 156+156 & -22l+22l \\ -22l+22l & 4l^2+4l^2 \end{bmatrix} = \frac{\rho A l}{420} \begin{bmatrix} 312 & 0 \\ 0 & 8l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

Eigenvalue problem:

$$[-\omega^2 [M] + [K]] \vec{W} = \vec{0}$$

For natural frequencies,

$$|-\omega^2 [M] + [K]| = 0$$

i.e.,

$$\left| -\frac{\omega^2 \rho A l}{420} \begin{bmatrix} 312 & 0 \\ 0 & 8l^2 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} (24 - 312\lambda) & 0 \\ 0 & 8l^2(1 - \lambda) \end{vmatrix} = 0 \text{ where } \lambda = \left(\frac{\rho A l^4 \omega^2}{420 EI} \right)$$

i.e., $192 l^2 (1-\lambda)(1-13\lambda) = 0$

$\therefore \lambda_1 = \frac{1}{13}, \quad \omega_1 = 22.736 \sqrt{\frac{EI}{\rho A L^4}} \quad \text{with } L = 2l$

$\lambda_2 = 1, \quad \omega_2 = 81.9756 \sqrt{\frac{EI}{\rho A L^4}}$

For mode shapes,

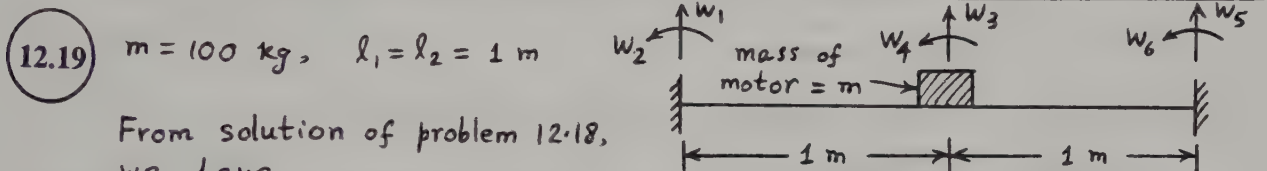
$$[-\omega_i^2 [M] + [K]] \vec{W}^{(i)} = \vec{0}; \quad i = 1, 2$$

i.e.,

$$\begin{bmatrix} (24 - 312 \lambda_i) & 0 \\ 0 & 8l^2(1 - \lambda_i) \end{bmatrix} \begin{Bmatrix} w_3^{(i)} \\ w_4^{(i)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For $\lambda_1 = \frac{1}{13}$, $w_3^{(1)}$ can have any value
(transverse displacement mode)

For $\lambda_2 = 1$, $w_4^{(2)}$ can have any value
(rotation or slope mode)



$$[K] = \frac{EI}{l^3} \begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix} \begin{Bmatrix} w_3 \\ w_4 \end{Bmatrix} \quad (E_1)$$

$$[M] = \frac{\rho A l}{420} \begin{bmatrix} 312 & 0 \\ 0 & 8l^2 \end{bmatrix} \begin{Bmatrix} w_3 \\ w_4 \end{Bmatrix} \quad (E_2)$$

When the mass of motor is added to d.o.f. w_3 , the mass matrix becomes

$$[M'] = \frac{\rho A l}{420} \begin{bmatrix} 312 + m \left(\frac{420}{\rho A l} \right) & 0 \\ 0 & 8l^2 \end{bmatrix} \quad (E_3)$$

Eqs. (E1) and (E3) yield the frequency equation

$$|-\omega^2 [M'] + [K]| = 0$$

$$\text{i.e.,} \quad \left| -\frac{\rho A l \omega^2}{420} \begin{bmatrix} \left(312 + \frac{42000}{\rho A l}\right) & 0 \\ 0 & 8l^2 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix} \right| = 0$$

i.e.,

$$\left| \begin{array}{cc} 24 - \left(312 + \frac{42000}{\rho A l}\right) \lambda & 0 \\ 0 & 8l^2(1-\lambda) \end{array} \right| = 0$$

i.e.,

$$192 l^2 (1-\lambda) \left\{ 1 - \left(13 + \frac{1750}{\rho A l}\right) \lambda \right\} = 0$$

where $\lambda = \left(\frac{\rho A l^4 \omega^2}{420 EI} \right)$.

$$\therefore \lambda_1 = 1 \Rightarrow \omega_1^2 = \frac{420 EI \lambda_1}{\rho A l^4} = \frac{420 EI}{\rho A l^4}$$

$$\lambda_2 = \frac{1}{\left(13 + \frac{1750}{\rho A l}\right)} \Rightarrow \omega_2^2 = \frac{420 EI}{\rho A l^4 \left(13 + \frac{1750}{\rho A l}\right)}$$

For the steel beam,

$$E = 2.1 \times 10^{11} \text{ Pa}, \quad l = 1 \text{ m}, \quad \rho = 7.8 \times 10^3 \text{ kg/m}^3$$

$$\omega_1^2 = \frac{420(2.1 \times 10^{11}) I}{(7.8 \times 10^3) A (1)^4} \Rightarrow \omega_1 = 10.6338 \times 10^4 \sqrt{\frac{I}{A}} \text{ rad/sec} \quad (E_4)$$

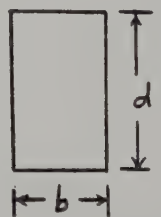
$$\omega_2^2 = \frac{420(2.1 \times 10^{11}) I}{13(7.8 \times 10^3) A (1)^4 + 1750(1)^3} = \frac{88.2 \times 10^{12} I}{101.4 \times 10^3 A + 1750}$$

$$\Rightarrow \omega_2 = \frac{9.3915 \times 10^6 \sqrt{I}}{(101400 A + 1750)^{1/2}} \text{ rad/sec} \quad (E_5)$$

Let depth = twice width for cross-section.

$$\text{Then } A = 2b^2 \text{ and } I = \frac{1}{12}(b)(2b)^3 = 0.6667 b^4$$

$$\begin{aligned} \text{operating speed of motor} = \Omega &= \frac{1800(2\pi)}{60} \\ &= 188.496 \text{ rad/sec} \end{aligned}$$



Assume $\omega_1 =$ smaller than ω_2 .

For $\omega_1 > 188.496$, we need to have

$$10^4 (10.6338) \sqrt{\frac{0.6667 b^4}{2b^2}} > 188.496$$

$$\text{i.e., } b > 3.07 \times 10^{-3} \text{ m}$$

Let $b = 5 \text{ mm}$ and $d = 10 \text{ mm}$ so that

$$A = 50 \times 10^{-6} \text{ m}^2 \text{ and } I = 416.6667 \times 10^{-12} \text{ m}^4.$$

This gives

$$\omega_2 = \frac{9.3915 \times 10^6 (416.6667 \times 10^{-12})^{1/2}}{(101400 \times 50 \times 10^{-6} + 1750)^{1/2}} = 4.5760 \frac{\text{rad}}{\text{sec}}$$

This violates the original assumption of $\omega_1 > \omega_2$.

Assume $\omega_2 =$ smaller than ω_1 .

For $\omega_2 > 188.496$, we need to have

$$\frac{9.3915 \times 10^6 (0.6667 b^4)^{1/2}}{\{101400 (2b^2) + 1750\}^{1/2}} > 188.496$$

i.e.,

$$1654.8624 b^4 - 0.2028 b^2 - 0.00175 > 0 \quad (E_6)$$

By setting the inequality in (E_6) to equality and solving for b^2 , we find

$$b^2 = 0.00109145 \text{ or } b = 0.03304 \text{ m}$$

Let $b = 35 \text{ mm}$ and $d = 70 \text{ mm}$, so that the left side of inequality, (E_6) becomes

$$(0.002483 - 0.000248 - 0.001750) \text{ which is positive.}$$

Hence the final design is given by

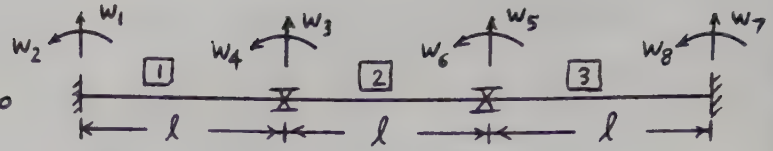
$$b = 0.035 \text{ m}$$

$$d = 0.070 \text{ m}$$

12.20

Boundary conditions:

$$w_1 = w_2 = w_3 = w_5 = w_7 = w_8 = 0$$



$$[K^{(i)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} w_1 & w_3 & w_5 \\ w_2 & w_4 & w_6 \\ w_3 & w_5 & w_7 \\ w_4 & w_6 & w_8 \end{matrix}$$

$$[M^{(i)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \begin{matrix} w_1 & w_3 & w_5 \\ w_2 & w_4 & w_6 \\ w_3 & w_5 & w_7 \\ w_4 & w_6 & w_8 \end{matrix}$$

Assembled stiffness matrix, after applying boundary conditions:

$$[K] = \frac{EI}{l^3} \begin{bmatrix} (4l^2 + 4l^2) & 2l^2 \\ 2l^2 & (4l^2 + 4l^2) \end{bmatrix} \begin{matrix} w_4 \\ w_6 \end{matrix}$$

$$[M] = \frac{\rho A l}{420} \begin{bmatrix} (4l^2 + 4l^2) & -3l^2 \\ -3l^2 & (4l^2 + 4l^2) \end{bmatrix} \begin{matrix} w_4 \\ w_6 \end{matrix}$$

Frequency equation:

$$\left| -\frac{\rho A l \omega^2}{420} \begin{bmatrix} 8l^2 & -3l^2 \\ -3l^2 & 8l^2 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 8l^2 & 2l^2 \\ 2l^2 & 8l^2 \end{bmatrix} \right| = 0 \quad (E_1)$$

Defining $\lambda = \left(\frac{\rho A l^4 \omega^2}{420 EI} \right),$

Eg. (E_1) can be expressed as

$$\begin{vmatrix} 8(1-\lambda) & (2+3\lambda) \\ (2+3\lambda) & 8(1-\lambda) \end{vmatrix} = 0$$

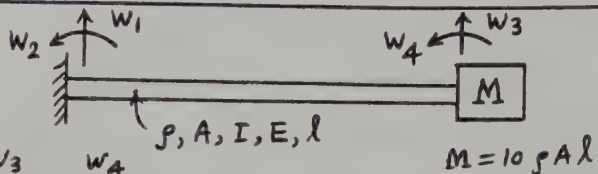
or $11\lambda^2 - 28\lambda + 12 = 0$

$$\therefore \lambda_{1,2} = \frac{28 \pm \sqrt{784 - 4(11)(12)}}{2(11)} = \frac{6}{11}, 2$$

i.e., $\omega_1 = 15.1357 \sqrt{\frac{EI}{\rho A l^4}}$ and $\omega_2 = 28.9828 \sqrt{\frac{EI}{\rho A l^4}}$

12.21

Element matrices:



$$[K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

$$[M^{(1)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

Assembled matrices (after applying boundary conditions

$w_1 = w_2 = 0$ and adding mass M at d.o.f. w_3):

$$[K] = \frac{EI}{l^3} \begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

$$[M] = \frac{\rho A l}{420} \begin{bmatrix} 156 + 4200 & -22l \\ -22l & 4l^2 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

Defining $\lambda = \left(\frac{\rho A l^4 \omega^2}{420 EI} \right)$, the frequency equation can be

written as

$$\begin{vmatrix} (12 - 4356 \lambda) & (-6l + 22l \lambda) \\ (-6l + 22l \lambda) & (4l^2 - 4l^2 \lambda) \end{vmatrix} = 0$$

or

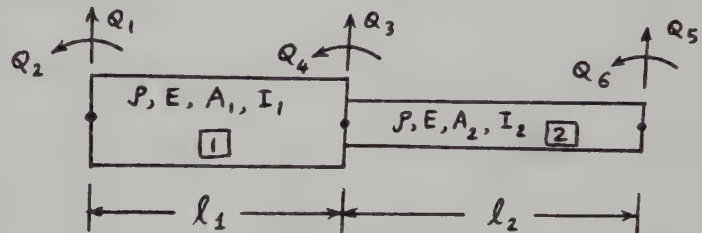
$$4235 \lambda^2 - 4302 \lambda + 3 = 0$$

$$\text{This gives } \lambda_{1,2} = \frac{4302 \pm (4302^2 - 4 \times 4235 \times 3)^{\frac{1}{2}}}{2(4235)}$$

$$= 0.0006978, 1.01512272$$

$$\therefore \omega_1 = 0.5414 \sqrt{\frac{EI}{\rho A l^4}}, \quad \omega_2 = 20.6483 \sqrt{\frac{EI}{\rho A l^4}}$$

12.22



$$[k^{(e)}] = [\bar{k}^{(e)}] = \frac{E^{(e)} I^{(e)}}{l^{(e)3}} \begin{bmatrix} 12 & 6l^{(e)} & -12 & 6l^{(e)} \\ 6l^{(e)} & 4l^{(e)2} & -6l^{(e)} & 2l^{(e)2} \\ -12 & -6l^{(e)} & 12 & -6l^{(e)} \\ 6l^{(e)} & 2l^{(e)2} & -6l^{(e)} & 4l^{(e)2} \end{bmatrix}; e=1,2$$

$$[m^{(e)}] = [\bar{m}^{(e)}] = \frac{\rho^{(e)} A^{(e)} l^{(e)}}{420} \begin{bmatrix} 156 & 22l^{(e)} & 54 & -13l^{(e)} \\ 22l^{(e)} & 4l^{(e)2} & 13l^{(e)} & -3l^{(e)2} \\ 54 & 13l^{(e)} & 156 & -22l^{(e)} \\ -13l^{(e)} & -3l^{(e)2} & -22l^{(e)} & 4l^{(e)2} \end{bmatrix}; e=1,2$$

$$[A^{(1)}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad [A^{(2)}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K] = \sum_{e=1}^2 [A^{(e)}]^T [\bar{k}^{(e)}] [A^{(e)}]$$

$$= E \begin{bmatrix} 12 I_1 / l_1^3 & 6 I_1 / l_1^2 & -12 I_1 / l_1^3 & 6 I_1 / l_1^2 & 0 & 0 \\ 6 I_1 / l_1^2 & 4 I_1 / l_1 & -6 I_1 / l_1^2 & 2 I_1 / l_1 & 0 & 0 \\ -12 I_1 / l_1^3 & -6 I_1 / l_1^2 & \left(\frac{12 I_1}{l_1^3} + \frac{12 I_2}{l_2^3} \right) & \left(-\frac{6 I_1}{l_1^2} + \frac{6 I_2}{l_2^2} \right) & -12 I_2 / l_2^3 & 6 I_2 / l_2^2 \\ 6 I_1 / l_1^2 & 2 I_1 / l_1 & \left(-\frac{6 I_1}{l_1^2} + \frac{6 I_2}{l_2^2} \right) & \left(\frac{4 I_1}{l_1} + \frac{4 I_2}{l_2} \right) & -6 I_2 / l_2^2 & 2 I_2 / l_2 \\ 0 & 0 & -12 I_2 / l_2^3 & -6 I_2 / l_2^2 & 12 I_2 / l_2^3 & -6 I_2 / l_2^2 \\ 0 & 0 & 6 I_2 / l_2^2 & 2 I_2 / l_2 & -6 I_2 / l_2^2 & 4 I_2 / l_2 \end{bmatrix}$$

$$[M] = \sum_{e=1}^2 [A^{(e)}]^T [\bar{m}^{(e)}] [A^{(e)}]$$

$$= \frac{P}{420} \begin{bmatrix} 156 A_1 l_1 & 22 A_1 l_1^2 & 54 A_1 l_1 & -13 A_1 l_1^2 & 0 & 0 \\ 22 A_1 l_1^2 & 4 A_1 l_1^3 & 13 A_1 l_1^2 & -3 A_1 l_1^3 & 0 & 0 \\ 54 A_1 l_1 & 13 A_1 l_1^2 & 156 (A_1 l_1 + A_2 l_2) & 22 (-A_1 l_1^2 + A_2 l_2^2) & 54 A_2 l_2 & -13 A_2 l_2^2 \\ -13 A_1 l_1^2 & -3 A_1 l_1^3 & 22 (-A_1 l_1^2 + A_2 l_2^2) & 4 (A_1 l_1^3 + A_2 l_2^3) & 13 A_2 l_2^2 & -3 A_2 l_2^3 \\ 0 & 0 & 54 A_2 l_2 & 13 A_2 l_2^2 & 156 A_2 l_2 & -22 A_2 l_2^2 \\ 0 & 0 & -13 A_2 l_2^2 & -3 A_2 l_2^3 & -22 A_2 l_2^2 & 4 A_2 l_2^3 \end{bmatrix}$$

where $I^{(e)} = I_e$ and $l^{(e)} = l_e$; $e = 1, 2$

Since $Q_1 = Q_2 = Q_5 = Q_6 = 0$, rows and columns 1, 2, 5 and 6 in $[K]$ and $[M]$ are deleted to obtain the frequency equation as

$$|[K] - \omega^2 [M]| = 0$$

$$\text{i.e.} \begin{vmatrix} \left\{ 12 E \left(\frac{I_1}{l_1^3} + \frac{I_2}{l_2^3} \right) - \frac{156 P \omega^2}{420} (A_1 l_1 + A_2 l_2) \right\} & \left\{ 6 E \left(-\frac{I_1}{l_1^2} + \frac{I_2}{l_2^2} \right) - \frac{22 P \omega^2}{420} (-A_1 l_1^2 + A_2 l_2^2) \right\} \\ \left\{ 6 E \left(-\frac{I_1}{l_1^2} + \frac{I_2}{l_2^2} \right) - \frac{22 P \omega^2}{420} (-A_1 l_1^2 + A_2 l_2^2) \right\} & \left\{ 4 E \left(\frac{I_1}{l_1} + \frac{I_2}{l_2} \right) - \frac{4 P \omega^2}{420} (A_1 l_1^3 + A_2 l_2^3) \right\} \end{vmatrix}$$

= 0

Roots of this equation give ω_1 and ω_2 .

Load vector:

$$\vec{f}^{(1)} = \vec{f}^{(2)} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{Bmatrix} \int_0^{l_1} f_1(x, t) N_1(x) dx \\ \int_0^{l_1} f_2(x, t) N_2(x) dx \\ \int_0^{l_1} f_3(x, t) N_3(x) dx \\ \int_0^{l_1} f_4(x, t) N_4(x) dx \end{Bmatrix} = P \begin{Bmatrix} \int_0^{l_1} N_1 dx \\ \int_0^{l_1} N_2 dx \\ \int_0^{l_1} N_3 dx \\ \int_0^{l_1} N_4 dx \end{Bmatrix}$$

$$= p \begin{Bmatrix} l/2 \\ l^2/12 \\ l/2 \\ l^2/12 \end{Bmatrix}$$

$$\vec{f}^{(2)} = \vec{f}^{(2)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

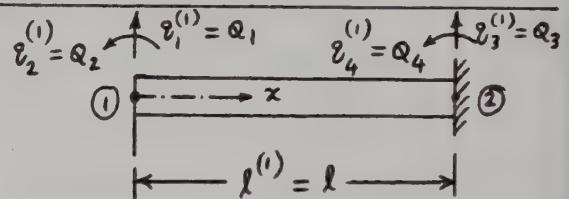
$$\vec{F} = \sum_{e=1}^2 [A^{(e)}]^T \vec{f}^{(e)} = p \begin{Bmatrix} l/2 \\ l^2/12 \\ l/2 \\ l^2/12 \\ 0 \\ 0 \end{Bmatrix}$$

After applying the boundary conditions, load vector becomes

$$\vec{F} = \begin{Bmatrix} p l/2 \\ p l^2/12 \end{Bmatrix}$$

12.23

$$[K] = [K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$



$$[M] = [M^{(1)}] = \frac{pAl}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

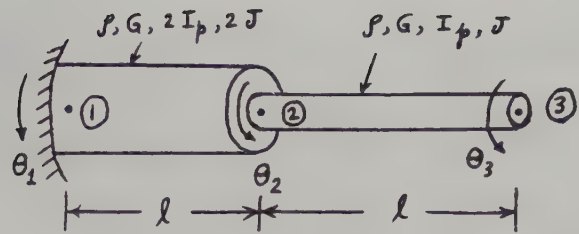
Since node ① is pin connected and node ② is fixed, $Q_1 = Q_3 = Q_4 = 0$. By deleting the corresponding rows and columns in $[K]$ and $[M]$, the eigenvalue problem becomes

$$([K] - \omega^2 [M]) \vec{Q} = \vec{0}$$

Frequency equation is

$$\left| \frac{EI}{l^3} (4l^2) - \omega^2 \frac{pAl}{420} (4l^2) \right| = 0$$

which gives $\omega_1^2 = \frac{420 EI}{pAl^4}$ or $\omega_1 = 20.4939 \sqrt{\frac{EI}{pAl^4}}$



$$[\bar{m}^{(1)}] = [m^{(1)}] = \frac{\rho I_p l_1}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho I_p l}{12} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$[\bar{m}^{(2)}] = [m^{(2)}] = \frac{\rho_2 I_{p2} l_2}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho I_p l}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[\bar{k}^{(1)}] = [k^{(1)}] = \frac{2G_1 J_1}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{GJ}{l} \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

$$[\bar{k}^{(2)}] = [k^{(2)}] = \frac{2G_2 J_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{GJ}{l} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$[A^{(1)}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad [A^{(2)}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\tilde{M}] = \sum_{e=1}^2 [A^{(e)}]^T [\bar{m}^{(e)}] [A^{(e)}] = \frac{\rho I_p l}{12} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$[\tilde{K}] = \sum_{e=1}^2 [A^{(e)}]^T [\bar{k}^{(e)}] [A^{(e)}] = \frac{2GJ}{l} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Since $\theta_1 = 0$, $[M] = \frac{\rho I_p l}{12} \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix}$ and $[K] = \frac{2GJ}{l} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$

Frequency equation is

$$\left| \frac{2GJ}{l} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\rho I_p l \omega^2}{12} \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \right| = 0$$

i.e. $\left| \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \right| = 0$ where $\lambda = \frac{\rho I_p l^2 \omega^2}{24 GJ}$

i.e. $11\lambda^2 - 14\lambda + 2 = 0$

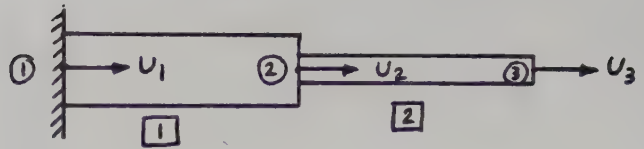
i.e. $\lambda_1 = 0.1640$, $\lambda_2 = 1.1087$

$$\omega_1^2 = 3.9360 GJ / (\rho I_p l^2), \quad \omega_2^2 = 26.6088 GJ / (\rho I_p l^2)$$

$$\therefore \omega_1 = 1.983935 \sqrt{\frac{GJ}{\rho I_p l^2}} \quad \text{and} \quad \omega_2 = 5.158372 \sqrt{\frac{GJ}{\rho I_p l^2}}$$

12.25

Element matrices:



$$[K^{(1)}] = \frac{A_1 E_1}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{4AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix}$$

$$[K^{(2)}] = \frac{A_2 E_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix}$$

$$[M^{(1)}] = \frac{\rho_1 A_1 l_1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{2\rho A l}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix}$$

$$[M^{(2)}] = \frac{\rho_2 A_2 l_2}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix}$$

Assembled matrices (with $U_1 = 0$):

$$[K] = \frac{AE}{l} \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix}, \quad [M] = \frac{\rho A l}{6} \begin{bmatrix} 10 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix}$$

Eigenvalue problem:

$$[-\omega^2 [M] + [K]] \vec{U} = \vec{0} \quad (E_1)$$

For natural frequencies,

$$\left| -\frac{\rho A l \omega^2}{6} \begin{bmatrix} 10 & 1 \\ 1 & 2 \end{bmatrix} + \frac{AE}{l} \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \right| = 0$$

$$\text{i.e.,} \quad \begin{vmatrix} 5 - 10\lambda & -1 - \lambda \\ -1 - \lambda & 1 - 2\lambda \end{vmatrix} = 0 \quad (E_2)$$

$$\text{Where } \lambda = \left(\frac{\rho l^2 \omega^2}{6E} \right) \quad (E_3)$$

$$\text{i.e.,} \quad 19\lambda^2 - 22\lambda + 4 = 0$$

This gives

$$\lambda_1 = 0.2259, \quad \omega_1 = 1.1642 \{E/(\rho l^2)\} \quad (E_4)$$

$$\lambda_2 = 0.9320, \quad \omega_2 = 2.3648 \{E/(\rho l^2)\} \quad (E_5)$$

For eigenvectors,

Use of (E_2) leads to

$$(5 - 10\lambda_1) U_2 - (1 + \lambda_1) U_3 = 0$$

or $U_3 = \left(\frac{5 - 10\lambda_1}{1 + \lambda_1} \right) U_2 = 2.2359 U_2$ with $\lambda_1 = 0.2259$

$$\therefore \vec{\tilde{U}}^{(1)} = \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1.0 \\ 2.2359 \end{Bmatrix} \quad (E_6)$$

Similarly,

$$U_3 = \left(\frac{5 - 10\lambda_2}{1 + \lambda_2} \right) U_2 = -2.2360 U_2 \text{ with } \lambda_2 = 0.9320$$

$$\therefore \vec{\tilde{U}}^{(2)} = \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1.0 \\ -2.2360 \end{Bmatrix} \quad (E_7)$$

Orthonormalization of normal modes with $[M]$ -matrix:

$$\text{Let } \vec{U}^{(1)} = a_1 \vec{\tilde{U}}^{(1)} \text{ and } \vec{U}^{(2)} = a_2 \vec{\tilde{U}}^{(2)} \quad (E_8)$$

where a_1 and a_2 are constants to be determined.

$$\vec{U}^{(1)T} [M] \vec{U}^{(1)} = 1 \text{ gives } a_1^2 = \frac{1}{\vec{\tilde{U}}^{(1)T} [M] \vec{\tilde{U}}^{(1)}}$$

$$\begin{aligned} \text{Since } \vec{\tilde{U}}^{(1)T} [M] \vec{\tilde{U}}^{(1)} &= (1.0 \quad 2.2359) \frac{\rho A l}{6} \begin{bmatrix} 10 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 2.2359 \end{Bmatrix} \\ &= 4.0784 \rho A l, \end{aligned}$$

$$a_1 = 0.4952 / \sqrt{\rho A l}, \text{ and}$$

$$\vec{U}^{(1)} = \frac{1}{\sqrt{\rho A l}} \begin{Bmatrix} 0.4952 \\ 1.1072 \end{Bmatrix} \quad (E_9)$$

$$\text{Similarly, } a_2^2 = \frac{1}{\vec{\tilde{U}}^{(2)T} [M] \vec{\tilde{U}}^{(2)}}$$

$$\begin{aligned} \text{where } \vec{\tilde{U}}^{(2)T} [M] \vec{\tilde{U}}^{(2)} &= (1.0 \quad -2.236) \frac{\rho A l}{6} \begin{bmatrix} 10 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 1.0 \\ -2.236 \end{Bmatrix} \\ &= 2.5879 \rho A l \end{aligned}$$

$$a_2 = 0.6216 / \sqrt{\rho A l}, \text{ and}$$

$$\vec{U}^{(2)} = \frac{1}{\sqrt{\rho A l}} \begin{Bmatrix} 0.6216 \\ -1.3900 \end{Bmatrix} \quad (E_{10})$$

Modal matrix:

$$[U] = \frac{1}{\sqrt{\rho A l}} \begin{bmatrix} 0.4952 & 0.6216 \\ 1.1072 & -1.3900 \end{bmatrix} \quad (E_{11})$$

with

$$[U]^T [M] [U] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } [U]^T [K] [U] = \frac{E}{\rho l^2} \begin{bmatrix} 1.3554 & 0 \\ 0 & 5.5921 \end{bmatrix}$$

Forced vibration equations:

$$\text{Equations of motion are } [M] \ddot{\vec{U}} + [K] \vec{U} = \vec{P} \quad (E_{12})$$

$$\text{Let } \vec{U}(t) = [U] \vec{\eta}(t) \quad (E_{13})$$

where $[U]$ = modal matrix and

$$\vec{\eta}(t) = \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} = \text{vector of generalized coordinates}$$

Substituting (E₁₃) into (E₁₂) and premultiplying by $[U]^T$ gives the uncoupled equations of motion:

$$\ddot{\eta}_i(t) + \omega_i^2 \eta_i(t) = Q_i(t) \quad ; \quad i = 1, 2 \quad (E_{14})$$

Where the generalized loads $Q_i(t)$ are given by

$$\begin{aligned} \vec{Q} &= [U]^T \vec{P} = \frac{1}{\sqrt{\rho A l}} \begin{bmatrix} 0.4952 & 0.6216 \\ 1.1072 & -1.3900 \end{bmatrix} \begin{Bmatrix} 0 \\ P(t) \end{Bmatrix} \\ &= \frac{1}{\sqrt{\rho A l}} \begin{Bmatrix} 0.6216 \\ -1.3900 \end{Bmatrix} P(t) \end{aligned} \quad (E_{15})$$

Hence equations of motion become

$$\ddot{\eta}_1 + 1.3554 \left(\frac{E}{\rho l^2} \right) \eta_1 = \frac{0.6216}{\sqrt{\rho A l}} P(t) \quad (E_{16})$$

$$\ddot{\eta}_2 + 5.5921 \left(\frac{E}{\rho l^2} \right) \eta_2 = - \frac{1.3900}{\sqrt{\rho A l}} P(t) \quad (E_{17})$$

Assume all initial conditions to be zero:

$$\left. \begin{aligned} \vec{U}(t=0) &= [U] \vec{\eta}(0) = 0 \\ \dot{\vec{U}}(t=0) &= [U] \dot{\vec{\eta}}(0) = 0 \end{aligned} \right\} \Rightarrow \vec{\eta}(0) = \dot{\vec{\eta}}(0) = \vec{0} \quad (E_{18})$$

Thus the solution of Eqs. (E₁₆) and (E₁₇) can be expressed as

$$\begin{aligned} \eta_1(t) &= \frac{1}{\omega_1} \int_0^t Q_1(\tau) \sin \omega_1(t-\tau) d\tau \\ &= \sqrt{\frac{l}{AE}} (0.5339) \int_0^t P(\tau) \sin \left\{ 1.1642 \sqrt{\frac{E}{\rho l^2}} (t-\tau) \right\} d\tau \end{aligned} \quad (E_{19})$$

$$\eta_2(t) = \frac{1}{\omega_2} \int_0^t Q_2(\tau) \sin \omega_2(t-\tau) d\tau$$

$$= -\sqrt{\frac{l}{AE}} (0.5878) \int_0^t P(\tau) \cdot \sin \left\{ \sqrt{\frac{E}{\rho l^2}} 2.3648 (t-\tau) \right\} d\tau \quad (E_{20})$$

Since

$$P(t) = \begin{cases} P_0 & \text{for } 0 \leq t \leq t_0 \\ 0 & \text{for } t > t_0 \end{cases} \quad (E_{21})$$

We can express the solution as

$$\vec{U}(t) = \begin{Bmatrix} U_2(t) \\ U_3(t) \end{Bmatrix} = [U] \vec{\eta}(t) = \frac{1}{\sqrt{\rho A l}} \begin{Bmatrix} 0.4952 \eta_1(t) + 0.6216 \eta_2(t) \\ 1.1072 \eta_1(t) - 1.3900 \eta_2(t) \end{Bmatrix}$$

Which becomes, in view of Eqs. (E₁₉) - (E₂₁):

For $t \leq t_0$:

$$\begin{aligned} U_2(t) &= \frac{P_0 l}{AE} \left\{ 0.3816 - 0.2271 \cos 1.1642 \sqrt{\frac{E}{\rho l^2}} t \right. \\ &\quad \left. - 0.1545 \cos 2.3648 \sqrt{\frac{E}{\rho l^2}} t \right\} \\ U_3(t) &= \frac{P_0 l}{AE} \left\{ 0.1622 - 0.5078 \cos 1.1642 \sqrt{\frac{E}{\rho l^2}} t \right. \\ &\quad \left. + 0.3456 \cos 2.3648 \sqrt{\frac{E}{\rho l^2}} t \right\} \quad (E_{22}) \end{aligned}$$

For $t > t_0$:

$$\begin{aligned} U_2(t) &= \frac{P_0 l}{AE} \left\{ 0.2271 \cos 1.1642 \sqrt{\frac{E}{\rho l^2}} (t-t_0) \right. \\ &\quad - 0.2271 \cos 1.1642 \sqrt{\frac{E}{\rho l^2}} t + 0.1547 \cos 2.3648 \sqrt{\frac{E}{\rho l^2}} (t-t_0) \\ &\quad \left. - 0.1547 \cos 2.3648 \sqrt{\frac{E}{\rho l^2}} t \right\} \\ U_3(t) &= \frac{P_0 l}{AE} \left\{ 0.5078 \cos 1.1642 \sqrt{\frac{E}{\rho l^2}} (t-t_0) \right. \\ &\quad - 0.5078 \cos 1.1642 \sqrt{\frac{E}{\rho l^2}} t - 0.3456 \cos 2.3648 \sqrt{\frac{E}{\rho l^2}} (t-t_0) \\ &\quad \left. + 0.3456 \cos 2.3648 \sqrt{\frac{E}{\rho l^2}} t \right\} \quad (E_{23}) \end{aligned}$$

12.26

$$E^{(1)} = E^{(2)} = 30 \times 10^6 \text{ psi}$$

$$A^{(1)} = A^{(2)} = 1 \text{ in}^2$$

$$l^{(1)} = 25''$$

$$l^{(2)} = \sqrt{25^2 + 10^2} = 26.9258''$$

$$\cos \theta_1 = \cos 0 = 1$$

$$\sin \theta_1 = 0$$

$$\cos \theta_2 = (x_3 - x_2) / l^{(2)} = 25 / 26.9258 = 0.9285$$

$$\sin \theta_2 = (y_3 - y_2) / l^{(2)} = (0 - 10) / 26.9258 = -0.3714$$

Element stiffness matrices:

$$[k^{(1)}] = \frac{A^{(1)} E^{(1)}}{l^{(1)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [\lambda^{(1)}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[K^{(1)}] = [\lambda^{(1)}]^T [k^{(1)}] [\lambda^{(1)}] = 12 \times 10^5 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_5 \\ u_6 \end{matrix}$$

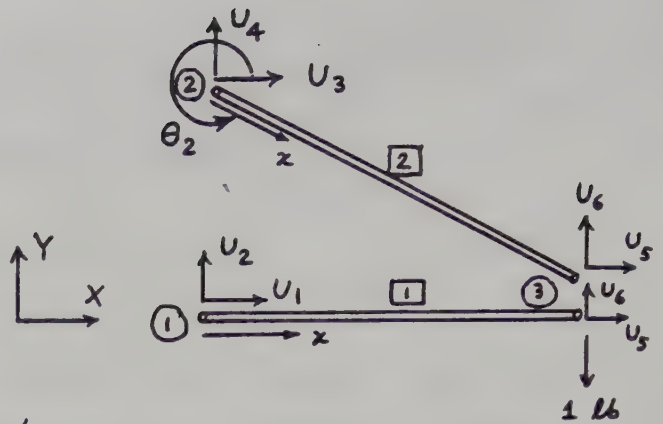
$$[k^{(2)}] = \frac{A^{(2)} E^{(2)}}{l^{(2)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 11.1417 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\lambda^{(2)}] = \begin{bmatrix} 0.9285 & -0.3714 & 0 & 0 \\ 0 & 0 & 0.9285 & -0.3714 \end{bmatrix}$$

$$[K^{(2)}] = [\lambda^{(2)}]^T [k^{(2)}] [\lambda^{(2)}]$$

$$= 11.1417 \times 10^5 \begin{bmatrix} 0.8621 & -0.3448 & -0.8621 & 0.3448 \\ -0.3448 & 0.1379 & 0.3448 & -0.1379 \\ -0.8621 & 0.3448 & 0.8621 & -0.3448 \\ 0.3448 & -0.1379 & -0.3448 & 0.1379 \end{bmatrix} \begin{matrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix}$$

Assembled stiffness matrix is:



$$[K] = \begin{bmatrix} (12 \times 10^5 + 0.8621 \times 11.1417 \times 10^5) & (0 - 0.3448 \times 11.1417 \times 10^5) \\ (0 - 0.3448 \times 11.1417 \times 10^5) & (0 + 0.1379 \times 11.1417 \times 10^5) \end{bmatrix}$$

$$= \begin{bmatrix} 21.6053 & -3.8417 \\ -3.8417 & 1.5364 \end{bmatrix} \times 10^5$$

Load vector: $\vec{F} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$

Equilibrium equations:

$$[K] \vec{U} = \vec{F}$$

i.e.,

$$\left. \begin{aligned} 21.6053 U_5 - 3.8417 U_6 &= 0 \\ -3.8417 U_5 + 1.5364 U_6 &= -1 \times 10^{-5} \end{aligned} \right\} \quad (E_1)$$

Solution is:

$$U_5 = -0.20838 \times 10^{-5} \text{ in.}, \quad U_6 = -1.1719 \times 10^{-5} \text{ in.}$$

Axial displacements of elements:

$$\begin{aligned} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^{(1)} &= [\lambda^{(1)}] \vec{U}^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} U_1=0 \\ U_2=0 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ U_5 \end{Bmatrix} \\ &= \begin{Bmatrix} 0 \\ -0.20838 \times 10^{-5} \end{Bmatrix} \text{ in.} \end{aligned}$$

$$\begin{aligned} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^{(2)} &= [\lambda^{(2)}] \vec{U}^{(2)} = \begin{bmatrix} 0.9285 & -0.3714 & 0 & 0 \\ 0 & 0 & 0.9285 & -0.3714 \end{bmatrix} \begin{Bmatrix} U_3=0 \\ U_4=0 \\ U_5 \\ U_6 \end{Bmatrix} \\ &= \begin{Bmatrix} 0 \\ 0.24176 \times 10^{-5} \end{Bmatrix} \text{ in.} \end{aligned}$$

Stresses in elements:

$$\sigma^{(1)} = E^{(1)} \epsilon^{(1)} = E^{(1)} (u_2^{(1)} - u_1^{(1)}) / l^{(1)}$$

$$= (30 \times 10^6) (-0.20838 \times 10^{-5}) / 25 = -2.5056 \text{ psi}$$

$$\sigma^{(2)} = E^{(2)} \epsilon^{(2)} = E^{(2)} (u_2^{(2)} - u_1^{(2)}) / l^{(2)}$$

$$= (30 \times 10^6) (0.24176 \times 10^{-5}) / 26.9258 = +2.6936 \text{ psi}$$

12.27

$$E_i = 2.1 \times 10^{11} \text{ Pa}; \quad i = 1, 2, 3$$

$$l_1 = 2.0 \text{ m}, \quad l_2 = 0.4 \text{ m}, \quad l_3 = 2.4 \text{ m}$$

$$I_1 = I_2 = \frac{\pi}{64} \left[\left(\frac{830}{1000} \right)^4 - \left(\frac{800}{1000} \right)^4 \right]$$

$$= 3.189853 \times 10^{-3} \text{ m}^4$$

$$I_3 = \frac{1}{12} [350(550)^3 - 320(520)^3] \times 10^{-12}$$

$$= 1.103058 \times 10^{-3} \text{ m}^4$$

$$\rho_i = 7.8 \times 10^3 \text{ kg/m}^3; \quad i = 1, 2, 3$$

$$A_1 = A_2 = \frac{\pi}{4} \left[\left(\frac{830}{1000} \right)^2 - \left(\frac{800}{1000} \right)^2 \right]$$

$$= 0.038406 \text{ m}^2$$

$$A_3 = (350 \times 550 - 320 \times 520) \times 10^{-6}$$

$$= 0.0261 \text{ m}^2$$

$$\frac{E_1 I_1}{l_1^3} = 8373.3641 \times 10^4; \quad \frac{E_2 I_2}{l_2^3} = 1046670.516 \times 10^4;$$

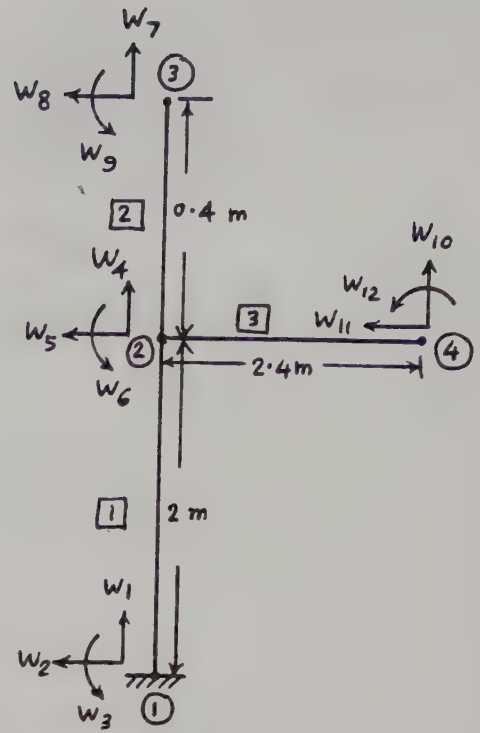
$$\frac{E_3 I_3}{l_3^3} = 1675.6523 \times 10^4$$

Element stiffness matrices:

$$[K^{(e)}] = \frac{E_e I_e}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$[K^{(1)}] = \begin{bmatrix} & w_2 & & & & \\ & 10.0480 & & & & \\ & 10.0480 & & & & \\ -10.0480 & & & & & \\ 10.0480 & & & & & \end{bmatrix} \times 10^8$$

$$\begin{matrix} & & w_3 & & w_5 & & w_6 \\ & & 10.0480 & & -10.0480 & & 10.0480 \\ & & 13.3974 & & -10.0480 & & 6.6987 \\ & & -10.0480 & & 10.0480 & & -10.0480 \\ & & 6.6987 & & -10.0480 & & 13.3974 \end{matrix}$$



$$[K^{(2)}] = \begin{bmatrix} & w_5 & w_6 & w_8 & w_9 \\ 1256.0052 & 251.2010 & -1256.0052 & 251.2010 \\ 251.2010 & 66.9869 & -251.2010 & 33.4935 \\ -1256.0052 & -251.2010 & 1256.0052 & -251.2010 \\ 251.2010 & 33.4935 & -251.2010 & 66.9869 \end{bmatrix} \times 10^8 \begin{matrix} w_5 \\ w_6 \\ w_8 \\ w_9 \end{matrix}$$

$$[K^{(3)}] = \begin{bmatrix} & w_4 & w_6 & w_{10} & w_{12} \\ 2.0108 & 2.4129 & -2.0108 & 2.4129 \\ 2.4129 & 3.8607 & -2.4129 & 1.9303 \\ -2.0108 & -2.4129 & 2.0108 & -2.4129 \\ 2.4129 & 1.9303 & -2.4129 & 3.8607 \end{bmatrix} \times 10^8 \begin{matrix} w_4 \\ w_6 \\ w_{10} \\ w_{12} \end{matrix}$$

Axial stiffnesses are given by

$$\frac{A_1 E_1}{l_1} = 40.3263 \times 10^8, \quad \frac{A_2 E_2}{l_2} = 201.6315 \times 10^8, \quad \frac{A_3 E_3}{l_3} = 22.8375 \times 10^8$$

Considering the axial stiffnesses of elements 1 and 2 at degree of freedom w_4 , the assembled stiffness matrix can be derived as

$$[K] = 10^8 \begin{bmatrix} & w_4 & w_5 & w_6 & w_8 & w_9 & w_{10} & w_{12} \\ 243.9686 & 0 & 2.4129 & 0 & 0 & -2.0108 & 2.4129 \\ 0 & 1266.0532 & 241.1530 & -1256.0052 & 251.2010 & 0 & 0 \\ 2.4129 & 241.1530 & 84.245 & -251.2010 & 33.4935 & -2.4129 & 1.9303 \\ 0 & -1256.0052 & -251.2010 & 1256.0052 & -251.2010 & 0 & 0 \\ 0 & 251.2010 & 33.4935 & -251.2010 & 66.9869 & 0 & 0 \\ -2.0108 & 0 & -2.4129 & 0 & 0 & 2.0108 & -2.4129 \\ 2.4129 & 0 & 1.9303 & 0 & 0 & -2.4129 & 3.8607 \end{bmatrix} \begin{matrix} w_4 \\ w_5 \\ w_6 \\ w_8 \\ w_9 \\ w_{10} \\ w_{12} \end{matrix}$$

$$\frac{P_1 A_1 l_1}{420} = 1.4265, \quad \frac{P_2 A_2 l_2}{420} = 0.2853, \quad \frac{P_3 A_3 l_3}{420} = 1.1633$$

Element mass matrices:

$$[M^{(e)}] = \frac{\rho_e A_e l_e}{420} \begin{bmatrix} 156 & 22 l_e & 54 & -13 l_e \\ 22 l_e & 4 l_e^2 & 13 l_e & -3 l_e^2 \\ 54 & 13 l_e & 156 & -22 l_e \\ -13 l_e & -3 l_e^2 & -22 l_e & 4 l_e^2 \end{bmatrix}$$

$$[M^{(1)}] = \begin{bmatrix} w_2 & w_3 & w_5 & w_6 \\ 222.534 & 62.766 & 77.031 & -37.089 \\ 62.766 & 22.824 & 37.089 & -17.118 \\ 77.031 & 37.089 & 222.534 & -62.766 \\ -37.089 & -17.118 & -62.766 & 22.824 \end{bmatrix} \begin{matrix} w_2 \\ w_3 \\ w_5 \\ w_6 \end{matrix}$$

$$[M^{(2)}] = \begin{bmatrix} w_5 & w_6 & w_8 & w_9 \\ 44.5068 & 2.5106 & 15.4062 & -1.4836 \\ 2.5106 & 0.1826 & 1.4836 & -0.1369 \\ 15.4062 & 1.4836 & 44.5068 & -2.5106 \\ -1.4836 & -0.1369 & -2.5106 & 0.1826 \end{bmatrix} \begin{matrix} w_5 \\ w_6 \\ w_8 \\ w_9 \end{matrix}$$

$$[M^{(3)}] = \begin{bmatrix} w_4 & w_6 & w_{10} & w_{12} \\ 181.4748 & 61.4222 & 62.8182 & -36.2950 \\ 61.4222 & 26.8024 & 36.2950 & -20.1018 \\ 62.8182 & 36.2950 & 181.4748 & -61.4222 \\ -36.2950 & -20.1018 & -61.4222 & 26.8024 \end{bmatrix} \begin{matrix} w_4 \\ w_6 \\ w_{10} \\ w_{12} \end{matrix}$$

For axial motion,

$$\frac{\rho_1 A_1 l_1}{3} = 199.71, \quad \frac{\rho_2 A_2 l_2}{3} = 39.942, \quad \frac{\rho_3 A_3 l_3}{3} = 162.864.$$

Considering mass matrix terms (corresponding to axial motion of elements 1 and 2) at degree of freedom w_4 , the assembled mass matrix can be obtained as:

$$[M] = \begin{bmatrix} w_4 & w_5 & w_6 & w_8 & w_9 & w_{10} & w_{12} \\ 421.1268 & 0 & 61.4222 & 0 & 0 & 62.8182 & -36.2950 \\ 0 & 267.0408 & -60.2554 & 15.4062 & -1.4836 & 0 & 0 \\ 61.4222 & -60.2554 & 49.8090 & 1.4836 & -0.1369 & 36.2950 & -20.1018 \\ 0 & 15.4062 & 1.4836 & 44.5068 & -2.5106 & 0 & 0 \\ 0 & -1.4836 & -0.1369 & -2.5106 & 0.1826 & 0 & 0 \\ 62.8182 & 0 & 36.2950 & 0 & 0 & 181.4748 & -61.4222 \\ -36.2950 & 0 & -20.1018 & 0 & 0 & -61.4222 & 26.8024 \end{bmatrix} \begin{matrix} w_4 \\ w_5 \\ w_6 \\ w_8 \\ w_9 \\ w_{10} \\ w_{12} \end{matrix}$$

Once $[K]$ and $[M]$ are known, the natural frequencies can be found by solving the eigenvalue problem.

12.28

Force vector:

$$\vec{F}^T = \{F_4, F_5, F_6, F_8, F_9, F_{10}, F_{10}\} = \{0, 0, 0, 0, 0, 5000, 500\}$$

Equilibrium equations:

$$[K] \vec{W} = \vec{F} \quad \dots (E_1)$$

where $[K]$ is given in the solution of Problem 12.27. The solution of Eqs. (E₁) is given by

$$\begin{aligned} w_4 &= 0.2066 \times 10^{-6} \text{ m}, & w_5 &= 0.3732 \times 10^{-4} \text{ m}, & w_6 &= 0.3731 \times 10^{-4} \text{ rad}, \\ w_8 &= 0.5224 \times 10^{-4} \text{ m}, & w_9 &= 0.3731 \times 10^{-4} \text{ rad}, & w_{10} &= 0.1954 \times 10^{-3} \text{ m}, \\ w_{12} &= 0.1046 \times 10^{-3} \text{ rad}. \end{aligned}$$

Bending stress in element "e":

$$\begin{aligned} \sigma_{\max} &= \left(E_e I_e \frac{d^2 w}{dx^2} \right)_e \cdot y_{\max, e} / I_e = E_e y_{\max, e} \frac{d^2 w}{dx^2} \bigg|_e \\ &= E_e y_{\max, e} \sum_{i=1}^4 W_i^{(e)} \frac{d^2 N_i^{(e)}}{dx^2} \\ &= E_e y_{\max, e} \left[W_1^{(e)} \left(-\frac{6}{l^2} + \frac{12}{l^3} x \right) + W_2^{(e)} \left(-\frac{4}{l} + \frac{6}{l^2} x \right) \right. \\ &\quad \left. + W_3^{(e)} \left(\frac{6}{l^2} - \frac{12}{l^3} x \right) + W_4^{(e)} \left(-\frac{2}{l} + \frac{6}{l^2} x \right) \right] \dots (E_2) \end{aligned}$$

Bending stress in element 1 ($e=1$):

$$\begin{aligned} w_1^{(1)} &= w_2 = 0, & w_2^{(1)} &= w_3 = 0, & w_3^{(1)} &= w_5 = 0.3732 \times 10^{-4} \text{ m}, \\ w_4^{(1)} &= w_6 = 0.3731 \times 10^{-4} \text{ rad} \end{aligned}$$

$$l_1 = 2 \text{ m}, \quad y_{\max, 1} = 0.415 \text{ m}, \quad E_1 = 2.1 \times 10^{11} \text{ Pa}$$

At $x=0$ (at node ①):

$$\begin{aligned} \sigma_{\max} &= (2.1 \times 10^{11}) (0.415) \left[0.3732 \times 10^{-4} \left(\frac{6}{4} \right) + 0.3731 \times 10^{-4} \left(-\frac{2}{2} \right) \right] \\ &= 1.6262 \times 10^6 \text{ Pa} \end{aligned}$$

At $x=l_1=2 \text{ m}$ (at node ②):

$$\begin{aligned} \sigma_{\max} &= (2.1 \times 10^{11}) (0.415) \left[0.3732 \times 10^{-4} \left(\frac{6}{4} - \frac{12}{4} \right) + 0.3731 \times 10^{-4} \left(-\frac{2}{2} + \frac{6}{2} \right) \right] \\ &= 1.6245 \times 10^6 \text{ Pa} \end{aligned}$$

Axial stress in element 1 (at node ①):

$$\begin{aligned} \sigma &= \left(\frac{w_4 - w_1}{l_1} \right) E_1 = E_1 E_1 = \left[(0.2066 \times 10^{-6} - 0) / 2 \right] (2.1 \times 10^{11}) \\ &= 0.0217 \times 10^6 \text{ Pa} \end{aligned}$$

$$\text{Total stress in element 1} = (1.6262 + 0.0217)10^6 = 1.6479 \times 10^6 \text{ Pa}$$

Bending stress in element 2 ($e=2$):

$$w_1^{(2)} = w_5 = 0.3732 \times 10^{-4} \text{ m}, \quad w_2^{(2)} = w_6 = 0.3731 \times 10^{-4} \text{ rad},$$

$$w_3^{(2)} = w_8 = 0.5224 \times 10^{-4} \text{ m}, \quad w_4^{(2)} = w_9 = 0.3731 \times 10^{-4} \text{ rad},$$

$$l_2 = 0.4 \text{ m}, \quad y_{\max,2} = 0.415 \text{ m}, \quad E_2 = 2.1 \times 10^{11} \text{ Pa}$$

At $x=0$:

$$\begin{aligned} \sigma_{\max} = (2.1 \times 10^{11})(0.415) & \left[0.3732 \left(-\frac{6}{0.16} \right) + 0.3731 \left(-\frac{4}{0.4} \right) \right. \\ & \left. + 0.5224 \left(\frac{6}{0.16} \right) - 0.3731 \left(-\frac{2}{0.4} \right) \right] 10^{-4} = 32.503 \times 10^6 \text{ Pa} \end{aligned}$$

Bending stress in element 3 ($e=3$):

$$w_1^{(3)} = w_4 = 0.2066 \times 10^{-6} \text{ m}, \quad w_2^{(3)} = w_6 = 0.3731 \times 10^{-4} \text{ rad},$$

$$w_3^{(3)} = w_{10} = 0.1954 \times 10^{-3} \text{ m}, \quad w_4^{(3)} = w_{12} = 0.1046 \times 10^{-3} \text{ rad}$$

$$l_3 = 2.4 \text{ m}, \quad y_{\max,3} = 0.275 \text{ m}, \quad E_3 = 2.1 \times 10^{11} \text{ Pa}$$

At $x=0$ (node ②):

$$\begin{aligned} \sigma_{\max} = (2.1 \times 10^{11})(0.275) & \left[0.2066 \times 10^{-6} \left(-\frac{6}{5.76} \right) \right. \\ & + 0.3731 \times 10^{-4} \left(-\frac{4}{2.4} \right) + 0.1954 \times 10^{-3} \left(\frac{6}{5.76} \right) \\ & \left. + 0.1046 \times 10^{-3} \left(-\frac{2}{2.4} \right) \right] \\ & = 3.1172 \times 10^6 \text{ Pa} \end{aligned}$$

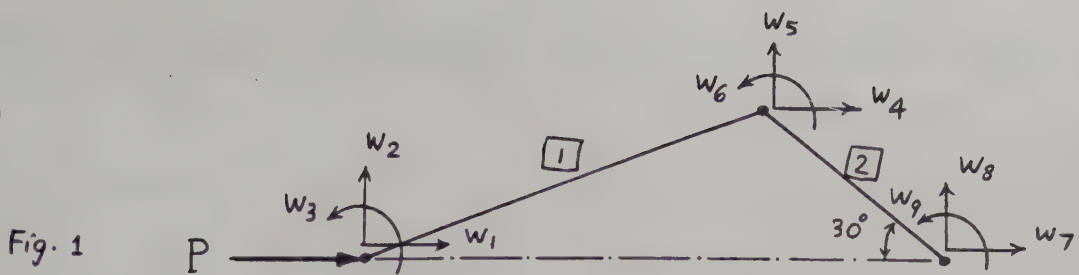
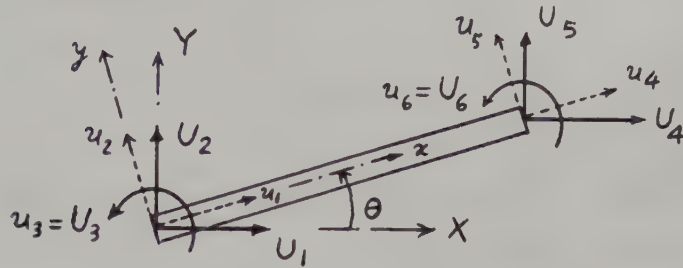


Fig. 2
General beam
element in a
plane



For a general beam element in XY plane, we consider two axial nodal displacements u_1 and u_4 , and four bending nodal displacements u_2 , u_3 , u_5 and u_6 . By superposing the stiffness matrices of a bar element and a beam element, the stiffness matrix of the element shown in Fig. 2 can be found as

$$[k] = \frac{EI}{l^3} \begin{bmatrix} \frac{Al^2}{I} & 0 & 0 & -\frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -\frac{Al^2}{I} & 0 & 0 & \frac{Al^2}{I} & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} \quad (E_1)$$

If U_i ; $i=1, \dots, 6$ denote the global nodal displacements, we find from Fig. 2,

$$\begin{aligned} u_1 &= U_1 \cos \theta + U_2 \sin \theta \\ u_2 &= -U_1 \sin \theta + U_2 \cos \theta \\ u_3 &= U_3 \\ u_4 &= U_4 \cos \theta + U_5 \sin \theta \\ u_5 &= -U_4 \sin \theta + U_5 \cos \theta \\ u_6 &= U_6 \end{aligned} \quad (E_2)$$

Defining $\lambda = \cos \theta$ and $\mu = \sin \theta$, $E_2 \cdot (E_2)$ can be expressed as

$$\vec{u} = [\tilde{\lambda}] \vec{U}$$

where

$$\vec{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}, \quad \vec{U} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} \quad \text{and} \quad [\tilde{\lambda}] = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (E_3)$$

Global stiffness matrix of the general beam element can be found as

$$[K^{(e)}] = [\tilde{\lambda}]^T [k] [\tilde{\lambda}] \quad (E_4)$$

For element [1]:

$$E = 30 \times 10^6 \text{ psi}, \quad l = 48''$$

$$I = 2 \left[\frac{1}{12} (2) (0.5)^3 + (2 \times 0.5) (1.5 + 0.25)^2 \right]$$

$$+ \frac{1}{12} (0.5) (3)^3 = 7.2917 \text{ in}^4$$

$$A = 2(2 \times 0.5) + 0.5 \times 3 = 3.5 \text{ in}^2$$

$$\lambda = \cos 7.1808^\circ = 0.9922$$

$$\mu = \sin 7.1808^\circ = 0.1250$$

For element [2]:

$$E = 30 \times 10^6 \text{ psi}, \quad I = 7.2917 \text{ in}^4,$$

$$A = 3.5 \text{ in}^2, \quad l = 12'', \quad \lambda = \cos 33.0^\circ = 0.866, \quad \mu = \sin 33.0^\circ = -0.5$$

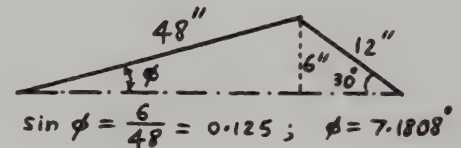
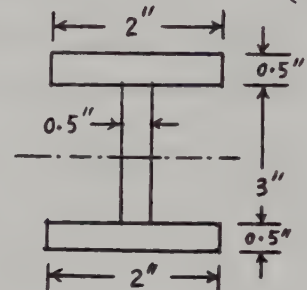
Boundary conditions:

$$w_2 = w_7 = w_8 = 0$$

$$\text{Load on piston} = \frac{\pi}{4} (12)^2 (200) = 22619.52 \text{ lb}$$

Load vector:

$$\vec{F} = \begin{Bmatrix} F_1 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_9 \end{Bmatrix} = \begin{Bmatrix} 22619.52 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$



Element matrices in global system (given by Eg. (E₄)):

$$[K^{(1)}] = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\ \cdot 2154 E 7 & & & & & \\ \cdot 2684 E 6 & \cdot 5755 E 5 & & & & \\ -\cdot 7121 E 5 & \cdot 5652 E 6 & \cdot 1823 E 8 & & & \\ -\cdot 2154 E 7 & -\cdot 2684 E 6 & \cdot 7121 E 5 & \cdot 2154 E 7 & & \\ -\cdot 2684 E 6 & -\cdot 5755 E 5 & -\cdot 5652 E 6 & \cdot 2684 E 6 & \cdot 5755 E 5 & \\ -\cdot 7121 E 5 & \cdot 5652 E 6 & \cdot 9115 E 7 & \cdot 7121 E 5 & -\cdot 5652 E 6 & \cdot 1823 E 8 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{matrix}$$

Symmetric

$$[K^{(2)}] = \begin{bmatrix} w_4 & w_5 & w_6 & w_7 & w_8 & w_9 \\ \cdot 6942 E 7 & & & & & \\ -\cdot 3131 E 7 & \cdot 3327 E 7 & & & & \\ \cdot 4557 E 7 & \cdot 7893 E 7 & \cdot 7292 E 8 & & & \\ -\cdot 6942 E 7 & \cdot 3131 E 7 & -\cdot 4557 E 7 & \cdot 6942 E 7 & & \\ \cdot 3131 E 7 & -\cdot 3327 E 7 & -\cdot 7893 E 7 & -\cdot 3131 E 7 & \cdot 3327 E 7 & \\ \cdot 4557 E 7 & \cdot 7893 E 7 & \cdot 3646 E 8 & -\cdot 4557 E 7 & -\cdot 7893 E 7 & \cdot 7292 E 8 \end{bmatrix} \begin{matrix} w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{matrix}$$

Symmetric

Assembled stiffness matrix (after incorporating boundary conditions):

$$[K] = \begin{bmatrix} w_1 & w_3 & w_4 & w_5 & w_6 & w_9 \\ 2 \cdot 154 & & & & & \\ -\cdot 0712 & 18 \cdot 23 & & & & \\ -2 \cdot 154 & \cdot 0712 & (2 \cdot 154 + 6 \cdot 942) & & & \\ -\cdot 2684 & -\cdot 5652 & (\cdot 2684 - 3 \cdot 131) & (\cdot 0575 + 3 \cdot 327) & & \\ -\cdot 0712 & 9 \cdot 115 & (\cdot 0712 + 4 \cdot 557) & (-\cdot 5652 + 7 \cdot 893) & (18 \cdot 23 + 72 \cdot 92) & \\ 0 & 0 & 4 \cdot 557 & 7 \cdot 893 & 36 \cdot 46 & 72 \cdot 92 \end{bmatrix} \begin{matrix} w_1 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_9 \end{matrix}$$

Symmetric

Solution of the equilibrium equations $[K] \vec{W} = \vec{F}$ gives the following:

$$\begin{aligned} w_1 &= 0 \cdot 0866 && \text{in} \\ w_3 &= 0 \cdot 007192 && \text{rad} \\ w_4 &= 0 \cdot 06306 && \text{in} \\ w_5 &= 0 \cdot 1047 && \text{in} \\ w_6 &= -0 \cdot 007704 && \text{rad} \\ w_9 &= -0 \cdot 01143 && \text{rad} \end{aligned}$$

Element displacements in local coordinate system:

Element 1: $\lambda = 0.9922, \mu = 0.1250$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}^{(1)} = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{(1)} \begin{Bmatrix} W_1 = 0.0866 \\ W_2 = 0 \\ W_3 = 0.00719 \\ W_4 = 0.06306 \\ W_5 = 0.1047 \\ W_6 = -0.0077 \end{Bmatrix} = \begin{Bmatrix} 0.08592 \text{ in} \\ -0.01083 \text{ in} \\ 0.00719 \text{ rad} \\ 0.07566 \text{ in} \\ 0.09599 \text{ in} \\ -0.0077 \text{ rad} \end{Bmatrix}$$

Element 2: $\lambda = 0.866, \mu = -0.5$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}^{(2)} = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{(2)} \begin{Bmatrix} W_4 = 0.06306 \\ W_5 = 0.1047 \\ W_6 = -0.0077 \\ W_7 = 0 \\ W_8 = 0 \\ W_9 = -0.01143 \end{Bmatrix} = \begin{Bmatrix} 0.00226 \text{ in} \\ 0.1222 \text{ in} \\ -0.0077 \text{ rad} \\ 0 \\ 0 \\ -0.01143 \text{ rad} \end{Bmatrix}$$

Axial stresses:

Element 1:

$$\sigma(x) = E \left(-\frac{1}{l} \quad \frac{1}{l} \right) \begin{Bmatrix} u_1 \\ u_4 \end{Bmatrix}^{(1)} = \frac{E}{l} (-u_1 + u_4)^{(1)} = \frac{30 \times 10^6}{48} (-0.08592 + 0.07566) \\ = -6411 \text{ psi}$$

Element 2:

$$\sigma(x) = \frac{E}{l} (-u_1 + u_4)^{(2)} = \frac{30 \times 10^6}{12} (-0.00226 + 0) = -5649 \text{ psi}$$

Bending stresses:

Element 1: $y_{\max} = 2''$, $l = 48''$

$$\sigma(x) = E \frac{d^2 w(x)}{dx^2} y_{\max} = E y_{\max} \left\{ \left(-\frac{6}{l^2} + \frac{12x}{l^3} \right) u_2 + \left(-\frac{4}{l^2} + \frac{6x}{l^3} \right) l u_3 \right. \\ \left. + \left(\frac{6}{l^2} - \frac{12x}{l^3} \right) u_5 + \left(-\frac{2}{l^2} + \frac{6x}{l^3} \right) l u_6 \right\}$$

at $x = 0$,

$$\sigma(0) = -12 \text{ psi}$$

$$\text{at } x = 48, \quad \sigma(48) = -37218 \text{ psi}$$

Element 2:

$$y_{\max} = 2'', \quad l = 12''$$

$$\sigma(x) = E y_{\max} \left\{ \left(-\frac{6}{l^2} + \frac{12x}{l^3} \right) u_2 + \left(-\frac{4}{l^2} + \frac{6x}{l^3} \right) l u_3 \right. \\ \left. + \left(\frac{6}{l^2} - \frac{12x}{l^3} \right) u_5 + \left(-\frac{2}{l^2} + \frac{6x}{l^3} \right) l u_6 \right\}$$

at $x=0$, $\sigma(0) = -37206 \text{ psi}$

at $x=12$, $\sigma(12) = -114 \text{ psi}$

Maximum stresses:

In element 1,

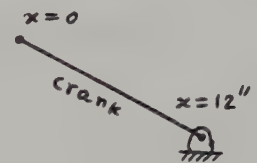
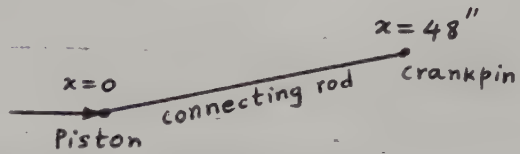
$$\sigma_{\max} = \max(\text{axial} \pm \text{bending})$$

$$= -6411 \pm 37218 = -43629 \text{ psi at } x=48''$$

In element 2,

$$\sigma_{\max} = \max(\text{axial} \pm \text{bending})$$

$$= -5649 \pm 37206 = -42855 \text{ psi at } x=0.$$



12.30

$$l = 480'', \quad E = 30 \times 10^6 \text{ psi}$$

$$I = \frac{\pi}{64} \{ (d+2t)^4 - d^4 \} = \frac{\pi}{64} \{ (24+2)^4 - 24^4 \} \\ = 6145.755 \text{ in}^4$$

$$\rho = 0.283/386.4 = 7.324 \times 10^{-4} \text{ lb-s}^2/\text{in}^4$$

$$A = \frac{\pi}{4} \{ (d+2t)^2 - d^2 \} = \frac{\pi}{4} (26^2 - 24^2) \\ = 78.54 \text{ in}^2$$

$$m = 10000/386.4 = 25.8799 \text{ lb-s}^2/\text{in}, \quad p_{\max} = 100 \text{ psi}$$

pressure at $x = \left(\frac{x}{l} p_{\max} \right) \text{ psi}$

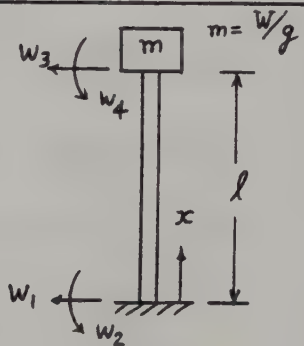
load at $x = \frac{x}{l} p_{\max} (d+2t) \text{ lb/in} = 5.4167 x \text{ lb/in}$

Stiffness matrix:

$$[K] = [K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

where $\frac{EI}{l^3} = \frac{(30 \times 10^6)(6145.755)}{(480)^3} = 1667.1427 \text{ lb/in}$

stiffness matrix, after incorporating the conditions $w_1 = w_2 = 0$, is



$$[K] = \frac{EI}{l^3} \begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix} = 1667.1427 \begin{bmatrix} 12 & -2880 \\ -2880 & 921600 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

Mass matrix:

$$[M] = [M^{(1)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

where $\frac{\rho A l}{420} = \frac{(7.324 \times 10^{-4})(78.54)(480)}{420} = 657.4022 \times 10^{-4} \text{ lb-s}^2/\text{in}$

Mass matrix, after incorporating $w_1 = w_2 = 0$,

$$[M] = \frac{\rho A l}{420} \begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix} = 657.4022 \times 10^{-4} \begin{bmatrix} 156 & -10560 \\ -10560 & 921600 \end{bmatrix} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

Consistent load vector:

$$f_1 = \int_0^l f(x) N_1(x) dx \quad \text{where } f(x) = \text{distributed transverse load}$$

$$= \int_0^l (5.4167x) \left\{ 1 - 3 \frac{x^2}{l^2} + 2 \frac{x^3}{l^3} \right\} dx = 5.4167 \left(\frac{3}{20} l^2 \right)$$

$$= 187201.152 \text{ lb}$$

$$f_3 = \int_0^l f(x) N_3(x) dx = \int_0^l (5.4167x) \left\{ 3 \left(\frac{x}{l} \right)^2 - 2 \left(\frac{x}{l} \right)^3 \right\} dx$$

$$= 5.4167 \left(\frac{7}{20} l^2 \right) = 436802.688 \text{ lb}$$

As there is no distributed bending moment, $f_2 = f_4 = 0$.

(a) STATIC ANALYSIS

Equilibrium equations are $[K] \vec{w} = \vec{f}$

i.e.,

$$1667.1427 \begin{bmatrix} 12 & -2880 \\ -2880 & 921600 \end{bmatrix} \begin{Bmatrix} w_3 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} = \begin{Bmatrix} 436802.688 \\ 0 \end{Bmatrix}$$

solution is

$$w_3 = 87.3356 \text{ in}$$

$$w_4 = 0.2729 \text{ rad}$$

Since $w(x) = N_1(x) W_1 + N_2(x) W_2 + N_3(x) W_3 + N_4(x) W_4$,

$$\text{stress} = \sigma = \frac{M \cdot c}{I} = \left(EI \frac{d^2 w}{dx^2} \right) \frac{c}{I}$$

with

$$\frac{d^2 w}{dx^2} = \sum_{i=1}^4 \frac{d^2 N_i(x)}{dx^2} W_i$$

Since $W_1 = W_2 = 0$, $N_3 = 3\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3$ and $N_4 = -\frac{x^2}{l} + \frac{x^3}{l^2}$,

and stress will be maximum at $x=0$,

$$\begin{aligned} \sigma_{\max} \Big|_{x=0} &= E c \frac{d^2 w}{dx^2} \Big|_{x=0} = (30 \times 10^6) \left(\frac{d+2t}{2} \right) \frac{d^2 w}{dx^2} \Big|_{x=0} \\ &= (30 \times 10^6) (13) (1.1373 \times 10^{-3}) = 443547 \text{ psi} \end{aligned}$$

Direct compressive stress due to load W :

$$\sigma_c = \frac{W}{A} = \frac{10000}{78.54} = 127.3237 \text{ psi}$$

$\therefore$ Max. compressive stress $= \sigma_c + \sigma_{\max} = 443674.3237 \text{ psi}$

Max. tensile stress $= -\sigma_c + \sigma_{\max} = 443419.6763 \text{ psi}$

(b) NATURAL FREQUENCY ANALYSIS

By adding the mass of tank, the mass matrix becomes

$$\begin{aligned} [M] &= 657.4022 \times 10^{-4} \begin{bmatrix} 156 & -10560 \\ -10560 & 921600 \end{bmatrix} + \begin{bmatrix} 25.8799 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 36.1054 & -694.2167 \\ -694.2167 & 60586.1867 \end{bmatrix} \end{aligned}$$

Natural frequencies are given by:

$$| -\omega^2 [M] + [K] | = 0$$

i.e.,

$$\begin{vmatrix} (2.0006 - 36.1054 \lambda) & (-480.1371 + 694.2167 \lambda) \\ (-480.1371 + 694.2167 \lambda) & (153643.8712 - 60586.1867 \lambda) \end{vmatrix} = 0$$

where $\lambda = (\omega^2 / 10^4)$.

This gives

$$170.5552 \lambda^2 - 500.1944 \lambda + 7.6848 = 0$$

$$\text{Hence } \lambda_{1,2} = \frac{500.1944 \pm (250194.4378 - 5242.7304)^{1/2}}{341.1104}$$

$$\lambda_1 = 0.01544,$$

$$\omega_1 = 12.4258 \text{ rad/sec}$$

$$\lambda_2 = 2.9173,$$

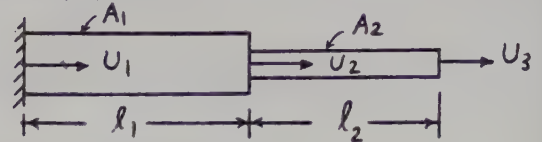
$$\omega_2 = 170.8011 \text{ rad/sec}$$

12.31

$$A_1 = 2 \text{ in}^2, \quad A_2 = 1 \text{ in}^2$$

$$E = 30 \times 10^6 \text{ psi}$$

$$\rho = 0.283 \text{ lb/in}^3, \quad l_1 = l_2 = 50 \text{ in}$$



Consistent mass matrices

$$\begin{aligned} [M^{(1)}]_c &= \frac{\rho_1 A_1 l_1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \left(\frac{0.283}{386.4} \right) (2) (50) \frac{1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ &= 0.01221 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix} \end{aligned}$$

$$\begin{aligned} [M^{(2)}]_c &= \frac{\rho_2 A_2 l_2}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \left(\frac{0.283}{386.4} \right) (1) (50) \frac{1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ &= 0.006105 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix} \end{aligned}$$

Lumped mass matrices

$$[M^{(1)}]_l = \frac{\rho_1 A_1 l_1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.03662 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix}$$

$$[M^{(2)}]_l = \frac{\rho_2 A_2 l_2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.01831 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix}$$

stiffness matrices

$$[K^{(1)}] = \frac{A_1 E_1}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2(30 \times 10^6)}{50} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1.2 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix}$$

$$[K^{(2)}] = \frac{A_2 E_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1(30 \times 10^6)}{50} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0.6 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix}$$

Assembled matrices (before applying boundary conditions)

$$[K] = 0.6 \times 10^6 \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix}$$

$$[M]_c = 0.006105 \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4+2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix}$$

$$[\tilde{M}]_l = 0.01831 \begin{bmatrix} u_1 & u_2 & u_3 \\ 2 & 0 & 0 \\ 0 & 2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}$$

Assembled matrices after applying boundary conditions

$$[K] = 0.6 \times 10^6 \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[M]_c = 0.006105 \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M]_l = 0.01831 \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Natural frequencies with consistent mass matrices:

$$\left| -\omega^2 (0.006105) \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} + 0.6 \times 10^6 \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} 3-6\lambda & -1-\lambda \\ -1-\lambda & 1-2\lambda \end{vmatrix} = 0$$

where $\lambda = \omega^2 (0.006105) / 0.6 \times 10^6 = 0.010175 \times 10^{-6} \omega^2$

i.e.,

$$11\lambda^2 - 14\lambda + 2 = 0$$

$$\therefore \lambda_{1,2} = \frac{14 \pm \sqrt{196 - 88}}{22} = 0.163986, 1.108741$$

or $\omega_1 = 4.0145 \times 10^3 \text{ rad/sec}$, $\omega_2 = 10.4387 \times 10^3 \text{ rad/sec}$

Natural frequencies with 'lumped mass matrices:

Here the frequency equation can be expressed as

$$\begin{vmatrix} (3-3\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0 \quad \text{with} \quad \lambda = \omega^2 (0.01831) / 0.6 \times 10^6 = 0.0305167 \times 10^{-6} \omega^2$$

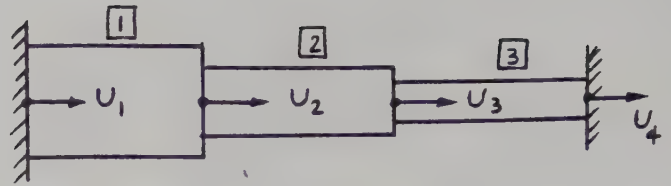
i.e., $3\lambda^2 - 6\lambda + 2 = 0$

$$\therefore \lambda_1 = 0.42265, \quad \lambda_2 = 1.57735$$

or $\omega_1 = 6.4450 \times 10^3 \text{ rad/sec}$, $\omega_2 = 12.4508 \times 10^3 \text{ rad/sec}$

12.32

With consistent mass matrices:



$$[M^{(e)}]_c = \frac{\rho^{(e)} A^{(e)} \ell^{(e)}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M^{(1)}]_c = \frac{(7.8 \times 10^3)(0.4 \times 10^{-3})(0.2)}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 0.104 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M^{(2)}]_c = \frac{(7.8 \times 10^3)(0.2 \times 10^{-3})(0.2)}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 0.052 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M^{(3)}]_c = \frac{(7.8 \times 10^3)(0.1 \times 10^{-3})(0.2)}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 0.026 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[\tilde{M}]_c = 0.026 \begin{bmatrix} U_1 & U_2 & U_3 & U_4 \\ 8 & 4 & 0 & 0 \\ 4 & 8+4 & 2 & 0 \\ 0 & 2 & 4+2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix}$$

$$[M]_c = 0.026 \begin{bmatrix} 12 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 0.312 & 0.052 \\ 0.052 & 0.156 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \end{matrix}$$

$$[K^{(e)}] = \frac{A^{(e)} E^{(e)}}{\ell^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K^{(1)}] = \frac{(0.4 \times 10^{-3})(2.1 \times 10^{11})}{0.2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 4.2 \times 10^8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K^{(2)}] = \frac{(0.2 \times 10^{-3})(2.1 \times 10^{11})}{0.2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2.1 \times 10^8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K^{(3)}] = \frac{(0.1 \times 10^{-3})(2.1 \times 10^{11})}{0.2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1.05 \times 10^8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\tilde{K}] = 1.05 \times 10^8 \begin{bmatrix} U_1 & U_2 & U_3 & U_4 \\ 4 & -4 & 0 & 0 \\ -4 & 4+2 & -2 & 0 \\ 0 & -2 & 2+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix}$$

$$[K] = 1.05 \times 10^8 \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} = 10^8 \begin{bmatrix} 6.3 & -2.1 \\ -2.1 & 3.15 \end{bmatrix} \begin{matrix} u_2 \\ u_3 \end{matrix}$$

Natural frequencies are given by

$$| -\omega^2 [M]_c + [K] | = 0$$

i.e.

$$\begin{vmatrix} 6.3 - 0.312 \lambda & -2.1 - 0.052 \lambda \\ -2.1 - 0.052 \lambda & 3.15 - 0.156 \lambda \end{vmatrix} = 0$$

$$\text{where } \lambda = 10^{-8} \omega^2$$

i.e.,

$$0.045968 \lambda^2 - 2.184 \lambda + 15.435 = 0$$

$$\text{This gives } \lambda_1 = 8.6372, \quad \lambda_2 = 38.8721$$

$$\text{or } \omega_1 = 2.9389 \times 10^4 \text{ rad/sec}, \quad \omega_2 = 6.2347 \times 10^4 \text{ rad/sec.}$$

with lumped
mass matrices:

$$[M^{(e)}]_l = \frac{\rho A l}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[M^{(1)}]_l = \frac{(7.8 \times 10^3)(0.4 \times 10^{-3})(0.2)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.312 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[M^{(2)}]_l = \frac{(7.8 \times 10^3)(0.2 \times 10^{-3})(0.2)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.156 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[M^{(3)}]_l = \frac{(7.8 \times 10^3)(0.1 \times 10^{-3})(0.2)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.078 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[M]_l = 0.078 \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ 4 & 0 & 0 & 0 \\ 0 & 4+2 & 0 & 0 \\ 0 & 0 & 2+1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}$$

$$[M]_l = 0.078 \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0.468 & 0 \\ 0 & 0.234 \end{bmatrix} \begin{matrix} u_2 \\ u_3 \end{matrix}$$

Natural frequencies are given by

$$|-\omega^2 [M]_l + [K]| = 0$$

i.e.,

$$\left| -\omega^2 \begin{bmatrix} 0.468 & 0 \\ 0 & 0.234 \end{bmatrix} + 10^8 \begin{bmatrix} 6.3 & -2.1 \\ -2.1 & 3.15 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} 6.3 - 0.468 \lambda & -2.1 \\ -2.1 & 3.15 - 0.234 \lambda \end{vmatrix} = 0$$

where $\lambda = 10^{-8} \omega^2$

i.e.,

$$0.109512 \lambda^2 - 2.9484 \lambda + 15.435 = 0$$

This gives $\lambda_1 = 7.1157$, $\lambda_2 = 19.8074$

or $\omega_1 = 2.6675 \times 10^4$ rad/sec , $\omega_2 = 4.4506 \times 10^4$ rad/sec.

12.33

```

C =====
C
C PROBLEM 12.33
C GENERATION OF ASSEMBLED STIFFNESS MATRIX OF A PLANAR TRUSS
C
C =====
C NE = NUMBER OF ELEMENTS
C N = TOTAL NUMBER OF DEGREES OF FREEDOM
C NN = TOTAL NUMBER OF NODES
C A(I) = AREA OF I TH ELEMENT
C EL(I) = LENGTH OF I TH ELEMENT
C DIMENSIONS:
C      BIGK(N,N),XK(4,4),A(NE),EL(NE),CX(NN),CY(NN),LOC(NE,2),AI(4,N),
C      AIT(N,4),XKA(4,N),AKA(N,N)
C
C ***** INPUT DATA *****
C      DIMENSION BIGK(8,8),XK(4,4),A(4),EL(4),CX(4),CY(4),LOC(4,2),
C      2 AI(4,8),AIT(8,4),XKA(4,8),AKA(8,8)
C      DATA NE,N,NN/4,8,4/
C      E=2.0E+6
C      DATA A/2.0,2.0,1.0,1.0/
C      DATA CX/0.0,50.0,100.0,200.0/
C      DATA CY/0.0,50.0,0.0,100.0/
C      DATA LOC/1,2,2,3,2,3,4,4/
C ***** END OF INPUT DATA *****
C      DO 10 I=1,N
C      DO 10 J=1,N
10    BIGK(I,J)=0.0
C      DO 100 I=1,NE
C      I1=LOC(I,1)
C      I2=LOC(I,2)
C      EL(I)=SQRT((CX(I2)-CX(I1))**2+(CY(I2)-CY(I1))**2)

```



```

XLIJ=(CX(I2)-CX(I1))/EL(I)
XMIJ=(CY(I2)-CY(I1))/EL(I)
XK(1,1)=XLIJ**2
XK(1,2)=XLIJ*XMIJ
XK(1,3)=-XK(1,1)
XK(1,4)=-XK(1,2)
XK(2,1)=XK(1,2)
XK(2,2)=XMIJ**2
XK(2,3)=-XK(2,1)
XK(2,4)=-XK(2,2)
XK(3,1)=-XK(1,1)
XK(3,2)=-XK(1,2)
XK(3,3)=XK(1,1)
XK(3,4)=XK(1,2)
XK(4,1)=-XK(2,1)
XK(4,2)=-XK(2,2)
XK(4,3)=XK(2,1)
XK(4,4)=XK(2,2)
DO 20 II=1,4
DO 20 JJ=1,4
20  XK(II,JJ)=XK(II,JJ)*A(I)*E/EL(I)
DO 30 II=1,4
DO 30 JJ=1,N
30  AI(II,JJ)=0.0
    I1=LOC(I,1)*2-1
    I2=LOC(I,1)*2
    I3=LOC(I,2)*2-1
    I4=LOC(I,2)*2
    AI(1,I1)=1.0
    AI(2,I2)=1.0
    AI(3,I3)=1.0
    AI(4,I4)=1.0
DO 40 II=1,N
DO 40 JJ=1,4
40  AIT(II,JJ)=AI(JJ,II)
    CALL MATMUL (XKA,XK,AI,4,4,N)
    CALL MATMUL (AKA,AIT,XKA,N,4,N)
DO 50 II=1,N
DO 50 JJ=1,N
50  BIGK(II,JJ)=BIGK(II,JJ)+AKA(II,JJ)
100 CONTINUE
DO 120 II=1,N
120 PRINT 130, (BIGK(II,JJ),JJ=1,N)
130 FORMAT (1X,10(1X,E8.2))
STOP
END

```

0.28E+05	0.28E+05	-.28E+05	-.28E+05	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.28E+05	0.28E+05	-.28E+05	-.28E+05	0.00E+00	0.00E+00	0.00E+00	0.00E+00
-.28E+05	-.28E+05	0.68E+05	0.38E+04	-.28E+05	0.28E+05	-.11E+05	-.38E+04
-.28E+05	-.28E+05	0.38E+04	0.58E+05	0.28E+05	-.28E+05	-.38E+04	-.13E+04
0.00E+00	0.00E+00	-.28E+05	0.28E+05	0.35E+05	-.21E+05	-.71E+04	-.71E+04
0.00E+00	0.00E+00	0.28E+05	-.28E+05	-.21E+05	0.35E+05	-.71E+04	-.71E+04
0.00E+00	0.00E+00	-.11E+05	-.38E+04	-.71E+04	-.71E+04	0.18E+05	0.11E+05
0.00E+00	0.00E+00	-.38E+04	-.13E+04	-.71E+04	-.71E+04	0.11E+05	0.83E+04

12.34

The general program is used to solve the example of section 12.8.
The program and output are given.

```

C =====
C
C PROBLEM 12.34
C PROGRAM FOR FINITE ELEMENT VIBRATION ANALYSIS OF STEPPED BEAM
C
C =====
C DIMENSIONS: XL(NE),XI(NE),A(NE),XMAS(NE),BJ(NE,4),XM(4,4),XK(4,4),
C              AI(4,N),AIT(N,4),XKA(4,N),XMA(4,N),AKA(N,N),AMA(N,N),
C              BIGM(N,N),BIGK(N,N),BM(ND,ND),BK(ND,ND)
C              U(ND,ND),UI(ND,ND),UTI(ND,ND),BMU(ND,ND),UMU(ND,ND),
C              XF(ND,ND),EV(ND,ND)
C NE = NUMBER OF STEPS = NUMBER OF ELEMENTS
C ND = NUMBER OF DEGREES OF FREEDOM (EXCLUDING THE FIXED D.O.F.)
C N  = TOTAL NUMBER OF DEGREES OF FREEDOM
C XL(I) = LENGTH OF I TH STEP OR ELEMENT
C XI(I) = AREA MOMENT OF INERTIA OF I TH STEP OR ELEMENT
C A(I)  = CROSS-SECTIONAL AREA OF I TH STEP OR ELEMENT
C RHO = MASS DENSITY OF THE MATERIAL
C E = YOUNG'S MODULUS
C BJ(I,J) = GLOBAL D.O.F. NUMBER CORRESPONDING TO THE LOCAL J TH
C           D.O.F. OF ELEMENT I
C ***** INPUT DATA *****
C           DIMENSION XL(3),XI(3),A(3),XMAS(3),BJ(3,4),XM(4,4),XK(4,4),
2           AI(4,8),AIT(8,4),XKA(4,8),XMA(4,8),AKA(8,8),AMA(8,8),
3           BIGM(8,8),BIGK(8,8),BM(4,4),BK(4,4)
C           DIMENSION U(4,4),UI(4,4),UTI(4,4),BMU(4,4),UMU(4,4),XF(4,4),
2           EV(4,4)
C           INTEGER BJ
C           NE=3
C           DATA XL/40.0,32.0,24.0/
C           DATA XI/1.333333,6.75,0.083333/
C           DATA A/4.0,9.0,1.0/
C           DO 2 I=1,NE
C             BJ(I,1)=I*2-1
C             BJ(I,2)=I*2
C             BJ(I,3)=I*2+1
C             BJ(I,4)=I*2+2
2           CONTINUE
C           E=30.0E+06
C           RHO=0.283/386.4
C ***** END OF INPUT DATA *****
C           N=NE*2+2
C           ND=NE*2-2
C           DO 10 I=1,NE
10          XMAS(I)=A(I)*RHO
C           DO 20 I=1,N
C           DO 20 J=1,N
C           BIGM(I,J)=0.0
20          BIGK(I,J)=0.0
C           DO 100 II=1,NE
C           DO 30 I=1,4
C           DO 30 J=1,N
30          AI(I,J)=0.0
C           I1=BJ(II,1)
C           I2=BJ(II,2)
C           I3=BJ(II,3)

```

```

      I4=BJ(II,4)
      AI(1,I1)=1.0
      AI(2,I2)=1.0
      AI(3,I3)=1.0
      AI(4,I4)=1.0
      XM(1,1)=156.0
      XM(1,2)=22.0*XL(II)
      XM(1,3)=54.0
      XM(1,4)=-13.0*XL(II)
      XM(2,2)=4.0*(XL(II)**2)
      XM(2,3)=13.0*XL(II)
      XM(2,4)=-3.0*(XL(II)**2)
      XM(3,3)=156.0
      XM(3,4)=-22.0*XL(II)
      XM(4,4)=4.0*(XL(II)**2)
      XK(1,1)=12.0
      XK(1,2)=6.0*XL(II)
      XK(1,3)=-12.0
      XK(1,4)=6.0*XL(II)
      XK(2,2)=4.0*(XL(II)**2)
      XK(2,3)=-6.0*XL(II)
      XK(2,4)=2.0*(XL(II)**2)
      XK(3,3)=12.0
      XK(3,4)=-6.0*XL(II)
      XK(4,4)=4.0*(XL(II)**2)
      DO 40 I=1,4
      DO 40 J=1,4
      XM(J,I)=XM(I,J)
40    XK(J,I)=XK(I,J)
      DO 50 I=1,4
      DO 50 J=1,4
      XM(I,J)=(XMAS(II)*XL(II)/420.0)*XM(I,J)
50    XK(I,J)=(E*XI(II)/(XL(II)**3))*XK(I,J)
      DO 60 I=1,N
      DO 60 J=1,4
60    AIT(I,J)=AI(J,I)
      CALL MATMUL (XKA,XK,AI,4,4,N)
      CALL MATMUL (XMA,XM,AI,4,4,N)
      CALL MATMUL (AKA,AIT,XKA,N,4,N)
      CALL MATMUL (AMA,AIT,XMA,N,4,N)
      DO 70 I=1,N
      DO 70 J=1,N
      BIGM(I,J)=BIGM(I,J)+AMA(I,J)
70    BIGK(I,J)=BIGK(I,J)+AKA(I,J)
100   CONTINUE
C APPLICATION OF BOUNDARY CONDITIONS
C ROWS AND COLUMNS CORRESPONDING TO ZERO DISPLACEMENTS ARE DELETED
      DO 110 I=1,ND
      DO 110 J=1,ND
      BM(I,J)=BIGM(I+2,J+2)
110   BK(I,J)=BIGK(I+2,J+2)
C DECOMPOSING THE MATRIX [BK] INTO TRIANGULAR MATRICES
      CALL DECOMP (BK,U,ND)
C FINDING THE INVERSE OF THE UPPER TRIANGULAR MATRIX [U]
      DO 120 I=1,ND

```

```

DO 120 J=1,ND
120  UI(I,J)=0.0
DO 130 I=1,ND
130  UI(I,I)=1.0/U(I,I)
DO 150 J=1,ND
DO 150 II=1,ND
I=ND-II+1
IF (I .GE. J) GO TO 150
IP=I+1
SUM=0.0
DO 140 K=IP,J
140  SUM=SUM+U(I,K)*UI(K,J)
UI(I,J)=-SUM/U(I,I)
150  CONTINUE
DO 160 I=1,ND
DO 160 J=1,ND
160  UTI(I,J)=UI(J,I)
CALL MATMUL (BMU,BM,UI,ND,ND,ND)
CALL MATMUL (UMU,UTI,BMU,ND,ND,ND)
CALL JACOBI (UMU,ND,EV,1.0E-05,200)
CALL MATMUL (XF,UI,EV,ND,ND,ND)
DO 170 I=1,ND
170  UMU(I,I)=SQRT(1.0/UMU(I,I))
PRINT 180, (UMU(I,I),I=1,ND)
180  FORMAT (//,40H NATURAL FREQUENCIES OF THE STEPPED BEAM,//,
2  4(1X,E15.6))
PRINT 190
190  FORMAT (//,12H MODE SHAPES)
DO 200 J=1,ND
200  PRINT 210, J,(XF(I,J),I=1,ND)
210  FORMAT (/ ,I4,5X,4(1X,E15.6))
STOP
END

```

NATURAL FREQUENCIES OF THE STEPPED BEAM

0.160083E+03 0.617460E+03 0.225198E+04 0.712653E+04

MODE SHAPES

1	0.103333E-01	0.189147E-03	0.141626E-01	0.445258E-04
2	-0.376593E-02	0.202976E-03	0.471097E-02	0.259495E-03
3	0.167902E-03	-0.181687E-03	0.135672E-02	0.207580E-03
4	0.182648E-03	0.607247E-04	0.373775E-03	0.163869E-03

Program 22 is used by replacing three lines (data XL, XI and A).
The new lines are

```
DATA XL/10.0,10.0,10.0/
DATA XI/0.0833333,0.0833333,0.0833333/
DATA A/1.0,1.0,1.0/
```

The results are given below.

NATURAL FREQUENCIES OF THE STEPPED BEAM

0.165450E+04 0.398424E+04 0.108074E+05 0.170605E+05

MODE SHAPES

1	0.515650E-02	0.413653E-03	0.515651E-02	-0.413601E-03
2	-0.250018E-02	0.249780E-03	0.250010E-02	0.249788E-03
3	0.277784E-03	-0.478346E-03	0.277633E-03	0.478551E-03
4	0.313508E-03	0.472654E-03	-0.314146E-03	0.472486E-03

Exact natural frequencies (from chapter 8):

$\omega_1 = 1452.3880$, $\omega_2 = 4003.5633$, $\omega_3 = 7848.5844$, $\omega_4 = 12974.1074$ rad/s.

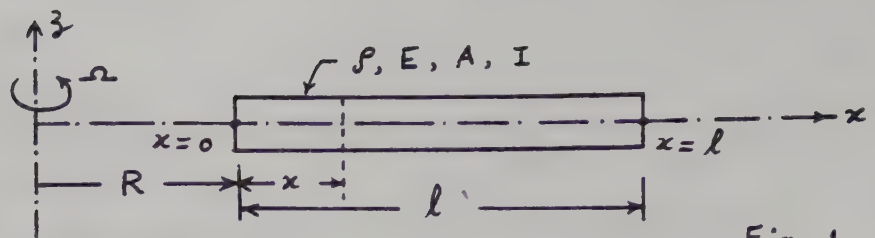


Fig. 1

(i) Rotating beam element

Strain energy due to rotation:

The rotation of the beam induces an axial force P in the beam due to centrifugal action. If the beam bends in xz -plane as shown in Fig. 2, the change in the horizontal projection

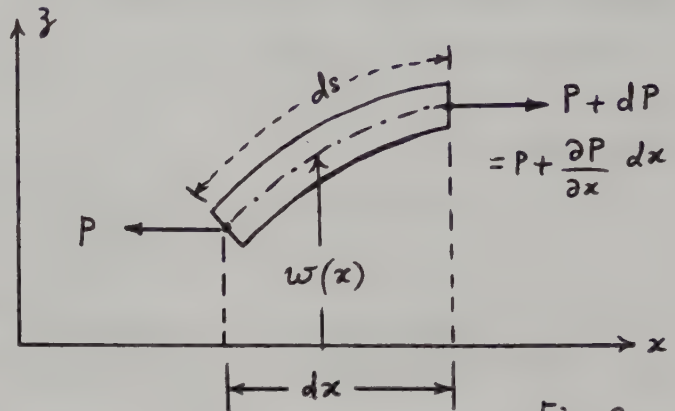


Fig. 2

of an element of length " ds " is given by

$$ds - dx = \left\{ (dx)^2 + \left(\frac{\partial w}{\partial x} dx \right)^2 \right\}^{\frac{1}{2}} - dx \approx \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 dx \quad \dots (E_1)$$

Since the axial force P acts against the change in the horizontal projection, the work done by P is

$$\frac{1}{2} \int_0^l P(x) \left(\frac{\partial w}{\partial x} \right)^2 dx$$

where

$$P(x) = \int_{x_e+x}^{l+x_e} \frac{\rho A}{g} \Omega^2 \xi d\xi \quad \text{where } \rho = \text{weight density}$$

$$\approx \frac{\rho A \Omega^2}{2g} \left[(l+R)^2 - (R+x)^2 \right] \quad (E_2)$$

Work done by the transverse distributed force

$p_w(x)$ can be expressed as

$$\int_0^l p_w(x) w(x) dx \quad \text{where } p_w(x) = \frac{\rho A \Omega^2}{g} w(x) \quad (E_3)$$

If w_i , $i=1,2,3,4$ denote the nodal displacements of the beam element, the transverse displacement $w(x)$ can be expressed as:

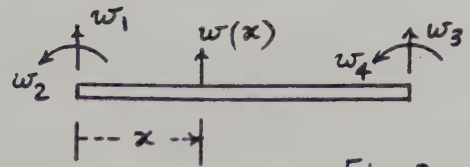


Fig. 3

$$w(x,t) = \sum_{i=1}^4 N_i(x) w_i(t) \quad (E_4)$$

Where $N_i(x)$ are the shape functions given by Eqs. (12.33) - (12.36). Introducing time variation of displacements, we get

kinetic energy of element is:

$$T(t) = \frac{1}{2} \int_0^l \rho A \left\{ \frac{\partial w(x,t)}{\partial t} \right\}^2 dx \equiv \frac{1}{2} \dot{\vec{w}}^T [m] \dot{\vec{w}} \quad (E_5)$$

Total strain energy of the element is

$$V(t) = \frac{1}{2} \int_0^l EI \left\{ \frac{\partial^2 w(x,t)}{\partial x^2} \right\}^2 dx + \frac{1}{2} \int_0^l p(x) \left\{ \frac{\partial w(x,t)}{\partial x} \right\}^2 dx - \int_0^l p_w(x,t) \cdot w(x,t) dx \equiv \frac{1}{2} \vec{w}^T [k] \vec{w} \quad (E_6)$$

Total virtual work of element is

$$\delta W(t) = \int_0^l f(x,t) \delta w(x,t) dx \equiv \vec{f}(t)^T \delta \vec{w}(t) \quad (E_7)$$

where $f(x,t)$ denotes the distributed force (which is zero in the present case).

Evaluation of integrals in Eqs. (E5) - (E7) enables us find the mass matrix, stiffness matrix and load vector.

(ii) For the helicopter blade, we can model it as one beam element for simplicity.

$$\omega = 300(2\pi)/60 = 31.416 \text{ rad/sec}$$

$$A = 12 \text{ in}^2, \quad I = \frac{1}{12} (12)(1)^3 = 1 \text{ in}^4, \quad l = 48 \text{ in},$$

$$E = 10.3 \times 10^6 \text{ psi}, \quad \rho = 0.098 \text{ lb/in}^3 \text{ (for aluminum)}.$$

Boundary conditions:

$$w_1 = w_2 = 0$$

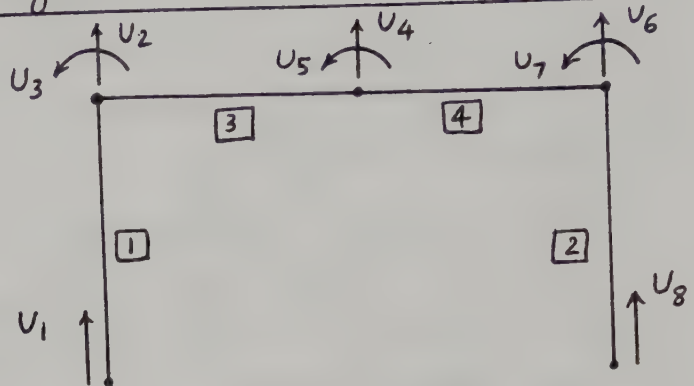
Solve the eigenvalue problem

$$[-\omega^2 [M] + [K]] \vec{w} = \vec{0} \quad \text{where } \vec{w} = \begin{Bmatrix} w_3 \\ w_4 \end{Bmatrix}$$

for the natural frequencies and mode shapes.



12.37



(a) Generate element matrices

$$[K^{(1)}] = \frac{A_1 E_1}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix}, \quad [K^{(2)}] = \frac{A_2 E_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} U_8 \\ U_6 \end{matrix}$$

$$[K^{(3)}] = \frac{E_3 I_3}{l_3^3} \begin{bmatrix} 12 & 6l_3 & -12 & 6l_3 \\ 6l_3 & 4l_3^2 & -6l_3 & 2l_3^2 \\ -12 & -6l_3 & 12 & -6l_3 \\ 6l_3 & 2l_3^2 & -6l_3 & 4l_3^2 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{matrix}$$

$$[K^{(4)}] = \frac{E_4 I_4}{l_4^3} \begin{bmatrix} 12 & 6l_4 & -12 & 6l_4 \\ 6l_4 & 4l_4^2 & -6l_4 & 2l_4^2 \\ -12 & -6l_4 & 12 & -6l_4 \\ 6l_4 & 2l_4^2 & -6l_4 & 4l_4^2 \end{bmatrix} \begin{matrix} U_4 \\ U_5 \\ U_6 \\ U_7 \end{matrix}$$

$$[M^{(1)}] = \frac{\rho_1 A_1 l_1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix}, \quad [M^{(2)}] = \frac{\rho_2 A_2 l_2}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} U_8 \\ U_6 \end{matrix}$$

$$[M^{(3)}] = \frac{\rho_3 A_3 l_3}{420} \begin{bmatrix} U_2 & U_3 & U_4 & U_5 \\ 156 & 22l_3 & 54 & -13l_3 \\ 22l_3 & 4l_3^2 & 13l_3 & -3l_3^2 \\ 54 & 13l_3 & 156 & -22l_3 \\ -13l_3 & -3l_3^2 & -22l_3 & 4l_3^2 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{matrix}$$

$$[M^{(4)}] = \frac{\rho_4 A_4 l_4}{420} \begin{bmatrix} U_4 & U_5 & U_6 & U_7 \\ 156 & 22l_4 & 54 & -13l_4 \\ 22l_4 & 4l_4^2 & 13l_4 & -3l_4^2 \\ 54 & 13l_4 & 156 & -22l_4 \\ -13l_4 & -3l_4^2 & -22l_4 & 4l_4^2 \end{bmatrix} \begin{matrix} U_4 \\ U_5 \\ U_6 \\ U_7 \end{matrix}$$

where $\rho_1 = \rho_2 = 2.7 \times 10^{-3} \text{ lbm/in}^3$, $l_1 = l_2 = 108 \text{ in}$,

$E_1 = E_2 = 4 \times 10^6 \text{ psi}$, $A_1 = A_2 = \frac{\pi d^2}{4}$, $\rho_3 = \rho_4 = 8.8 \times 10^{-3} \text{ lbm/in}^3$,

$l_3 = l_4 = 108 \text{ in}$, $E_3 = E_4 = 30 \times 10^6 \text{ psi}$, $A_3 = A_4 = bh = 2b^2$,

$I_3 = I_4 = \frac{1}{12} bh^3 = \frac{2}{3} b^4$.

(b) Find the assembled stiffness and mass matrices after applying the boundary conditions $U_1 = U_8 = 0$.

(c) Select trial values of d and b and find the fundamental natural frequency (ω_1) by solving the eigenvalue problem

$$|-\omega^2 [M] + [K]| = 0$$

(d) change d and b until $\omega_1 > \frac{1500(2\pi)}{60} = 157.08 \text{ rad/sec.}$

Chapter 13

Nonlinear Vibration

13.1 $\ddot{\theta}_1 + \omega_0^2 \theta_1 = f$, $\ddot{\theta}_2 + \omega_0^2 \theta_2 = \frac{\omega_0^2}{6} \theta_2^3$

Use $\theta = \theta_1 + \theta_2$ in $E_0 \cdot (E_2)$:

$$\ddot{\theta}_1 + \ddot{\theta}_2 + \omega_0^2 \theta_1 + \omega_0^2 \theta_2 = f + \frac{\omega_0^2}{6} \theta_1^3 + \frac{\omega_0^2}{6} \theta_2^3 + \frac{\omega_0^2}{2} \theta_1^2 \theta_2 + \frac{\omega_0^2}{2} \theta_1 \theta_2^2$$

Left hand side is not equal to the right hand side.

Thus superposition principle is not valid.

13.2 $T = \text{kinetic energy at time zero} = \frac{1}{2} m (\dot{x}_0)^2$

Let $x_2 = \text{maximum displacement on right side.}$

$V = \text{potential energy in spring at displacement}$

$$x_2 = \frac{1}{2} k_2 x_2^2 \quad (\dot{x} \text{ is zero at } x_2)$$

Since $T = V$, $x_2 = \sqrt{\frac{m(\dot{x}_0)^2}{k_2}} = \sqrt{\frac{m}{k_2}} \dot{x}_0$

Let $x_1 = \text{maximum displacement to left side.}$ $V = \frac{1}{2} k_1 x_1^2$

$T = V$ gives $x_1 = \sqrt{\frac{m}{k_1} (\dot{x}_0)^2} = \sqrt{\frac{m}{k_1}} \dot{x}_0$

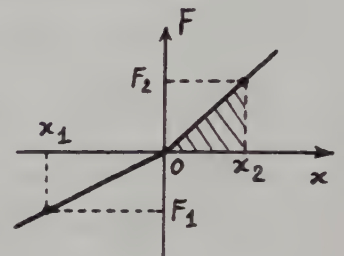
(a) Since $k_1 < k_2$, maximum deflection $= x_1 = \sqrt{\frac{m}{k_1}} \dot{x}_0$

(b) Period of vibration for a spring-mass system is $\tau_n = 2\pi \sqrt{\frac{m}{k}}$.

In the present case, $\tau_n = (\text{time for } m \text{ to go to } x = x_1 \text{ from } x = 0 \text{ and return to } x = 0) +$

$(\text{time for } m \text{ to go to } x = x_2 \text{ from } x = 0 \text{ and return to } x = 0)$

$$\therefore \tau_n = \pi \left(\sqrt{\frac{m}{k_1}} + \sqrt{\frac{m}{k_2}} \right)$$



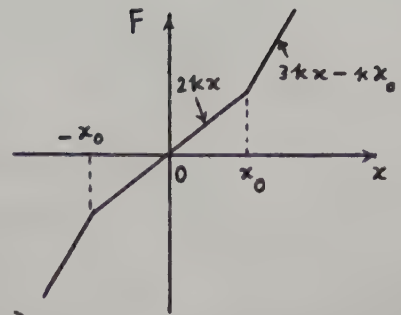
13.3

Restoring force = $F = \begin{cases} 2kx & ; 0 \leq x \leq x_0 \\ 2kx + k(x - x_0) & ; x > x_0 \end{cases}$

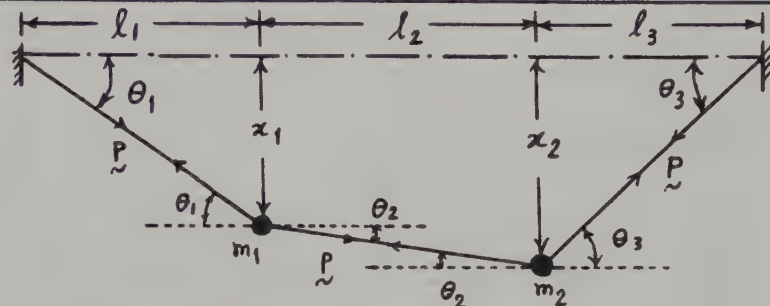
i.e. $F = \begin{cases} 2kx & ; 0 \leq x \leq x_0 \\ 3kx - kx_0 & ; x > x_0 \end{cases} \quad \text{--- (E}_1\text{)}$

Equation of motion is

$m \ddot{x} + F = P \sin \omega t$ where F is given by (E₁).



13.4



Let

$\tilde{P}$ = tension in wire after displacement of masses

P = initial tension in wire

$\tilde{l}_i$ = length of i^{th} segment of wire after displacement of masses

x_i = transverse displacement of mass i ($i = 1, 2$)

$$\tilde{l}_1 = \sqrt{l_1^2 + x_1^2}, \quad \tilde{l}_2 = \sqrt{l_2^2 + (x_2 - x_1)^2}, \quad \tilde{l}_3 = \sqrt{l_3^2 + x_2^2}$$

$$\epsilon = \text{strain in wire} = \left(\frac{\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 - l_1 - l_2 - l_3}{l_1 + l_2 + l_3} \right) \quad (\text{E}_1)$$

$$\text{New tension in wire} = \tilde{P} = (P + AE\epsilon)$$

where A = cross-sectional area of wire and E = Young's modulus.

Equations of motion of masses:

$$m_1 \ddot{x}_1 + \tilde{P} \sin \theta_1 + \tilde{P} \sin \theta_2 = 0 \quad (\text{E}_2)$$

$$m_2 \ddot{x}_2 + \tilde{P} \sin \theta_2 + \tilde{P} \sin \theta_3 = 0 \quad (\text{E}_3)$$

$$\text{i.e., } m_1 \ddot{x}_1 + (P + AE\epsilon) \left(\frac{x_1}{\sqrt{l_1^2 + x_1^2}} + \frac{x_2 - x_1}{\sqrt{l_2^2 + (x_2 - x_1)^2}} \right) = 0 \quad (\text{E}_4)$$

$$m_2 \ddot{x}_2 + (P + AE\epsilon) \left(\frac{x_2 - x_1}{\sqrt{l_2^2 + (x_2 - x_1)^2}} + \frac{x_2}{\sqrt{l_3^2 + x_2^2}} \right) = 0 \quad (\text{E}_5)$$

$$\text{with } \frac{x_1}{\sqrt{l_1^2 + x_2^2}} = \frac{x_1}{l_1 \sqrt{1 + \left(\frac{x_2}{l_1}\right)^2}} = \frac{x_1}{l_1} \left[1 + \left(\frac{x_2}{l_1}\right)^2 \right]^{-\frac{1}{2}} \approx \frac{x_1}{l_1} \left[1 - \frac{1}{2} \left(\frac{x_2}{l_1}\right)^2 \right] \quad (\text{E}_6)$$

$$\frac{x_2 - x_1}{\sqrt{l_2^2 + (x_2 - x_1)^2}} \approx \frac{x_2 - x_1}{l_2} \left[1 - \frac{1}{2} \left(\frac{x_2 - x_1}{l_2} \right)^2 \right] \quad (E_7)$$

$$\text{and } \frac{x_2}{\sqrt{l_3^2 + x_2^2}} \approx \frac{x_2}{l_3} \left[1 - \frac{1}{2} \left(\frac{x_2}{l_3} \right)^2 \right] \quad (E_8)$$

$$\text{similarly, } l_1 - l_1 = \sqrt{l_1^2 + x_1^2} - l_1 \approx l_1 \left[1 + \frac{1}{2} \left(\frac{x_1}{l_1} \right)^2 \right] - l_1 \approx \frac{1}{2} \frac{x_1^2}{l_1}$$

$$l_2 - l_2 = \sqrt{l_2^2 + (x_2 - x_1)^2} - l_2 \approx \frac{1}{2} \frac{(x_2 - x_1)^2}{l_2}$$

$$l_3 - l_3 \approx \frac{1}{2} \frac{x_2^2}{l_3}$$

$$\text{and hence } \epsilon = \frac{\left\{ \frac{x_1^2}{l_1} + \frac{(x_2 - x_1)^2}{l_2} + \frac{x_2^2}{l_3} \right\}}{2(l_1 + l_2 + l_3)} \quad (E_9)$$

Substitution of Eqs. (E₆) to (E₉) into Eqs. (E₄) and (E₅) yields the nonlinear equations of motion of the masses m_1 and m_2 .

13.5

Using x and θ as the coordinates, the kinetic and potential energies of the system can be expressed as

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 \quad (1)$$

where $J_0 = m(\ell + x)^2$ and

$$V = \frac{1}{2} k (x + \delta_{st})^2 - m g (\ell + x) \cos \theta \quad (2)$$

where $\delta_{st} = \frac{mg}{k}$. Equations (1) and (2) give

$$\frac{\partial T}{\partial \dot{x}} = m \dot{x} ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) = m \ddot{x}$$

$$\frac{\partial T}{\partial \dot{\theta}} = J_0 \dot{\theta} ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) = \dot{J}_0 \dot{\theta} + J_0 \ddot{\theta} = 2 m (\ell + x) \dot{x} \dot{\theta} + J_0 \ddot{\theta}$$

$$\frac{\partial T}{\partial x} = m (\ell + x) \dot{\theta}^2 ; \quad \frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial V}{\partial x} = k (x + \delta_{st}) - m g \cos \theta ; \quad \frac{\partial V}{\partial \theta} = m g (\ell + x) \sin \theta$$

The equations of motion can be derived using Lagrange's equations, Eq. (6.44), as:

$$m \ddot{x} - m (\ell + x) \dot{\theta}^2 + k x + m g - m g \cos \theta = 0 \quad (3)$$

$$m (\ell + x)^2 \ddot{\theta} + 2 m (\ell + x) \dot{x} \dot{\theta} + m g (\ell + x) \sin \theta = 0 \quad (4)$$

Using $\sin \theta \approx \theta$, $\cos \theta \approx 1$, and neglecting nonlinear terms involving $\ddot{\theta}$, $\dot{\theta}^2$, $\dot{x}$, and $\dot{\theta}$, Eqs. (3) and (4) can be reduced (linearized) to obtain:

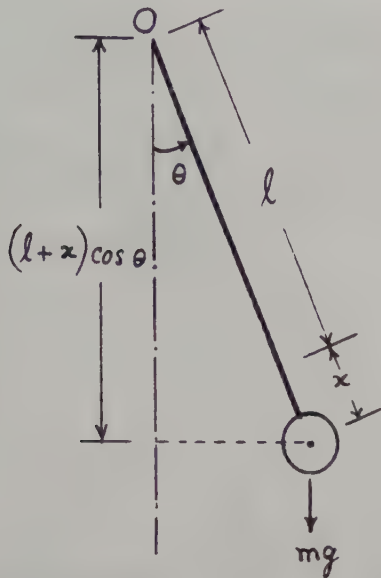
$$m \ddot{x} + k x = 0 \quad (5)$$

$$m \ell^2 \ddot{\theta} + m g \ell \theta = 0 \quad (6)$$

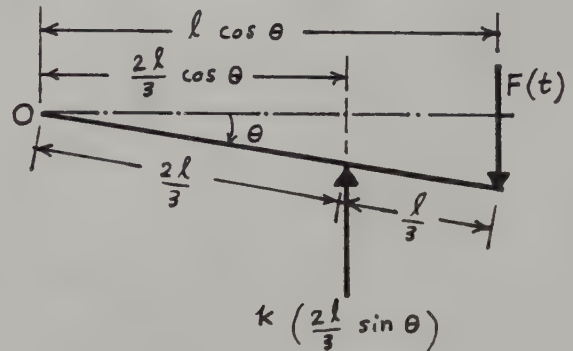
Equations (5) and (6) correspond to the natural frequencies:

$$\omega_{n1} = \sqrt{\frac{k}{m}} \quad (7)$$

$$\omega_{n2} = \sqrt{\frac{g}{\ell}} \quad (8)$$



13.6 $J_0 = \frac{1}{3} m \ell^2$



$$\sum M_0 = J_0 \ddot{\theta}$$

$$\text{or } J_0 \ddot{\theta} = -k \left(\frac{2\ell}{3} \sin \theta \right) \left(\frac{2\ell}{3} \cos \theta \right) + (\ell \cos \theta) F(t)$$

$$\text{or } \left(\frac{1}{3} m \ell^2 \right) \ddot{\theta} + \frac{4}{9} k \ell^2 \sin \theta \cos \theta = \ell \cos \theta F(t) \quad (1)$$

Eq. (1) can be approximated using

$$\sin \theta \approx \theta - \frac{\theta^3}{6} \quad \text{and} \quad \cos \theta \approx 1 - \frac{\theta^2}{2}$$

which yields

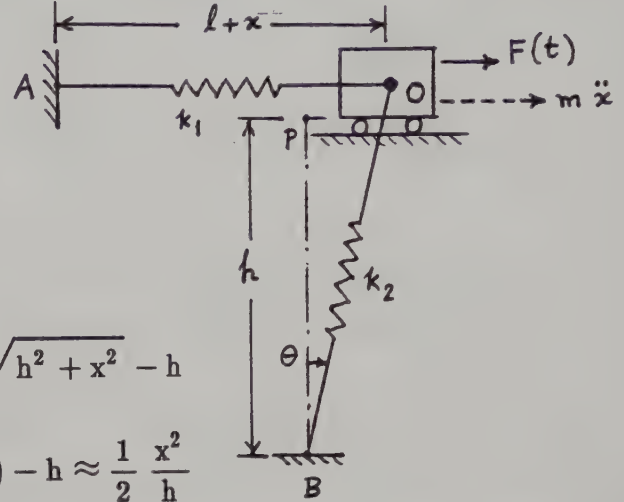
$$\left(\frac{1}{3} m \ell^2 \right) \ddot{\theta} + \frac{4}{9} k \ell^2 \left(\theta - \frac{\theta^3}{6} \right) \left(1 - \frac{\theta^2}{2} \right) = \ell \left(1 - \frac{\theta^2}{2} \right) F(t)$$

$$\text{or } \frac{1}{3} m \ell^2 \ddot{\theta} + \frac{4}{9} k \ell^2 \left(\theta - \frac{2}{3} \theta^3 \right) = \left(\ell - \frac{\ell \theta^2}{2} \right) F(t) \quad (2)$$

Neglecting the nonlinear term, Eq. (2) can be reduced to

$$\frac{1}{3} m \ell^2 \ddot{\theta} + \frac{4}{9} k \ell^2 \theta = \ell F(t) \quad (3)$$

13.7



Extension of k_2 :

$$\begin{aligned} OB - PB &= \sqrt{BP^2 + PO^2} - PB = \sqrt{h^2 + x^2} - h \\ &= h \left(1 + \frac{x^2}{h^2} \right)^{\frac{1}{2}} - h \approx h \left(1 + \frac{1}{2} \frac{x^2}{h^2} \right) - h \approx \frac{1}{2} \frac{x^2}{h} \end{aligned}$$

$$\sin \theta = \frac{x}{BO} = \frac{x}{h + \frac{1}{2} \frac{x^2}{h}}$$

$$\sum F = m \ddot{x}$$

$$\text{or } m \ddot{x} = -k_1 x - \left(\frac{1}{2} \frac{x^2}{h} \right) k_2 \sin \theta + F(t)$$

$$\text{or } m \ddot{x} + k_1 x + \left(\frac{\frac{1}{2} \frac{x^3 k_2}{h}}{h + \frac{1}{2} \frac{x^2}{h}} \right) = F(t) \quad (1)$$

Note that

$$\frac{x^3 k_2}{2 h \left(h + \frac{1}{2} \frac{x^2}{h} \right)} \approx \frac{x^3 k_2}{2 h^2} - \frac{x^5 k_2}{4 h^4} \quad (2)$$

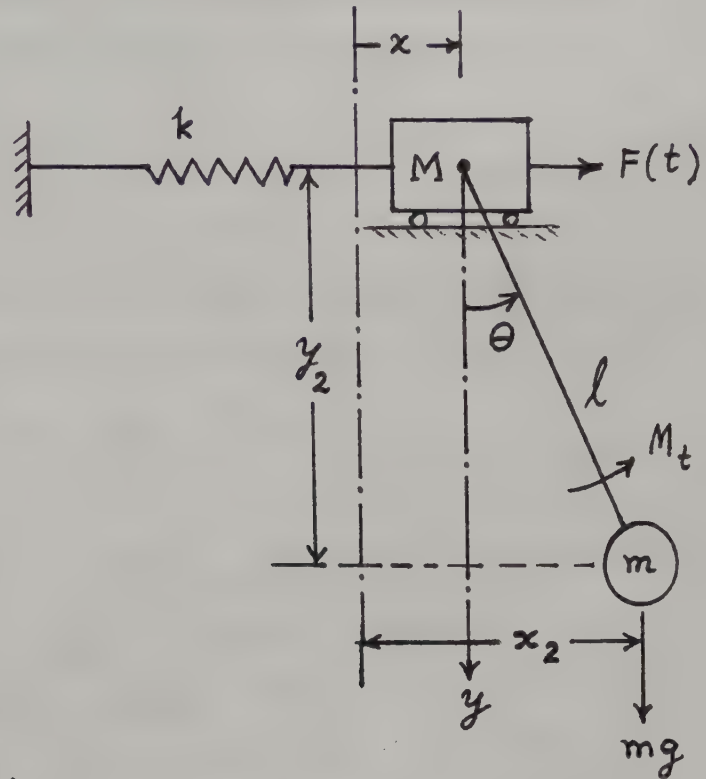
Substituting Eq. (2) into (1), we obtain

$$m \ddot{x} + k_1 x + \frac{x^3 k_2}{2 h^2} - \frac{x^5 k_2}{4 h^4} = F(t) \quad (3)$$

Neglecting the terms involving x^5 , Eq. (3) can be reduced to

$$m \ddot{x} + k_1 x + \frac{k_2}{2 h^2} x^3 = F(t) \quad (4)$$

13.8



$$x_2 = x + \ell \sin \theta ; \dot{x}_2 = \dot{x} + \ell \dot{\theta} \cos \theta$$

$$y_2 = \ell \cos \theta ; \dot{y}_2 = -\ell \dot{\theta} \sin \theta$$

$$\begin{aligned} T &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[(\dot{x} + \ell \dot{\theta} \cos \theta)^2 + (-\ell \dot{\theta} \sin \theta)^2 \right] \\ &= \frac{1}{2} (M + m) \dot{x}^2 + \frac{1}{2} m \ell^2 \dot{\theta}^2 + m \ell \dot{x} \dot{\theta} \cos \theta \end{aligned}$$

$$V = \frac{1}{2} k x^2 + m g \ell (1 - \cos \theta)$$

$$Q_x = F(t) ; Q_\theta = M_t(t)$$

Equations of motion:

$$(M + m) \ddot{x} + m \ell \ddot{\theta} \cos \theta - m \ell \dot{\theta}^2 \sin \theta + k x = F(t) \quad (1)$$

$$m \ell^2 \ddot{\theta} + m \ell \ddot{x} \cos \theta - m \ell \dot{x} \dot{\theta} \sin \theta + m g \ell \sin \theta = M_t(t) \quad (2)$$

Using the approximations

$$\cos \theta \approx 1 - \frac{\theta^2}{2} ; \sin \theta \approx \theta - \frac{\theta^3}{6}$$

Eqs. (1) and (2) can be expressed as

$$(M + m) \ddot{x} + m \ell \ddot{\theta} - \frac{1}{2} m \ell \theta^2 \ddot{\theta} - m \ell \theta \dot{\theta}^2 + \frac{1}{6} m \ell \theta^3 \dot{\theta}^2 + k x = F(t) \quad (3)$$

$$m \ell^2 \ddot{\theta} + m \ell \ddot{x} - \frac{1}{2} m \ell \theta^2 \ddot{x} - m \ell \theta \dot{\theta} \dot{x} + \frac{1}{6} m \ell \theta^3 \dot{\theta} \dot{x} + m g \ell \theta - \frac{1}{6} m g \ell \theta^3 = M_t(t) \quad (4)$$

By neglecting the nonlinear terms, the linearized equations of motion can be written as

$$(M + m) \ddot{x} + m \ell \ddot{\theta} + k x = F(t) \quad (5)$$

$$m \ell^2 \ddot{\theta} + m \ell \ddot{x} + m g \ell \theta = M_t(t) \quad (6)$$

13.9

Eq. (13.1) is $\ddot{\theta} + \frac{g}{l} \sin \theta = 0 \quad \dots (E_1)$

But $\frac{d^2 \theta}{dt^2} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta} \frac{d\theta}{dt} = \dot{\theta} \frac{d\dot{\theta}}{d\theta}$

Eq. (E₁) can be rewritten as $\dot{\theta} \frac{d\dot{\theta}}{d\theta} + \frac{g}{l} \sin \theta = 0$

Integrating this, we get $\frac{(\dot{\theta})^2}{2} - \frac{g}{l} \cos \theta = c \quad \dots (E_2)$

At $t=0$, $\dot{\theta}=0$ and $\theta = \theta_0 \Rightarrow c = -\left(\frac{g}{l}\right) \cos \theta_0$

Eq. (E₂) gives $\left(\frac{d\theta}{dt}\right)^2 = (\dot{\theta})^2 = 2 \left(c + \frac{g}{l} \cos \theta\right)$

$$dt = \frac{d\theta}{\sqrt{2 \left(c + \frac{g}{l} \cos \theta\right)}} = \frac{d\theta}{\sqrt{2 \frac{g}{l} (\cos \theta - \cos \theta_0)}}$$

$$t = \int \frac{d\theta}{\sqrt{2 \frac{g}{l} (\cos \theta - \cos \theta_0)}} \quad \dots (E_3)$$

But $\cos \theta - \cos \theta_0 = \left(1 - 2 \sin^2 \frac{\theta}{2}\right) - \left(1 - 2 \sin^2 \frac{\theta_0}{2}\right)$
 $= 2 \sin^2 \frac{\theta_0}{2} \left(1 - \frac{\sin^2 \frac{\theta}{2}}{\sin^2 \frac{\theta_0}{2}}\right) \quad \dots (E_4)$

If $\frac{\sin \frac{\theta}{2}}{\sin \frac{\theta_0}{2}} = \frac{\sin \frac{\theta}{2}}{k} = \sin \phi$ where $k = \sin \frac{\theta_0}{2}$,

differentiation gives

$$\frac{1}{k} \cos \frac{\theta}{2} \cdot \frac{d\theta}{2} = \cos \phi \cdot d\phi \Rightarrow d\theta = \frac{2k \cos \phi d\phi}{\cos \frac{\theta}{2}} \quad \dots (E_5)$$

Eg. (E₃) to (E₅) give

$$t = \int \left(\frac{2\kappa \cos \phi \, d\phi}{\cos \frac{\theta}{2}} \right) / \left(2\sqrt{\frac{g}{l}} \sin \frac{\theta_0}{2} \cos \phi \right) = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\cos \frac{\theta}{2}}$$

$$= \sqrt{\frac{l}{g}} \int \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\theta}{2}}} = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\sqrt{1 - \kappa^2 \sin^2 \phi}}$$

$$\tau = \text{period} = 4 \sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \kappa^2 \sin^2 \phi}} \quad \text{where } \kappa = \sin \frac{\theta_0}{2}.$$

The integral in τ is the complete elliptic integral of the first kind. Hence the numerical value of τ corresponding to any θ_0 can be obtained from the elliptic integral tables.

13.10

Equation of motion :

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0$$

or $\frac{d}{d\theta} (\dot{\theta}^2) + \frac{2g}{l} \sin \theta = 0$

Let $\theta = \theta_0$ at $\frac{d\theta}{dt} = 0$. $\int_{\dot{\theta}=0}^{\dot{\theta}} \frac{d}{d\theta} (\dot{\theta}^2) d\theta + \int_{\theta=\theta_0}^{\theta} \frac{2g}{l} \sin \theta d\theta = 0$

i.e., $(\dot{\theta})^2 + \frac{2g}{l} (-\cos \theta) \Big|_{\theta=\theta_0}^{\theta} = 0$

i.e., $(\dot{\theta})^2 = \frac{2g}{l} (\cos \theta - \cos \theta_0)$

i.e., $\frac{d\theta}{dt} = \pm \sqrt{\frac{2g}{l}} \sqrt{\cos \theta - \cos \theta_0}$

i.e., $dt = \pm \sqrt{\frac{l}{2g}} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}}$

Time taken by pendulum to reach vertical position can be found as

$$t = \int_0^t dt = \sqrt{\frac{l}{2g}} \int_{\theta=\theta_0}^0 \frac{d\theta}{(\cos \theta - \cos \theta_0)^{\frac{1}{2}}} \quad (E_1)$$

Using the relation

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2},$$

Eg. (E₁) can be rewritten as

$$t = \frac{1}{2} \sqrt{\frac{l}{g}} \int_{\theta_0}^0 \frac{d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}} \quad (E_2)$$

Let $a = \sin \frac{\theta_0}{2}$ and $\sin \frac{\theta}{2} = a \sin \phi$ so that

When $\theta = 0$, $\phi = 0$ and when $\theta = \theta_0$, $\phi = \frac{\pi}{2}$.

$$\cos \frac{\theta}{2} \cdot \frac{d\theta}{2} = a \cos \phi \cdot d\phi \quad ; \quad d\theta = \left(\frac{2a \cos \phi}{\cos \frac{\theta}{2}} \right) d\phi$$

$$\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} = a^2 - a^2 \sin^2 \phi = a^2 \cos^2 \phi$$

E_2 . (E_2) becomes

$$\begin{aligned} t &= \pm \frac{1}{2} \sqrt{\frac{l}{g}} \int_{\pi/2}^0 \frac{2a \cos \phi \, d\phi}{\cos \frac{\theta}{2} \cdot a \cos \phi} = \pm \sqrt{\frac{l}{g}} \int_{\pi/2}^0 \frac{1}{\sqrt{1 - \cos^2 \frac{\theta}{2}}} d\phi \\ &= \sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - a^2 \sin^2 \phi}} = \sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\theta_0}{2} \cdot \sin^2 \phi}} \\ &= \sqrt{\frac{l}{g}} \cdot F(a, \frac{\pi}{2}) \end{aligned} \quad (E_3)$$

where F is called the incomplete elliptic integral of the first kind. Here $\sqrt{\frac{l}{g}} = \sqrt{\frac{30}{386.4}} = 0.2786$

and $a = \sin \frac{\theta_0}{2} = \sin 40^\circ = 0.6428$.

From CRC Standard Mathematical Tables, we find for

$$F(a, \frac{\pi}{2}) = F(\sin \frac{\theta_0}{2}, \frac{\pi}{2}) \text{ with } \theta_0 = 80^\circ, F = 1.7868$$

$$\therefore t = 0.2786 (1.7868) = 0.4978 \text{ second.}$$

Alternatively,

$$l = 30'' , \quad \theta_0 = 80^\circ = 1.3963 \text{ rad and } g = 386.4 \text{ in/sec}^2.$$

Time taken by pendulum to reach vertical position (one-quarter of the time period) can be found from E_2 . (E_{15}) in solution of problem 13.7 as

$$t = \frac{\tau}{4} = \frac{1}{\omega_0 (1 - \frac{\theta_0^2}{12})} F(a, \frac{\pi}{2}) \quad (E_4)$$

$$\text{Here } \omega_0 = \sqrt{g/l} = \sqrt{386.4/30} = 3.5889, \quad \theta_0 = 1.3963 \text{ rad,}$$

$$\begin{aligned} a^2 &= \frac{\theta_0^2}{12 (1 - \frac{1}{12} \theta_0^2)} = \frac{1.9496}{12 (0.8375)} = 0.1940 = (0.4405)^2 \\ &= (\sin^{-1} 26.1358^\circ)^2 \end{aligned}$$

$$F(a, \frac{\pi}{2}) \approx 1.6490 \text{ from CRC Standard Mathematical Tables.}$$

$$t \approx \frac{1.6490}{(3.5889)(0.8375)} = 0.5486 \text{ sec.}$$

Note that this result is only approximate due to the approximation $\sin \theta \approx \theta - \frac{1}{6} \theta^3$ used in the solution of problem 13.7.

13.11

$$\ddot{\theta} + \omega_0^2 \left(\theta - \frac{1}{6} \theta^3 \right) = 0 \quad (E_1)$$

This equation is similar to Eq. (13.9) with

$$x = \theta, \quad a = \omega_0, \quad F(x) = F(\theta) = \theta - \frac{1}{6} \theta^3.$$

Eq. (E₁) can be rewritten as

$$\frac{d}{d\theta} (\dot{\theta}^2) + 2\omega_0^2 \left(\theta - \frac{1}{6} \theta^3 \right) = 0 \quad (E_2)$$

which upon integration gives

$$\dot{\theta}^2 = 2\omega_0^2 \int_{\theta}^{\theta_0} F(\eta) \cdot d\eta = 2\omega_0^2 \int_{\theta}^{\theta_0} \left(\eta - \frac{1}{6} \eta^3 \right) \cdot d\eta$$

$$= 2\omega_0^2 \left(\frac{1}{2} \eta^2 - \frac{1}{24} \eta^4 \right)_{\theta}^{\theta_0} = \omega_0^2 \left(\theta_0^2 - \frac{1}{12} \theta_0^4 - \theta^2 + \frac{1}{12} \theta^4 \right) \quad (E_3)$$

$$= \omega_0^2 (\theta_0^2 - \theta^2) \left\{ 1 - \frac{1}{12} (\theta_0^2 + \theta^2) \right\} \quad (E_4)$$

Since the maximum value of θ is θ_0 , we assume

$$\theta(t) = \theta_0 \sin \beta \quad (E_5)$$

$$\text{Thus } \theta_0^2 - \theta^2 = \theta_0^2 - \theta_0^2 \sin^2 \beta = \theta_0^2 \cos^2 \beta \quad (E_6)$$

$$\theta_0^2 + \theta^2 = \theta_0^2 (1 + \sin^2 \beta) \quad (E_7)$$

$$\text{and } \dot{\theta} = A_0 \cos \beta \frac{d\beta}{dt} \quad (E_8)$$

Substitution of Eqs. (E₆) to (E₈) into (E₄) gives

$$\theta_0^2 \cos^2 \beta \left(\frac{d\beta}{dt} \right)^2 = \omega_0^2 \theta_0^2 \cos^2 \beta \left\{ 1 - \frac{1}{12} \theta_0^2 (1 + \sin^2 \beta) \right\}$$

i.e.,

$$\left(\frac{d\beta}{dt} \right)^2 = \omega_0^2 \left(1 - \frac{1}{12} \theta_0^2 \right) \left\{ 1 - \frac{\theta_0^2 \sin^2 \beta}{12 \left(1 - \frac{1}{12} \theta_0^2 \right)} \right\} \quad (E_9)$$

Defining

$$a^2 = \frac{\theta_0^2}{12 \left(1 - \frac{1}{12} \theta_0^2 \right)} \quad (E_{10})$$

Eq. (E₉) can be used to express (taking positive root):

$$\frac{d\beta}{dt} = \omega_0 \left(1 - \frac{1}{12} \theta_0^2 \right)^{\frac{1}{2}} \left(1 - a^2 \sin^2 \beta \right)^{\frac{1}{2}} \quad (E_{11})$$

i.e.,

$$\omega_0 \left(1 - \frac{1}{12} \theta_0^2 \right)^{\frac{1}{2}} dt = \int \frac{d\beta}{\sqrt{1 - a^2 \sin^2 \beta}} \quad (E_{12})$$

Integration of (E₁₂) yields

$$\omega_0 \left(1 - \frac{1}{12} \theta_0^2 \right)^{\frac{1}{2}} (t - t_0) = \int_{\beta_0}^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2 \beta}} \quad (E_{13})$$

Using the initial conditions $\beta_0 = 0$ at $t_0 = 0$, Eq. (E₁₃) can be reduced to

$$\omega_0 \left(1 - \frac{1}{12} \theta_0^2\right)^{\frac{1}{2}} \cdot t = \int_0^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2 \beta}} = F(a, \beta) \quad (E_{14})$$

where $F(a, \beta)$ is an incomplete elliptic integral of the first kind. Using $\beta = \frac{\pi}{2}$ when $\theta = \theta_0$ and $\beta = 0$ when $\theta = 0$, we get for one-quarter period,

$$\frac{\tau}{4} = t = \frac{1}{\omega_0 \left(1 - \frac{1}{12} \theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \quad (E_{15})$$

Thus the time period of the pendulum is given by

$$\tau = \frac{4}{\omega_0 \left(1 - \frac{1}{12} \theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \quad (E_{16})$$

13.12

Equation: $E[x] = \ddot{x} + \omega_0^2 x - \frac{1}{6} \omega_0^2 x^3 = 0 \quad (E_1)$

Assumed solution: $\tilde{x}(t) = A_0 \sin \omega t + A_3 \sin 3\omega t \quad (E_2)$

$$\ddot{\tilde{x}}(t) = -A_0 \omega^2 \sin \omega t - A_3 (3\omega)^2 \sin 3\omega t$$

$$\tilde{x}^3 = A_0^3 \sin^3 \omega t + A_3^3 \sin^3 3\omega t \quad (\text{neglecting cross product terms})$$

$$= \sin \omega t \left(\frac{3A_0^3}{4}\right) + \sin 3\omega t \left(-\frac{1}{4}A_0^3 + \frac{3}{4}A_3^3\right) + \sin 9\omega t \left(-\frac{1}{4}A_3^3\right)$$

This gives

$$\begin{aligned} E = E[\tilde{x}] &= -A_0 \omega^2 \sin \omega t - A_3 (3\omega)^2 \sin 3\omega t + A_0 \omega_0^2 \sin \omega t \\ &\quad + A_3 \omega_0^2 \sin 3\omega t - \frac{1}{8} \omega_0^2 A_0^3 \sin \omega t \\ &\quad + \frac{1}{24} \omega_0^2 (A_0^3 - 3A_3^3) \sin 3\omega t + \frac{1}{24} \omega_0^2 A_3^3 \sin 9\omega t \\ &= B_1 \sin \omega t + B_2 \sin 3\omega t + B_3 \sin 9\omega t \end{aligned} \quad (E_3)$$

Where $B_1 = -A_0 \omega^2 + A_0 \omega_0^2 - \frac{1}{8} A_0^3 \omega_0^2$

$$B_2 = -9A_3 \omega^2 + A_3 \omega_0^2 + \frac{1}{24} A_0^3 \omega_0^2 - \frac{1}{8} A_3^3 \omega_0^2$$

$$B_3 = \frac{1}{24} A_3^3 \omega_0^2$$

In Ritz - Galerkin method,

$$\int_0^{\tau} \tilde{E} \frac{\partial \tilde{E}}{\partial A_0} dt = 0 \quad (E_4)$$

$$\int_0^{\tau} \tilde{E} \frac{\partial \tilde{E}}{\partial A_3} dt = 0 \quad (E_5)$$

$$\text{i.e., } \int_0^{2\pi} (B_1 \sin \omega t + B_2 \sin 3\omega t + B_3 \sin 9\omega t)(B_4 \sin \omega t + B_5 \sin 3\omega t) dt = 0$$

$$\text{with } B_4 = -\omega^2 + \omega_0^2 - \frac{3}{8} A_0^2 \omega_0^2 \quad \text{and} \quad B_5 = \frac{1}{8} A_0^2 \omega_0^2.$$

$$\text{i.e., } \int_0^{2\pi} (B_1 B_4 \sin^2 \omega t + B_2 B_4 \sin \omega t \sin 3\omega t + B_3 B_4 \sin \omega t \sin 9\omega t + B_1 B_5 \sin \omega t \sin 3\omega t + B_2 B_5 \sin^2 3\omega t + B_3 B_5 \sin 3\omega t \sin 9\omega t) dt = 0$$

$$\text{i.e., } B_1 B_4 + B_2 B_5 = 0 \quad (E_6)$$

and Eq. (E5) leads to

$$\int_0^{2\pi} (B_1 \sin \omega t + B_2 \sin 3\omega t + B_3 \sin 9\omega t)(B_6 \sin 3\omega t + B_7 \sin 9\omega t) dt = 0$$

$$\text{with } B_6 = -9\omega^2 + \omega_0^2 - \frac{3}{8} A_3^2 \omega_0^2 \quad \text{and} \quad B_7 = \frac{1}{8} A_3^2 \omega_0^2.$$

$$\text{i.e., } B_2 B_6 + B_3 B_7 = 0 \quad (E_7)$$

Eqs. (E6) and (E7) can be rewritten as

$$(-A_0 \omega^2 + A_0 \omega_0^2 - \frac{1}{8} A_0^3 \omega_0^2) \left(-\omega^2 + \omega_0^2 - \frac{3}{8} A_0^2 \omega_0^2 \right) + (-A_3 \cdot 9\omega^2 + A_3 \omega_0^2 + \frac{1}{24} A_0^3 \omega_0^2 - \frac{1}{8} A_3^3 \omega_0^2) \left(\frac{1}{8} A_0^2 \omega_0^2 \right) = 0 \quad (E_8)$$

and

$$(-A_3 \cdot 9\omega^2 + A_3 \omega_0^2 + \frac{1}{24} A_0^3 \omega_0^2 - \frac{1}{8} A_3^3 \omega_0^2) \left(-9\omega^2 + \omega_0^2 - \frac{3}{8} A_3^2 \omega_0^2 \right) + \left(\frac{1}{24} A_3^3 \omega_0^2 \right) \left(\frac{1}{8} A_3^2 \omega_0^2 \right) = 0 \quad (E_9)$$

The solution of Eqs. (E8) and (E9) gives the values of A_0 and A_3 .

Note that (E8) and (E9) are two simultaneous algebraic equations in A_0 and A_3 for which general closed-form solution is not possible.

Particular case :

If $A_3 = 0$, Eq. (E) yields

$$A_0 \left(-\omega^2 + \omega_0^2 - \frac{1}{8} A_0^2 \omega_0^2 \right) \left(-\omega^2 + \omega_0^2 - \frac{3}{8} A_0^2 \omega_0^2 \right) + \frac{1}{192} A_0^5 \omega_0^4 = 0$$

$$\text{i.e., } A_0 \left\{ \omega^4 + \omega^2 \left(-2\omega_0^2 + \frac{1}{2} A_0^2 \omega_0^2 \right) + \omega_0^4 \left(1 - \frac{1}{2} A_0^2 + \frac{5}{96} A_0^4 \right) \right\} = 0$$

Since $A_0 \neq 0$, we get

$$\omega^4 - \omega^2 \left(2\omega_0^2 - \frac{1}{2} A_0^2 \omega_0^2 \right) + \omega_0^4 \left(1 - \frac{1}{2} A_0^2 + \frac{5}{96} A_0^4 \right) = 0 \quad (E_{10})$$

which can be seen to be identical to Eq. (E.6) of Example 13.1.

13.13

Equation: $\ddot{x} + \omega_0^2 x + \alpha x^3 = 0$ (E₁)

we assume

$$x(t) = x_0(t) + \alpha x_1(t) + \alpha^2 x_2(t) \quad (E_2)$$

and $\omega^2 = \omega_0^2 + \alpha \omega_1(A_0) + \alpha^2 \omega_2(A_0)$ (E₃)

or $\omega_0^2 = \omega^2 - \alpha \omega_1(A_0) - \alpha^2 \omega_2(A_0)$ (E₄)

where A_0 = amplitude and ω = true fundamental frequency.

Substitution of (E₂) and (E₄) into (E₁) gives

$$\begin{aligned} &\ddot{x}_0 + \alpha \ddot{x}_1 + \alpha^2 \ddot{x}_2 + \omega^2 x_0 - \alpha \omega_1 x_0 - \alpha^2 \omega_2 x_0 + \omega^2 \alpha x_1 \\ &- \alpha^2 \omega_1 x_1 - \alpha^3 \omega_2 x_1 + \alpha^2 \omega^2 x_2 - \alpha^3 \omega_1 x_2 - \alpha^4 \omega_2 x_2 + \alpha x_0^3 \\ &+ \alpha^4 x_1^3 + \alpha^7 x_2^3 + 3\alpha^2 x_0^2 x_1 + 3\alpha^3 x_0 x_1^2 + 3\alpha^3 x_0^2 x_2 \\ &+ 3\alpha^5 x_0 x_2^2 + 3\alpha^5 x_1^2 x_2 + 3\alpha^6 x_1 x_2^2 + 6\alpha^4 x_0 x_1 x_2 = 0 \end{aligned} \quad (E_5)$$

Neglecting terms involving $\alpha^3, \alpha^4, \dots$, (E₅) can be rewritten as

$$\begin{aligned} &\alpha^0 (\ddot{x}_0 + \omega^2 x_0) + \alpha^1 (\ddot{x}_1 - \omega_1 x_0 + \omega^2 x_1 + x_0^3) \\ &+ \alpha^2 (\ddot{x}_2 - \omega_2 x_0 - \omega_1 x_1 + \omega^2 x_2 + 3x_0^2 x_1) = 0 \end{aligned} \quad (E_6)$$

Setting the coefficient of α^0 to zero in (E₆), we get

$$\ddot{x}_0 + \omega^2 x_0 = 0 \quad (E_7)$$

whose solution is

$$x_0(t) = D_1 \cos \omega t + D_2 \sin \omega t \quad (E_8)$$

and hence $\dot{x}_0(t) = -D_1 \omega \sin \omega t + D_2 \omega \cos \omega t$ (E₉)

Let the initial conditions of the system be

$$x_0(t=0) = A_0 \quad \text{and} \quad \dot{x}_0(t=0) = 0 \quad (E_{10})$$

Eqs. (E₈) to (E₁₀) yield $D_1 = A_0$ and $D_2 = 0$ so that

$$x_0(t) = A_0 \cos \omega t \quad (E_{11})$$

Thus, if $\alpha = 0$, the solution becomes

$$x_0(t) = A_0 \cos \omega t, \quad \omega = \omega_0 \quad (E_{12})$$

By setting the coefficient of α^1 to zero in (E₆), we get

$$\ddot{x}_1 + \omega^2 x_1 = \omega_1 x_0 - x_0^3 \quad (E_{13})$$

Substituting Eq. (E₁₁) for x_0 , (E₁₃) leads to

$$\ddot{x}_1 + \omega^2 x_1 = (\omega_1 A_0 - \frac{3}{4} A_0^3) \cos \omega t - \frac{1}{4} A_0^3 \cos 3\omega t \quad (E_{14})$$

The complete solution of (E₁₄) can be expressed as (can be verified by substitution):

$$x_1(t) = D_1 \cos \omega t + D_2 \sin \omega t + (\omega_1 A_0 - \frac{3}{4} A_0^3) \frac{1}{2} \omega t \cdot \sin \omega t + \frac{1}{32} \cdot \frac{1}{\omega^2} \cdot A_0^3 \cos 3\omega t \quad (E_{15})$$

Note that the third term on the r.h.s. of (E₁₅) is a secular term which can be eliminated by setting

$$\omega_1 A_0 - \frac{3}{4} A_0^3 = 0 \quad \text{i.e.,} \quad A_0 = 0 \quad \text{or} \quad \omega_1 = \frac{3}{4} A_0^2 \quad (E_{16})$$

Since $A_0 \neq 0$, we have $\omega_1 = \frac{3}{4} A_0^2 \quad (E_{17})$

Using the initial conditions $x_1(t=0) = \dot{x}_1(t=0) = 0$ in (E₁₅), we get

$$D_1 = -\frac{1}{32\omega^2} A_0^3, \quad D_2 = 0 \quad (E_{18})$$

Hence the first-order correction is given by

$$x_1(t) = -\frac{A_0^3}{32\omega^2} (\cos \omega t - \cos 3\omega t); \quad \omega_1 = \frac{3}{4} A_0^2 \quad (E_{19})$$

Finally, by setting the coefficient of α^2 in (E₆) to zero, we get

$$\ddot{x}_2 + \omega^2 x_2 = \omega_2 x_0 + x_1 \omega_1 - 3 x_0^2 x_1 \quad (E_{20})$$

Substituting for x_0 , x_1 and ω_1 , Eq. (E₂₀) leads to

$$\ddot{x}_2 + \omega^2 x_2 = (\omega_2 A_0 - \frac{3}{128} \frac{A_0^5}{\omega^2}) \cos \omega t + (\frac{3}{128} \frac{A_0^5}{\omega^2}) \cos 3\omega t + \frac{3}{32} \frac{A_0^5}{\omega^2} (\cos^3 \omega t - \cos^2 \omega t \cos 3\omega t) \quad (E_{21})$$

Here $\cos^3 \omega t = \frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t \quad (E_{22})$

and $\cos^2 \omega t \cdot \cos 3\omega t = \frac{1}{4} \cos \omega t + \frac{1}{2} \cos 3\omega t + \frac{1}{4} \cos 5\omega t \quad (E_{23})$

Eqs. (E₂₁) to (E₂₃) lead to

$$\ddot{x}_2 + \omega^2 x_2 = (\omega_2 A_0 + \frac{3}{128} \frac{A_0^5}{\omega^2}) \cos \omega t - \frac{3}{128} \frac{A_0^5}{\omega^2} \cos 5\omega t \quad (E_{24})$$

Since the first term on the r.h.s. of (E₂₄) leads to the secular term, we eliminate it by setting

$$\omega_2 A_0 + \frac{3}{128} \frac{A_0^5}{\omega^2} = 0 \quad \text{or} \quad \omega_2 = -\frac{3}{128} \frac{A_0^4}{\omega^2} \quad (E_{25})$$

With this, the total solution of (E₂₄), after applying the initial conditions $x_2(t=0) = 0$ and $\dot{x}_2(t=0) = 0$, can be obtained as

$$x_2(t) = -\frac{A_0^5}{1024 \omega^4} (\cos \omega t - \cos 5 \omega t); \quad \omega_2 = -\frac{3}{128} \frac{A_0^4}{\omega^2} \quad (E_{26})$$

Thus the total solution becomes

$$x(t) = A_0 \cos \omega t - \frac{1}{32} \frac{A_0^3 \alpha}{\omega^2} (\cos \omega t - \cos 3 \omega t) \\ - \frac{1}{1024} \cdot \frac{A_0^5 \alpha^2}{\omega^4} (\cos \omega t - \cos 5 \omega t)$$

and

$$\omega^2 = \omega_0^2 + \frac{3}{4} A_0^2 \cdot \alpha - \frac{3}{128} \frac{A_0^4}{\omega^2} \cdot \alpha^2 \quad (E_{27})$$

13.14

Equation: $\ddot{x} + c \dot{x} + k_1 x + k_2 x^3 = a_1 \cos 3 \omega t - a_2 \sin 3 \omega t \quad (E_1)$

Assume

$$x(t) = A \cos \omega t + B \cos 3 \omega t + C \sin 3 \omega t \quad (E_2)$$

so that $\dot{x}(t) = -\omega A \sin \omega t - 3 \omega B \sin 3 \omega t + 3 \omega C \cos 3 \omega t \quad (E_3)$

$$\ddot{x}(t) = -\omega^2 A \cos \omega t - 9 \omega^2 B \cos 3 \omega t - 9 \omega^2 C \sin 3 \omega t \quad (E_4)$$

$$x^3 = A^3 \cos^3 \omega t + B^3 \cos^3 3 \omega t + C^3 \sin^3 3 \omega t \\ + 3 A^2 B \cos^2 \omega t \cos 3 \omega t + 3 A B^2 \cos \omega t \cos^2 3 \omega t \\ + 3 A^2 C \cos^2 \omega t \sin 3 \omega t + 3 A C^2 \cos \omega t \sin^2 3 \omega t \\ + 3 B^2 C \cos^2 3 \omega t \sin 3 \omega t + 3 B C^2 \cos 3 \omega t \sin^2 3 \omega t \\ + 6 A B C \cos \omega t \cos 3 \omega t \sin 3 \omega t \\ = \cos \omega t \left(\frac{3}{4} A^3 + \frac{3}{4} A^2 B + \frac{3}{2} A B^2 + \frac{3}{2} A C^2 \right) + \sin \omega t \left(\frac{3}{4} A^2 C \right) \\ + \cos 3 \omega t \left(\frac{1}{4} A^3 + \frac{3}{4} B^3 + \frac{3}{2} A^2 B + \frac{3}{2} B C^2 \right) \\ + \sin 3 \omega t \left(\frac{3}{4} C^3 + \frac{3}{2} A^2 C + \frac{3}{2} B^2 C - \frac{3}{4} B^2 C - \frac{3}{4} B C^2 \right) \\ + \cos 9 \omega t \left(\frac{1}{4} B^3 \right) + \sin 9 \omega t \left(-\frac{1}{4} C^3 + \frac{3}{4} B^2 C - \frac{3}{4} B C^2 \right) \\ + \cos 5 \omega t \left(\frac{3}{4} A^2 B + \frac{3}{4} A B^2 - \frac{3}{4} A C^2 \right) \\ + \cos 7 \omega t \left(\frac{3}{4} A B^2 - \frac{3}{4} A C^2 \right) + \sin 5 \omega t \left(\frac{3}{4} A^2 C + \frac{3}{2} A B C \right) \\ + \sin 7 \omega t \left(\frac{3}{2} A B C \right) \quad (E_5)$$

For convenience, we use the notation

$$\theta = \omega t \quad \text{so that} \quad \dot{x} = \frac{dx}{dt} = \frac{dx}{d\theta} \frac{d\theta}{dt} = \omega \frac{dx}{d\theta} = \omega x'$$

$$\text{and} \quad \ddot{x} = \frac{d}{dt}(\omega x') = \omega^2 x''$$

With this, Eqs. (E₁) to (E₅) become

$$\omega^2 x'' + c\omega x' + k_1 x + k_2 x^3 = a_1 \cos 3\theta - a_2 \sin 3\theta \quad (E_6)$$

$$x(\theta) = A \cos \theta + B \cos 3\theta + C \sin 3\theta \quad (E_7)$$

$$x'(\theta) = \dot{x}/\omega = -A \sin \theta - 3B \sin 3\theta + 3C \cos 3\theta \quad (E_8)$$

$$x''(\theta) = \ddot{x}/\omega^2 = -A \cos \theta - 9B \cos 3\theta - 9C \sin 3\theta \quad (E_9)$$

$$x^3(\theta) = \left(\frac{3}{4} A^3 + \frac{3}{4} A^2 B + \frac{3}{2} AB^2 + \frac{3}{2} AC^2 \right) \cos \theta \\ + \left(\frac{3}{4} A^2 C \right) \sin \theta + \dots + \left(\frac{3}{2} ABC \right) \sin 7\theta \quad (E_{10})$$

Terms containing $\cos 5\theta$, $\sin 5\theta$, $\cos 7\theta$, $\sin 7\theta$, $\cos 9\theta$ and $\sin 9\theta$ in Eq. (E₁₀) can be neglected for the present analysis.

Then the substitution of Eqs. (E₇) to (E₁₀) into (E₆) and equating the coefficients of $\cos \theta$, $\sin \theta$, $\cos 3\theta$ and $\sin 3\theta$ on both sides gives:

$$(k_1 - \omega^2) + \frac{3}{4} k_2 (A^2 + AB + 2B^2 + 2C^2) = 0 \quad (E_{11})$$

$$-c\omega + \frac{3}{4} k_2 AC = 0 \quad (E_{12})$$

$$(k_1 - 9\omega^2)B + 3c\omega C + \frac{1}{4} k_2 (A^3 + 6A^2 B + 3B^3 + 3BC^2) = a_1 \quad (E_{13})$$

$$(k_1 - 9\omega^2)C - 3c\omega B + \frac{3}{4} k_2 C (C^2 + 2A^2 + B^2) = -a_2 \quad (E_{14})$$

Eqs. (E₁₁) to (E₁₄) represent four simultaneous nonlinear algebraic equations in A , B , C and ω . For any specific case, these equations should yield non-zero real values for A , B and C for the existence of subharmonic oscillations of order 3.

More specifically, for A :

Eqs. (E₁₁) and (E₁₂) can be solved to find

$$\omega^2 = k_1 + \frac{3}{4} k_2 (A^2 + AB + 2B^2 + 2C^2) \quad (E_{15})$$

$$C = \frac{4c\omega}{3k_2 A} \quad (E_{16})$$

Eg. (E₁₅) can be solved to obtain

$$A = -\frac{B}{2} \pm \frac{1}{2} \left\{ \frac{16}{3k_2} (\omega^2 - k_1) - 7B^2 - 8C^2 \right\}^{\frac{1}{2}} \quad (E_{17})$$

This equation gives the condition for the existence of subharmonic of order 3 (nonzero A) as

$$\frac{16}{3k_2} (\omega^2 - k_1) - 7B^2 - 8C^2 > 0$$

i.e., $\omega^2 > k_1 + \frac{3k_2}{16} (7B^2 + 8C^2)$ (E₁₈)

13.15 Equation: $\ddot{x} + c\dot{x} + k_1x + k_2x^2 = a \cos 2\omega t$ (E₁)
 As in the solution of problem 13.14, we introduce

$$\theta = \omega t, \quad \dot{x} = \omega x' \quad \text{and} \quad \ddot{x} = \omega^2 x'' \quad \text{with} \quad x' = \frac{dx}{d\theta}.$$

Eg. (E₁) becomes

$$\omega^2 x'' + c\omega x' + k_1x + k_2x^2 = a \cos 2\theta \quad (E_2)$$

The solution of (E₂) must contain terms involving $\cos \theta$ and/or $\sin \theta$ in order to include subharmonics of order 2. If the nonlinear term k_2x^2 is assumed to be small with $k_2 > 0$, we can express c and a as $c = \epsilon k_2^2$ and $a = a_0 k_2$.

Thus (E₂) becomes

$$\omega^2 x'' + \epsilon \omega k_2^2 x' + k_1x + k_2x^2 = k_2 a_0 \cos 2\theta \quad (E_3)$$

Using perturbation method, we assume

$$x(t) = x_0(t) + k_2 x_1(t) + k_2^2 x_2(t) + \dots \quad (E_4)$$

$$\omega = \omega_0 + k_2 \omega_1 + k_2^2 \omega_2 + \dots \quad (E_5)$$

Substituting (E₄) and (E₅) into (E₃), we get

$$\begin{aligned} & (\omega_0 + k_2 \omega_1 + k_2^2 \omega_2)^2 (x_0'' + k_2 x_1'' + k_2^2 x_2'') + \epsilon k_2^2 (\omega_0 + k_2 \omega_1 \\ & + k_2^2 \omega_2) (x_0' + k_2 x_1' + k_2^2 x_2') + k_1 (x_0 + k_2 x_1 + k_2^2 x_2) \\ & + k_2 (x_0 + k_2 x_1 + k_2^2 x_2)^2 = k_2 a_0 \cos 2\theta \end{aligned}$$

i.e.,

$$\begin{aligned} & k_2^0 (x_0'' \omega_0^2 + k_1 x_0) + k_2^1 (x_1'' \omega_0^2 + 2\omega_0 \omega_1 x_0'' + k_1 x_1 + x_0'^2 - a_0 \cos 2\theta) \\ & + k_2^2 (x_2'' \omega_0^2 + x_0'' \omega_1^2 + 2\omega_0 \omega_1 x_1'' + 2\omega_0 \omega_2 x_0'' + \epsilon \omega_0 x_0' + k_1 x_2 \\ & + 2x_0 x_1) + k_2^3 (\dots) + \dots = 0 \end{aligned} \quad (E_6)$$

By setting the coefficients of various powers of k_2 equal to zero in (E₆), we get the following.

coefficient of κ_2^0 : $x_0'' \omega_0^2 + \kappa_1 x_0 = 0$ (E7)

Solution of (E7) is $x_0(t) = A_1 \cos \theta + A_2 \sin \theta$ (E8)

and $\omega_0^2 = \kappa_1$ (E9)

coefficient of κ_2^1 :

$$x_1'' \omega_0^2 + \kappa_1 x_1 = -2\omega_0 \omega_1 x_0'' - x_0^2 + \omega_0 \cos 2\theta$$
 (E10)

Substituting (E8) and (E9), Eq. (E10) becomes

$$x_1'' \omega_0^2 + \kappa_1 x_1 = -2\omega_0 \omega_1 (-A_1 \cos \theta - A_2 \sin \theta) - A_1^2 \cos^2 \theta - A_2^2 \sin^2 \theta - A_1 A_2 \sin 2\theta + \omega_0 \cos 2\theta$$

or,

$$x_1'' + x_1 = \left(\frac{2\omega_1 A_1}{\omega_0} \right) \cos \theta + \left(\frac{2\omega_1 A_2}{\omega_0} \right) \sin \theta + \frac{1}{\kappa_1} \left(-\frac{A_1^2}{2} + \frac{A_2^2}{2} + \omega_0 \right) \cos 2\theta - \left(\frac{A_1 A_2}{\kappa_1} \right) \sin 2\theta - \left(\frac{A_1^2 + A_2^2}{2\kappa_1} \right)$$
 (E11)

Since the first two terms on the r.h.s. of (E11) lead to secular terms, we should have

$$\omega_1 = 0$$
 (E12)

to avoid those terms. Thus (E11) reduces to

$$x_1'' + x_1 = \frac{1}{\kappa_1} \left(-\frac{A_1^2}{2} + \frac{A_2^2}{2} + \omega_0 \right) \cos 2\theta - \frac{A_1 A_2}{\kappa_1} \sin 2\theta - \frac{A_1^2 + A_2^2}{2\kappa_1}$$
 (E13)

The particular integral of (E13) can be written as

$$x_1(\theta) = -\left(\frac{A_1^2 + A_2^2}{2\kappa_1} \right) + A_3 \cos 2\theta + A_4 \sin 2\theta$$
 (E14)

where A_3 and A_4 can be found by substituting (E14) into (E13):

$$A_3 = \frac{1}{3\kappa_1} \left(\frac{A_1^2 - A_2^2}{2} - \omega_0 \right), \quad A_4 = \frac{A_1 A_2}{3\kappa_1}$$
 (E15)

coefficient of κ_2^2 :

$$\omega_0^2 x_2'' + \kappa_1 x_2 = -\omega_1^2 x_0'' - 2\omega_0 \omega_1 x_1'' - 2\omega_0 \omega_2 x_0'' - \epsilon \omega_0 x_0' - 2x_0 x_1$$

or, $x_2'' + x_2 = -2 \frac{\omega_2}{\omega_0} x_0'' - \frac{\epsilon}{\omega_0} x_0' - \frac{2}{\kappa_1} x_0 x_1$ (E16)

since $\omega_1 = 0$ and $\omega_0^2 = \kappa_1$. By substituting (E8) and (E14) on the r.h.s. of (E16), we get

$$x_2'' + x_2 = \left(\frac{2\omega_2 A_1}{\omega_0} - \frac{\epsilon}{\omega_0} A_2 + \frac{A_1}{\kappa_1^2} (A_1^2 + A_2^2) - \frac{A_1 A_3}{\kappa_1} - \frac{A_2 A_4}{\kappa_1} \right) \cos \theta + \left(\frac{2\omega_2 A_2}{\omega_0} + \frac{\epsilon A_1}{\omega_0} + \frac{A_2}{\kappa_1^2} (A_1^2 + A_2^2) + \frac{A_2 A_3}{\kappa_1} - \frac{A_1 A_4}{\kappa_1} \right) \sin \theta$$

$$+ \left(-\frac{A_1 A_3}{k_1} + \frac{A_2 A_4}{k_1} \right) \cos 3\theta + \left(-\frac{A_2 A_3}{k_1} - \frac{A_1 A_4}{k_1} \right) \sin 3\theta \quad (E_{17})$$

To avoid secular terms, the coefficients of $\cos \theta$ and $\sin \theta$ in (E₁₇) must be zero. This gives,

$$\frac{2\omega_2 A_1}{\omega_0} - \frac{\epsilon A_2}{\omega_0} + \frac{A_1}{k_1^2} (A_1^2 + A_2^2) - \frac{A_1 A_3}{k_1} - \frac{A_2 A_4}{k_1} = 0 \quad (E_{18})$$

$$\frac{2\omega_2 A_2}{\omega_0} + \frac{\epsilon A_1}{\omega_0} + \frac{A_2}{k_1^2} (A_1^2 + A_2^2) + \frac{A_2 A_3}{k_1} - \frac{A_1 A_4}{k_1} = 0 \quad (E_{19})$$

Thus (E₁₇) reduces to

$$x_2'' + x_2 = (-A_1 A_3 + A_2 A_4) \frac{1}{k_1} \cos 3\theta - (A_2 A_3 + A_1 A_4) \frac{1}{k_1} \sin 3\theta \quad (E_{20})$$

Substituting (E₁₅) into (E₁₈) and (E₁₉), we get

$$\frac{2\omega_2 A_1}{\omega_0} - \frac{\epsilon A_2}{\omega_0} + \frac{A_1}{6k_1^2} (6A_1^2 + 6A_2^2 - A_1^2 + A_2^2 - 2A_2^2) + \frac{A_1 \omega_0}{3k_1^2} = 0 \quad \dots (E_{21})$$

$$\frac{2\omega_2 A_2}{\omega_0} + \frac{\epsilon A_1}{\omega_0} + \frac{A_2}{6k_1^2} (6A_1^2 + 6A_2^2 + A_1^2 - A_2^2 - 2A_1^2) - \frac{A_2 \omega_0}{3k_1^2} = 0 \quad \dots (E_{22})$$

Dividing (E₂₁) and (E₂₂) by A_1 and A_2 , respectively, we get

$$\frac{2\omega_2}{\omega_0} - \frac{\epsilon}{\omega_0} \frac{A_2}{A_1} + \frac{5}{6k_1^2} (A_1^2 + A_2^2) + \frac{\omega_0}{3k_1^2} = 0 \quad (E_{23})$$

$$\frac{2\omega_2}{\omega_0} + \frac{\epsilon}{\omega_0} \frac{A_1}{A_2} + \frac{5}{6k_1^2} (A_1^2 + A_2^2) - \frac{\omega_0}{3k_1^2} = 0 \quad (E_{24})$$

Addition and subtraction of (E₂₃) and (E₂₄) give

$$\omega_2 = \frac{\epsilon}{4} \left(\frac{A_2}{A_1} - \frac{A_1}{A_2} \right) - \frac{5\omega_0}{12k_1^2} (A_1^2 + A_2^2) \quad (E_{25})$$

$$\text{and } r + \frac{1}{r} = p \quad (E_{26})$$

$$\text{where } r = A_2/A_1 \text{ and } p = \left(\frac{2\omega_0}{3\epsilon k_1 \omega_0} \right) \quad (E_{27})$$

$$\text{Solution of (E}_{26}\text{) is: } r = \frac{A_2}{A_1} = \frac{p \pm \sqrt{p^2 - 4}}{2} \quad (E_{28})$$

Thus ω_2 becomes (Eg. (E₂₅)):

$$\omega_2 = \pm \frac{(\omega_0^2 - 9\epsilon^2 k_1^3)^{1/2}}{6k_1 \omega_0} - \frac{5(A_1^2 + A_2^2)}{12k_1 \omega_0} \quad (E_{29})$$

The solution of Eg. (E₂₀) can be determined as

$$x_2(\theta) = \left(\frac{A_1 A_3 - A_2 A_4}{8k_1} \right) \cos 3\theta + \left(\frac{A_2 A_3 + A_1 A_4}{8k_1} \right) \sin 3\theta \quad (E_{30})$$

Thus Egs. (E₄) and (E₅) become

$$x(t) = x_0(t) + k_2 x_1(t) + k_2^2 x_2(t) ; \quad \omega = \omega_0 + k_2 \omega_1 + k_2^2 \omega_2$$

along with Egs. (E₈), (E₉), (E₁₂), (E₁₄), (E₂₉) and (E₃₀).

13.16

Eq. (13.82) can be rewritten as

$$(\omega^2)^3 - \omega_0^2 (\omega^2)^2 - \frac{3}{4} \alpha A^2 (\omega^2)^2 + \frac{3}{32} \alpha F A (\omega^2) - \frac{3}{128} \alpha F^2 = 0 \quad (E_1)$$

Differentiation of (E₁) gives

$$d\omega^2 \left(3\omega^4 - 2\omega_0^2 \omega^2 - \frac{3}{2} \alpha A^2 \omega^2 + \frac{3}{32} \alpha F A \right) + dA \left(-\frac{3}{2} \alpha \omega^4 A + \frac{3}{32} \alpha F \omega^2 \right) = 0 \quad (E_2)$$

$$\text{Setting } \frac{d\omega^2}{dA} = 0, \text{ Eq. (E}_2\text{) gives } A = \frac{1}{16} \frac{F}{\omega^2} \quad (E_3)$$

With this Eq. (E₃), Eq. (E₁) becomes

$$\omega^6 - \omega_0^2 \omega^4 = \frac{3}{128} \alpha F^2 + \frac{3}{1024} \alpha F^2 - \frac{3}{512} \alpha F^2 = \frac{21}{1024} \alpha F^2 \quad (E_4)$$

Dividing (E₄) by ω_0^6 , we get

$$\left(\frac{\omega}{\omega_0} \right)^6 - \left(\frac{\omega}{\omega_0} \right)^4 = \frac{21}{1024} \frac{\alpha F^2}{\omega_0^6} \quad (E_5)$$

Letting $\frac{\omega}{\omega_0} = 1 + \delta$ with $\delta \ll 1$, Eq. (E₅) can be expressed as

$$(1 + \delta)^6 - (1 + \delta)^4 = \frac{21}{1024} \frac{\alpha F^2}{\omega_0^6} \quad (E_6)$$

Since $(1 + \delta)^6 \approx 1 + 6\delta$ and $(1 + \delta)^4 \approx 1 + 4\delta$,Eq. (E₆) becomes

$$2\delta = \frac{21}{1024} \frac{\alpha F^2}{\omega_0^6}$$

$$\text{or } \omega_{\min} = \omega_0 + \frac{21}{2048} \frac{\alpha F^2}{\omega_0^5} \quad (E_7)$$

13.17

Derivation of Eq. (13.113b):

Using $\alpha_0 = \frac{1}{4}$ and $y_0 = \sin \frac{t}{2}$, Eq. (13.103) gives

$$\ddot{y}_1 + \alpha_0 y_1 = -\alpha_1 y_0 - y_0 \cos t$$

$$\text{or } \ddot{y}_1 + \frac{1}{4} y_1 = \left(-\alpha_1 + \frac{1}{2} \right) \sin \frac{t}{2} - \frac{1}{2} \sin \frac{3t}{2} \quad (E_1)$$

Homogeneous solution of (E₁) is:

$$y_1(t) = A_1 \cos \frac{t}{2} + A_2 \sin \frac{t}{2} \quad (E_2)$$

Where A_1 and A_2 are constants of integration. To avoid secular term, the coefficient of $\sin \frac{t}{2}$ must be set equal to zero in (E₁). This gives $\alpha_1 = \frac{1}{2}$ and

$$\ddot{y}_1 + \frac{1}{4} y_1 = -\frac{1}{2} \sin \frac{3t}{2} \quad (E_3)$$

Using the particular solution

$$y_1(t) = A_2 \sin \frac{3t}{2} \quad (E_4)$$

in (E₃), we get $A_2 = \frac{1}{4}$ and $y_1(t) = \frac{1}{4} \sin \frac{3t}{2}$ (E₅)

With $\alpha_0 = \frac{1}{4}$, $\alpha_1 = \frac{1}{2}$, $y_0 = \sin \frac{t}{2}$ and $y_1 = \frac{1}{4} \sin \frac{3t}{2}$, E_g. (13.104) becomes

$$\ddot{y}_2 + \frac{1}{4} y_2 = (-\alpha_2 - \frac{1}{8}) \sin \frac{t}{2} - \frac{1}{8} \sin \frac{3t}{2} - \frac{1}{8} \sin \frac{5t}{2} \quad (E_6)$$

Again the coefficient of $\sin \frac{t}{2}$ in (E₆) must be set equal to zero to avoid the secular term. This gives $\alpha_2 = -\frac{1}{8}$. Thus E_g. (13.100) becomes

$$\alpha = \alpha_0 + \epsilon \alpha_1 + \epsilon^2 \alpha_2 = \frac{1}{4} + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} \quad (13.113b)$$

Derivation of E_g. (13.116b) :

With $\alpha_0 = 1$ and $y_0 = \sin t$, E_g. (13.103) becomes

$$\ddot{y}_1 + y_1 = -\alpha_1 y_0 - y_0 \cos t = -\alpha_1 \sin t - \frac{1}{2} \sin 2t \quad (E_7)$$

Coefficient of $\sin t$ in (E₇) must be zero to avoid the secular term. This gives $\alpha_1 = 0$ and hence (E₇) reduces to

$$\ddot{y}_1 + y_1 = -\frac{1}{2} \sin 2t \quad (E_8)$$

By substituting the particular solution

$$y_1(t) = A_2 \sin 2t \quad (E_9)$$

into (E₈), we get $A_2 = \frac{1}{6}$. Using $\alpha_0 = 1$, $y_0 = \sin t$, $\alpha_1 = 0$ and $y_1 = \frac{1}{6} \sin 2t$, E_g. (13.104) becomes

$$\begin{aligned} \ddot{y}_2 + \alpha_0 y_2 &= -\alpha_2 y_0 - \alpha_1 y_1 - y_1 \cos t \\ &= (-\alpha_2 - \frac{1}{12}) \sin t - \frac{1}{12} \sin 3t \end{aligned} \quad (E_{10})$$

To avoid secular terms, the coefficient of $\sin t$ on the r.h.s. of E_g. (E₁₀) must be zero. This gives $\alpha_2 = -\frac{1}{12}$. Thus E_g. (13.100) becomes

$$\alpha = \alpha_0 + \epsilon \alpha_1 + \epsilon^2 \alpha_2 = 1 - \frac{1}{12} \epsilon^2 \quad (13.116b)$$

13.18

$\ddot{x} + 0.4 \dot{x} + 0.8x = 0$. This is in the form $\ddot{x} + 2\gamma\omega_n \dot{x} + \omega_n^2 x = 0$
with $\omega_n = \sqrt{0.8} = 0.89443$ and $\gamma = 0.22361$.

$$\omega_d = \omega_n \sqrt{1 - \gamma^2} = 0.87178.$$

$$\text{Solution is } x(t) = e^{-\gamma\omega_n t} (A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

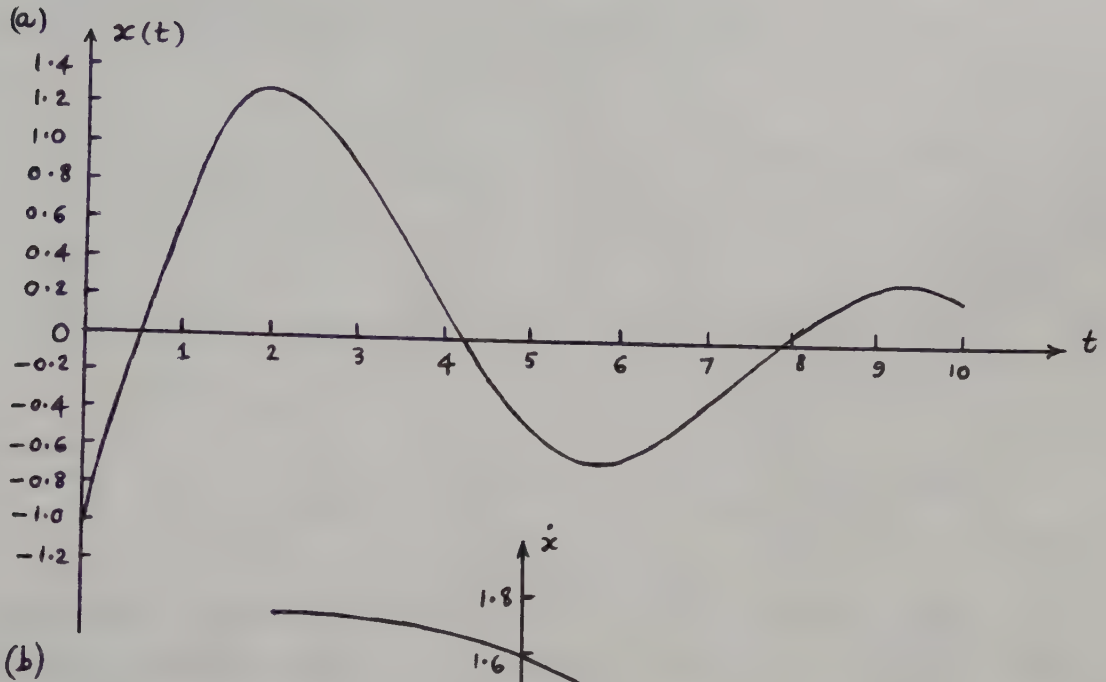
$$\dot{x}(t) = e^{-\gamma\omega_n t} [(-\gamma\omega_n A_1 + \omega_d A_2) \cos \omega_d t - (\omega_d A_1 + \gamma\omega_n A_2) \sin \omega_d t]$$

Using $x(0) = A_1 = -1$ and $\dot{x}(0) = -\gamma\omega_n A_1 + \omega_d A_2 = 2$, we obtain

$$A_1 = -1 \text{ and } A_2 = (2 + \gamma\omega_n A_1) / \omega_d = 1.77058$$

$$\therefore x(t) = e^{-0.2t} (-\cos 0.87178t + 1.77058 \sin 0.87178t)$$

and $\dot{x}(t) = e^{-0.2t} (1.74356 \cos 0.87178t + 0.51766 \sin 0.87178t)$



13.19

$$\ddot{x} + 0.1(x^2 - 1)\dot{x} + x = 0 \quad \text{or} \quad \ddot{x} = -[0.1(x^2 - 1)\dot{x} + x]$$

$$\text{Let } x = x_1 \quad \dot{x}_1 = x_2 = f_1(x_1, x_2)$$

$$\dot{x}_2 = -[0.1(x_1^2 - 1)x_2 + x_1] = f_2(x_1, x_2)$$

For equilibrium, from Eq. (13.129),

$$f_1 = 0 \Rightarrow x_2 = 0; \quad f_2 = 0 \Rightarrow x_1 = 0$$

Eq. (13.132) becomes

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

where

$$a_{11} = \left. \frac{\partial f_1}{\partial x_1} \right|_{(0,0)} = 0, \quad a_{12} = \left. \frac{\partial f_1}{\partial x_2} \right|_{(0,0)} = 1,$$

$$a_{21} = \left. \frac{\partial f_2}{\partial x_1} \right|_{(0,0)} = -[0.2x_1x_2 + 1]_{(0,0)} = -1,$$

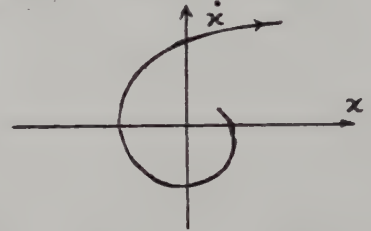
$$a_{22} = \left. \frac{\partial f_2}{\partial x_2} \right|_{(0,0)} = -[0.1(x_1^2 - 1)]_{(0,0)} = 0.1$$

From Eq. (13.135), we find $p = 0.1$, $q = 1$

$$\lambda_1, \lambda_2 = \frac{1}{2}(0.1 \pm \sqrt{0.01 - 4}) = \text{complex with positive real parts}$$

Since $p > 0$, the system is unstable at the equilibrium point $(x, \dot{x}) = (0, 0)$.

Hence the phase-plane trajectory in the neighborhood of the equilibrium position appears as shown in the figure.



13.20

$$\ddot{x} + 0.4\dot{x} + 0.8x = 0$$

$$\text{Let } x = x, \quad \dot{x} = y, \quad \dot{y} = -0.4y - 0.8x$$

Initial conditions:

$$x(0) = 2, \quad y(0) = 1$$

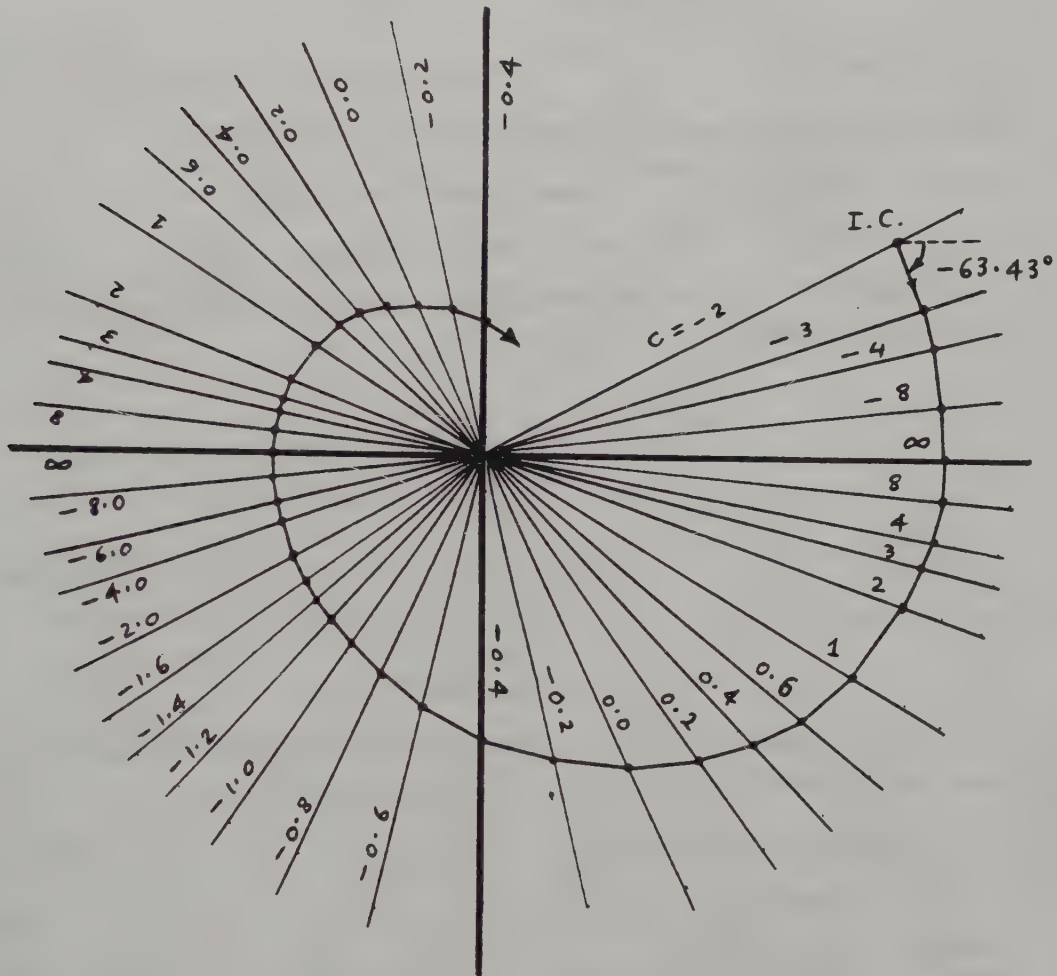
$$\frac{dy}{dx} = \frac{-0.4y - 0.8x}{y} = c = \text{slope of the trajectory}$$

$$\text{Equation of isocline is } -0.4y - 0.8x = cy \quad \text{or} \quad \frac{y}{x} = \frac{-0.8}{c + 0.4}$$

This equation is plotted for different values of c . Then small line segments are drawn and slopes are used for extrapolation.

c	-2	-3	-4	-8	∞	8	4
value of y/x	0.5	0.3077	0.2222	0.1053	0	-0.0952	-0.1818
Angle	26.5651°	17.10°	12.53°	6.01°	0°	-5.44°	-10.30°
slope of trajectory	-63.43°	-71.56°	-75.96°	-82.87°	-90°	82.87°	75.96°

3	2	1	0.6	0.4	0.2	0	-0.2	-0.4
-0.2353	-0.3333	-0.5714	-0.8	-1.0	-1.3333	-2.0	-4.0	$-\infty$
-13.24°	-18.43°	-29.74°	-38.65°	-45°	-53.13°	-63.43°	-75.96°	-90°
71.56°	63.43°	45°	30.96°	21.80°	11.31°	0°	-11.31°	-21.80°
-0.6	-0.8	-1.0	-1.2	-1.6	-2.0	-4	-8	∞
4.0	2.0	1.3333	1.0	0.6667	0.5	0.2222	0.1053	0
75.96°	63.43°	53.13°	45°	33.69°	26.57°	12.53°	6.01°	0°
-30.96°	-38.66°	-45.00°	-50.19°	-57.99°	-63.43°	-75.96°	-82.87°	-90°



13.21

Equation of motion: $\ddot{x} + 0.1 \dot{x} + x = 5$ (E₁)

Letting $\frac{dx}{dt} = y$, $\frac{dy}{dt} = 5 - x - 0.1 y$ (E₂)

we obtain

$$\frac{dy}{dx} = \frac{5 - x - 0.1 y}{y} \quad (E_3)$$

Instead of integrating (E₃) directly, we find the solution of (E₁) using the procedures of chapter 4.

Noting that $m=1$, $c=0.1$, $k=1$, $F_0=5$, we find

$$\omega_n = \sqrt{\frac{k}{m}} = 1 \text{ and } \zeta = \frac{c}{2\sqrt{km}} = \frac{0.1}{2(1)} = 0.05.$$

Hence the solution of (E₁), with the initial conditions $x(0) = \dot{x}(0) = 0$, is given by [see Example 4.3]:

$$x(t) = \frac{F_0}{k} \left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_d t - \phi) \right]$$

$$\text{where } \omega_d = \omega_n \sqrt{1-\zeta^2} = 1 \sqrt{1-0.05^2} = 0.9987$$

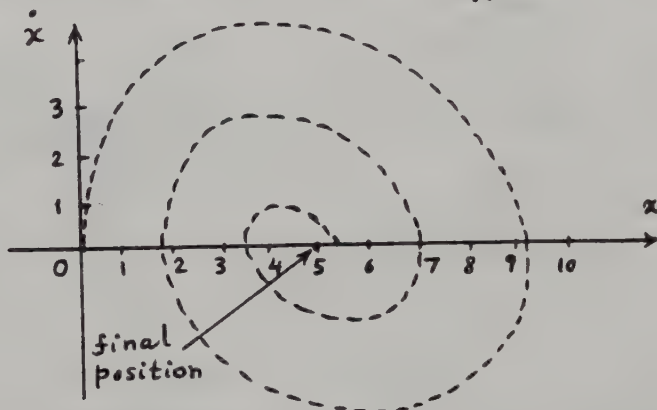
$$\text{and } \phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) = \tan^{-1}\left(\frac{0.05}{0.9987}\right) = 2.8681^\circ$$

$$\therefore x(t) = 5 \left[1 - 1.0013 e^{-0.05t} \cos(0.9987t - 2.8681^\circ) \right] \quad (E_4)$$

Velocity is given by (differentiating E₄·(E₄)):

$$\begin{aligned} \dot{x}(t) = & 0.2503 e^{-0.05t} \cos(0.9987t - 2.8681^\circ) \\ & + 5.0 e^{-0.05t} \sin(0.9987t - 2.8681^\circ) \end{aligned} \quad (E_5)$$

The trajectory in the phase-plane ($x-\dot{x}$ plane) appears as shown below.



13.22

$$\text{Equation of motion: } \ddot{x} + f \frac{\dot{x}}{|\dot{x}|} + \omega_n^2 x = 0 \quad (E_1)$$

$$\text{i.e. } \ddot{x} + \omega_n^2 (x+a) = 0 \quad \text{for } \dot{x} > 0 \quad (E_2)$$

$$\text{and } \ddot{x} + \omega_n^2 (x-a) = 0 \quad \text{for } \dot{x} < 0 \quad (E_3)$$

$$\text{where } a = f/\omega_n^2 \quad (E_4)$$

Multiplying by $2\dot{x}$ and integrating, (E₂) and (E₃) yield

$$\dot{x}^2 + \omega_n^2 (x+a)^2 = R_j^2 \quad \text{for } \dot{x} > 0 \quad (E_5)$$

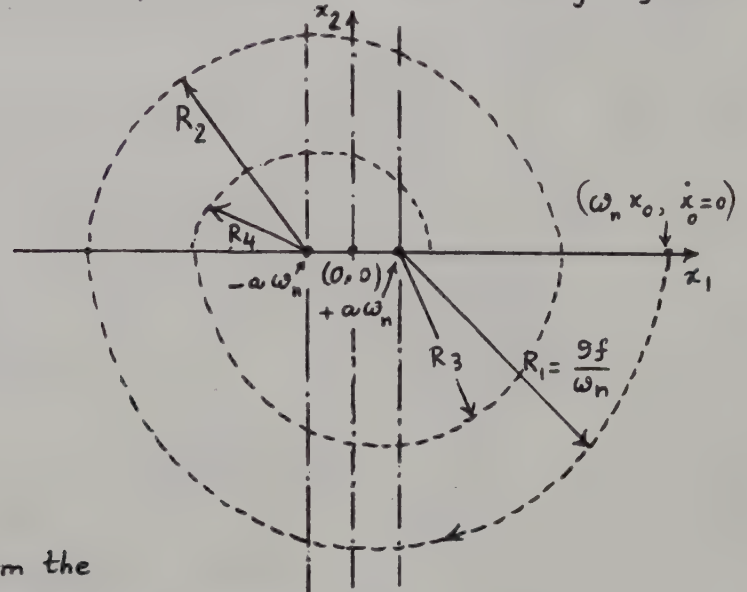
$$\dot{x}^2 + \omega_n^2 (x-a)^2 = R_{j+1}^2 \quad \text{for } \dot{x} < 0 \quad (E_6)$$

where R_j^2 and R_{j+1}^2 are integration constants which are to be computed at each switching of sign of $\dot{x}$.

We can plot the trajectories of a representative point whose coordinates are

$$x_1 = \omega_n x, \quad x_2 = \dot{x} \quad (E_7)$$

Egs. (E₅) and (E₆) show that the trajectory is made of semicircles whose centers are located at $x = -a$ (or $x_1 = -a\omega_n$) and $x = +a$ (or $x_1 = +a\omega_n$) as shown in the following figure.



R_1 can be obtained from the initial conditions using Eq. (E₆) as:

$$R_1^2 = 0^2 + \omega_n^2 \left(\frac{10f}{\omega_n^2} - \frac{f}{\omega_n^2} \right)^2 = \left(\frac{9f}{\omega_n} \right)^2 ; \quad R_1 = \frac{9f}{\omega_n} \quad (E_8)$$

Notice that the radii of the circles $R_1, R_2, \dots$ decrease according to the relation

$$R_j = R_{j-1} - 2a\omega_n ; \quad j = 1, 2, \dots$$

and the system will stop when

$$R_k \leq 2a\omega_n$$

$$\text{Here } R_1 = \frac{9f}{\omega_n}, \quad R_2 = R_1 - \frac{2f}{\omega_n} = \frac{7f}{\omega_n}, \quad R_3 = R_2 - \frac{2f}{\omega_n} = \frac{5f}{\omega_n},$$

$$R_4 = R_3 - \frac{2f}{\omega_n} = \frac{3f}{\omega_n}, \quad R_5 = R_4 - \frac{2f}{\omega_n} = \frac{f}{\omega_n},$$

and the motion stops at this point (after five half-cycles) since $R_6 < 2a\omega_n = \frac{2f}{\omega_n}$.

13.23

$$\ddot{\theta} + c\dot{\theta} + \sin \theta = 0 \quad \text{or} \quad \ddot{\theta} = -c\dot{\theta} - \sin \theta$$

$$\text{Let } x = \theta \text{ and } y = \frac{dx}{dt} = \dot{\theta}$$

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy - \sin x \quad (E_1)$$

Equilibrium or critical point (where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$) of this system is $(x=0, y=0)$. Linearization of Eqs. (E_1) about the equilibrium point (origin) leads to

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy - x$$

$$\text{or} \quad \begin{cases} dx/dt \\ dy/dt \end{cases} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \begin{cases} x \\ y \end{cases} \quad (E_2)$$

The eigenvalues of this system are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\text{i.e.} \quad \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda - c \end{vmatrix} = 0 \quad \text{i.e.,} \quad \lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$$

$$\text{i.e.,} \quad \lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - 1} \quad (E_3)$$

$$\text{If } c=0: \quad p=0; q=1; \lambda_{1,2} = \pm \sqrt{-1}$$

The origin will be a center.

$$\text{If } 0 < c < 2: \quad p > 0; q > 0; \lambda_{1,2} = \text{complex conjugates}$$

The origin will be a stable focal point (spiral point).

$$\text{If } c=2: \quad p > 0; q > 0; \lambda_{1,2} = \text{negative and equal.}$$

The origin will be a stable nodal point.

$$\text{If } c > 2: \quad p > 0; q > 0; \text{If } \lambda_{1,2} = \text{negative real, the origin will be a stable nodal point.}$$

$$\text{If } -2 < c < 0: \quad p < 0; q > 0; \lambda_{1,2} = \text{complex conjugates.}$$

The origin will be an unstable focal point (spiral point).

13.24

$$\text{Equation of motion:} \quad \ddot{\theta} + 0.5\dot{\theta} + \sin \theta = 0.8 \quad (E_1)$$

$$\text{Let } x = \theta \text{ and } y = dx/dt$$

$$\therefore \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x - 0.5y + 0.8 \quad (E_2)$$

$$\frac{dy}{dx} = \frac{-\sin x - 0.5y + 0.8}{y} \quad (E_3)$$

At $(x = \sin^{-1} 0.8, y = 0)$, $\frac{dy}{dx} = \frac{0}{0}$ and hence it will be an equilibrium point. To investigate the nature of singularity, we rewrite Eqs. (E₂) in linearized form as

$$\left. \begin{aligned} \frac{dx}{dt} &= (0)x + (1)y \\ \frac{dy}{dt} &= (0)x - 0.5y \end{aligned} \right\} \quad (E_4)$$

Thus the eigenvalues of the system are given by

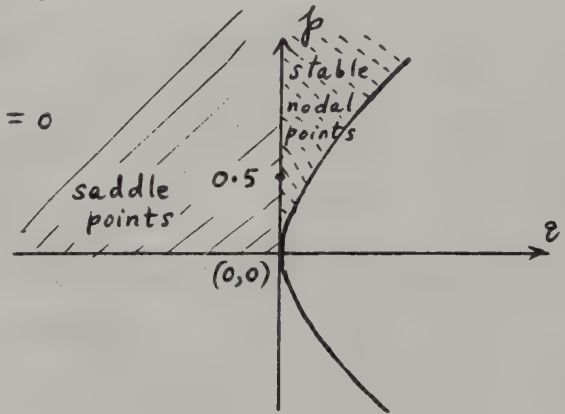
$$\left| \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \quad \text{or} \quad \begin{vmatrix} -\lambda & 1 \\ 0 & -0.5 - \lambda \end{vmatrix} = 0$$

$$\text{i.e., } \lambda^2 + 0.5\lambda \equiv \lambda^2 + p\lambda + q = 0$$

$$\therefore \lambda_1 = 0, \lambda_2 = \text{negative}$$

Here $p = \text{positive}$, $q = 0$, $\lambda_1 = 0$ and $\lambda_2 = \text{negative}$.

Thus the equilibrium point falls on the border of saddle points and stable nodal points as shown in the adjacent figure.



13.25

$$\frac{dx}{dt} = (0)x + (1)y \quad (E_1)$$

$$\frac{dy}{dt} = -1.x - c.y + (0.1)x^3 \quad (E_2)$$

Eqs. (E₁) and (E₂) are zero at $(x=0, y=0)$. Hence the origin $(0,0)$ will be equilibrium point (singularity). The eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -1 & -c-\lambda \end{vmatrix} = 0$$

$$\text{i.e., } \lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$$

$$\text{i.e., } \lambda_{1,2} = \left\{ \frac{-c \pm \sqrt{c^2 - 4}}{2} \right\}$$

For $c > 0$ and $c < 2$:

$p > 0$, $q > 0$ and $\lambda_{1,2} = \text{complex conjugates}$.

Hence the origin will be a stable focus (or spiral point).

For $c \geq 2$:

$p > 0$, $q > 0$; $\lambda_{1,2} = \text{negative real}$.

Hence the origin will be a stable nodal point.

13.26 $\ddot{x} - \alpha \dot{x} (1 - x^2) + x = 0$, $\alpha > 0$ (E₁)

This equation can be rewritten as

$$\left. \begin{aligned} dx/dt &= y \\ dy/dt &= -x + y \alpha (1 - x^2) \end{aligned} \right\} \quad (E_2)$$

$$\therefore \frac{dy}{dx} = \frac{-x + y \alpha (1 - x^2)}{y} \quad (E_3)$$

Thus the system has singularity at $(x=0, y=0)$. Near the origin, the nature of singularity can be investigated by considering the linearized equations:

$$\left. \begin{aligned} dx/dt &= (0)x + (1)y \\ dy/dt &= (-1)x + (\alpha)y \end{aligned} \right\} \quad (E_4)$$

The eigenvalues corresponding to Eqs. (E₄) are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -1 & \alpha - \lambda \end{vmatrix} = 0$$

$$\text{i.e., } \lambda^2 - \alpha \lambda + 1 \equiv \lambda^2 + p \lambda + q = 0$$

$$\text{i.e., } \lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$$

Here $p = -\alpha$, $q = 1$ and $\lambda_{1,2} = \text{complex conjugates}$ for $\alpha > 0$ and $\alpha = \text{small}$. Hence the origin will be an unstable focus (spiral point).

13.27 $\ddot{x} + \omega_n^2 x (1 + \kappa^2 x^2) = 0$ (E₁)

Let $dx/dt = y$, $dy/dt = -\omega_n^2 x - \omega_n^2 \kappa^2 x^3$ (E₂)

Singularity is at $(x=0, y=0)$. Eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -\omega_n^2 & -\lambda \end{vmatrix} = 0$$

$$\text{i.e., } \lambda^2 + \omega_n^2 \equiv \lambda^2 + p\lambda + q = 0$$

$$\text{i.e., } \lambda_{1,2} = \pm i\omega_n$$

Here $p=0$; $q = \omega_n^2 = \text{positive}$; $\lambda_{1,2} = \text{imaginary (conjugates)}$

$\therefore$ Equilibrium point is a center.

13.28

$$\ddot{x} + \omega_n^2 x (1 - k^2 x^2) = 0 \quad (E_1)$$

$$\text{with } dx/dt = y, \quad dy/dt = -\omega_n^2 x + \omega_n^2 k^2 x^3 \quad (E_2)$$

Singularity at $(x=0, y=0)$.

Linearized equations about origin are

$$dx/dt = y, \quad dy/dt = -\omega_n^2 x \quad (E_3)$$

Eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

i.e.,

$$\lambda^2 + \omega_n^2 \equiv \lambda^2 + p\lambda + q = 0$$

i.e.,

$$\lambda_{1,2} = \pm i\omega_n$$

$\therefore p=0$; $q = \text{positive} = \omega_n^2$; $\lambda_{1,2} = \text{conjugates (imaginary)}$

Hence equilibrium point is a center.

13.29

$$\ddot{\theta} + \omega_n^2 \sin \theta = 0 \quad (E_1)$$

Using $x = \theta$, E_1 can be expressed as

$$dx/dt = y, \quad dy/dt = -\omega_n^2 \sin x \quad (E_2)$$

Singularity is at $(x=0, y=0)$.

Linearized equations about origin yield

$$dx/dt = y, \quad dy/dt = -\omega_n^2 x$$

$$\text{Eigenvalues are given by } \left| \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\text{i.e., } \lambda^2 + \omega_n^2 \equiv \lambda^2 + p\lambda + q = 0; \quad \lambda_{1,2} = \pm i\omega_n$$

Here $p=0$; $q = \omega_n^2 = \text{positive}$; $\lambda_{1,2} = \text{imaginary (conjugates)}$

$\therefore$ Equilibrium point is a center.

13.30

$$(a) \quad \begin{aligned} \dot{x} &= x - y \\ \dot{y} &= x + 3y \end{aligned}$$

Assuming the solution as $\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} X \\ Y \end{Bmatrix} e^{\lambda t}$,

we get the following eigenvalue problem

$$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_1)$$

The eigenvalues are given by

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = 0$$

$$\text{i.e., } \lambda_{1,2} = \frac{4 \pm \sqrt{16-4}}{2} = 2, 2$$

Both eigenvalues are same.

Using $\lambda_1 = 2$ in the first equation of (E_1) , we get

$$Y = -X.$$

Letting $X = 1$ arbitrarily, we get $\begin{Bmatrix} X \\ Y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ for $\lambda_1 = 2$ (E₂)

Since the substitution of $\lambda_2 = 2$ gives the same eigenvector as in (E_2) , we seek a

second solution in the form $\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} X \\ Y \end{Bmatrix} t e^{2t}$ (E₃)

Substitution of (E_3) into the original equations gives

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} e^{2t} + 2 \begin{Bmatrix} X \\ Y \end{Bmatrix} t e^{2t} - \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} t e^{2t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_4)$$

It can be seen that the only way (E_4) can be satisfied for all values of t is to have $\begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$. This implies that solution of the form of E_3 cannot exist. Hence we seek a solution of the form

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} e^{2t} + \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} t e^{2t} \quad (E_5)$$

where $\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}$ and $\begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix}$ are to be determined. Substitution of

(E_5) into original equations gives

$$2 \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} t e^{2t} + \left(\begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} + 2 \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} \right) e^{2t} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \left(\begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} t e^{2t} + \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} e^{2t} \right) \quad (E_6)$$

By equating the coefficients of $t e^{2t}$ and e^{2t} on both sides of (E_6) , we get

$$2 \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} \quad (E_7)$$

$$\text{and } \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} + 2 \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} \quad (E_8)$$

It can be seen that (E_7) will be satisfied if

$$\begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \text{eigenvector corresponding to } \lambda_1 = 2.$$

Eg. (E_8) can be rewritten in scalar form as

$$\left. \begin{aligned} x_2 + 2x_1 &= x_1 - y_1 \\ y_2 + 2y_1 &= x_1 + 3y_1 \end{aligned} \right\} \text{ or, } \left. \begin{aligned} x_1 + y_1 &= -x_2 \\ -x_1 - y_1 &= -y_2 \end{aligned} \right\} \quad (E_9)$$

Thus for $x_2 = 1$ and $y_2 = -1$, (E_9) give $x_1 = c = \text{any constant}$ and $y_1 = -1 - c$.

$$\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} = \begin{Bmatrix} c \\ -1 - c \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} + c \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \quad (E_{10})$$

$\therefore$ The solution of (E_5) becomes

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{2t} + c \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{2t} + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} t e^{2t} \quad (E_{11})$$

We can ignore the middle term on the r.h.s. of (E_{11}) since it is same as $\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}^{(1)}$. Thus the second solution of original equations is

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{2t} + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} t e^{2t} \quad (E_{12})$$

$$(b) \quad \begin{cases} \dot{x} = x + y \\ \dot{y} = 4x + y \end{cases} \quad (E_1)$$

$$\text{Assuming the solution as } \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix} e^{\lambda t} \quad (E_2)$$

Eg. (E_1) give

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_3)$$

$$\text{Eigenvalues are given by } \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0 \quad (E_4)$$

$$\text{i.e., } \lambda_1 = -1, \quad \lambda_2 = 3$$

$$\text{Substitution of } \lambda_1 = -1 \text{ in } (E_3) \text{ gives } \left. \begin{aligned} 2x^{(1)} + y^{(1)} &= 0 \\ 4x^{(1)} + 2y^{(1)} &= 0 \end{aligned} \right\}$$

$$\text{or } y^{(1)} = -2x^{(1)}$$

choosing $x^{(1)} = 1$, arbitrarily, the first eigenvector becomes

$$\begin{Bmatrix} x \\ y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$

Next, by substituting $\lambda_2 = 3$ in (E_3) , we get $-2x^{(2)} + y^{(2)} = 0$

or $y^{(2)} = 2x^{(2)}$. By choosing $x^{(2)} = 1$ arbitrarily, the second eigenvector becomes $\begin{Bmatrix} x \\ y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$.

13.31

$$\begin{cases} \dot{x} = x - 2y \\ \dot{y} = 4x - 5y \end{cases}, \quad \frac{dy}{dx} = \frac{4x - 5y}{x - 2y} \quad (E_1)$$

Eigenvalues are defined by

$$\left[\begin{bmatrix} 1 & -2 \\ 4 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_2)$$

$$\text{This gives } \begin{vmatrix} 1-\lambda & -2 \\ 4 & -5-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0$$

$$\lambda_1 = -3, \quad \lambda_2 = -1.$$

Eigenvectors: For $\lambda_1 = -3$, $E_0(E_2)$ gives $4x = 2y$

For $\lambda_2 = -1$, $E_0(E_2)$ gives $2x = 2y$

$$\vec{x}^{(1)} = \begin{Bmatrix} x \\ y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \quad \text{and} \quad \vec{x}^{(2)} = \begin{Bmatrix} x \\ y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Thus the solutions are: $\vec{x}^{(1)} = \begin{Bmatrix} x \\ y \end{Bmatrix}^{(1)} e^{\lambda_1 t}$, $\vec{x}^{(2)} = \begin{Bmatrix} x \\ y \end{Bmatrix}^{(2)} e^{\lambda_2 t}$

General solution is

$$\vec{x} = \begin{Bmatrix} x(t) \\ y(t) \end{Bmatrix} = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) = c_1 \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} e^{-3t} + c_2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{-t} \quad (E_3)$$

where c_1 and c_2 are arbitrary constants.

The solution can be represented in the xy plane for various values of c_1 and c_2 .

First, we consider $c_2 = 0$ so that

$$x = c_1 e^{-3t}, \quad y = 2c_1 e^{-3t}$$

Eliminating t between these,

we find that the solution lies on the straight line $y = 2x$.

The solution $\vec{x}^{(1)}(t)$ approaches origin along the line $y = 2x$.

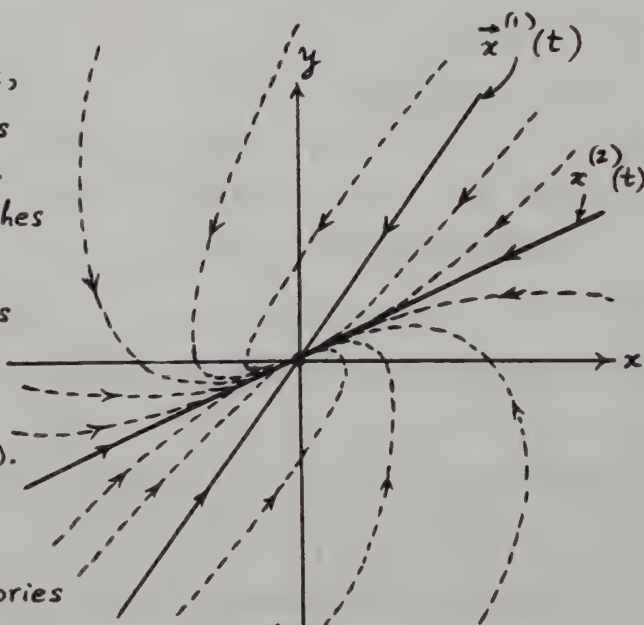
Similarly, $\vec{x}^{(2)}(t)$ approaches origin along the line $y = x$.

As $t \rightarrow \infty$, $\vec{x}^{(1)}(t)$ is negligible compared to $\vec{x}^{(2)}(t)$.

Hence $\vec{x}(t)$ of (E_3)

approaches origin tangential to line $y = x$. The trajectories

are shown in the figure. Here the origin is a node.



13.32

$$\dot{x} = x - y, \quad \dot{y} = x + 3y \quad (E_1)$$

$$\frac{dy}{dx} = \frac{x + 3y}{x - y} \quad (E_2)$$

The eigenvalues and eigenvectors of this system were found in Problem 13.30(a) as

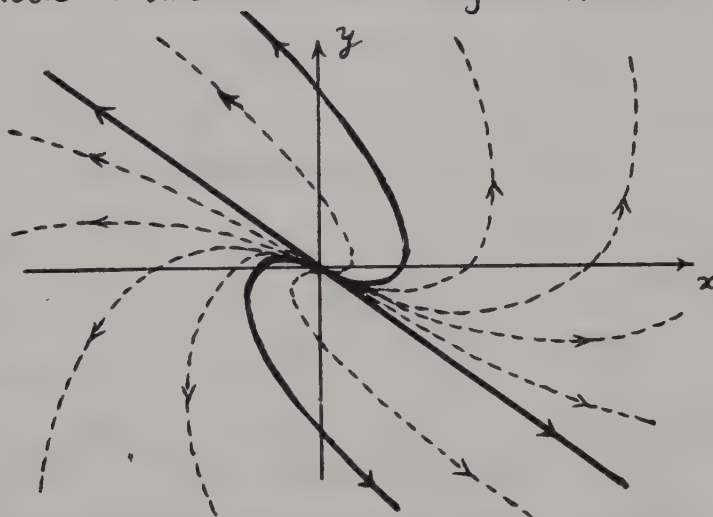
$$\lambda_1 = 2, \quad \lambda_2 = 2, \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \text{only one independent eigenvector}$$

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{2t}, \quad \vec{x}^{(2)}(t) = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} t e^{2t} + \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{2t} \quad (E_3)$$

The general solution of (E₁) is

$$\vec{x}(t) = c_1 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{2t} + c_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} t e^{2t} + c_2 \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{2t} \quad (E_4)$$

To draw the trajectories given by (E₄), we first observe the following behavior. As $t \rightarrow \infty$, $\vec{x}(t)$ becomes unbounded and as $t \rightarrow -\infty$, $\vec{x}(t) \rightarrow \vec{0}$. Also, as $t \rightarrow -\infty$, the solutions approach the origin $\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ tangentially to the line $y = -x$ (which denotes the eigenvector, $\vec{X}^{(1)}$). Further, as $t \rightarrow \infty$, the trajectories will lie asymptotic to the line $y = -x$. The origin will be a node in this case. The trajectories are shown below.



13.33

$$\dot{x} = 2x + y, \quad \dot{y} = -3x - 2y \quad (E_1)$$

$$\frac{dy}{dx} = \frac{-3x - 2y}{2x + y} \quad (E_2)$$

$$\text{Eigenvalue problem is: } \left[\begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_3)$$

$$\text{Eigenvalues are given by } \begin{vmatrix} 2-\lambda & 1 \\ -3 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = 1 \quad (E_4)$$

Eigenvectors: For $\lambda_1 = -1$; $(2 - \lambda_1)X + Y = 0 \Rightarrow Y = -3X$

For $\lambda_2 = 1$; $(2 - \lambda_2)X + Y = 0 \Rightarrow Y = -X$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ -3 \end{Bmatrix}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Hence the general solution is: $\vec{x}(t) = c_1 \begin{Bmatrix} 1 \\ -3 \end{Bmatrix} e^{-t} + c_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^t$ (E₅)
where c_1 and c_2 are arbitrary constants.

To plot the trajectories, first consider the case of $c_1 = 0$.

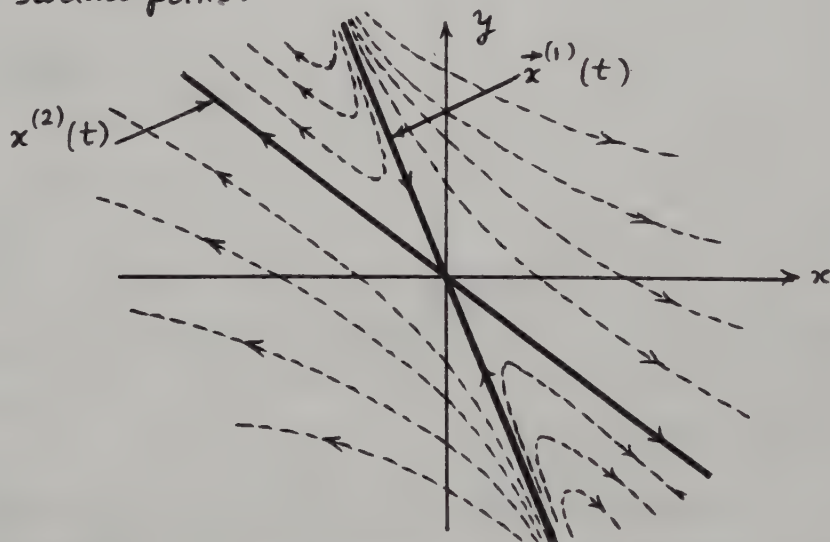
$$\vec{x}(t) = c_2 \vec{x}^{(2)}(t) \quad \text{i.e.,} \quad x(t) = c_2 e^t, \quad y(t) = -c_2 e^t \quad (E_6)$$

Eliminating t from (E₆), we get $y = -x$. This line denotes the eigenvector $\vec{X}^{(2)}$. If we consider the trajectory of a representative point, the particle moves away from origin as t increases. Next, we consider $c_2 = 0$. Eq. (E₅) gives

$$x(t) = c_1 e^{-t}, \quad y(t) = -3c_1 e^{-t} \quad (E_7)$$

This leads to the equation $y = -3x$, which represents the eigenvector, $\vec{X}^{(1)}$. A representative point in this case goes to origin as $t \rightarrow \infty$.

In the general solution of (E₅), the term $c_2 \vec{x}^{(2)}(t)$ dominates for large values of t . Thus the solutions with $c_2 \neq 0$ will be asymptotic to the line $y = -x$ as $t \rightarrow \infty$. Similarly, all solutions with $c_1 \neq 0$ will be asymptotic to the line $y = -3x$ as $t \rightarrow -\infty$. The trajectories are shown below, with origin denoting a saddle point.



13.34

Van der Pol's equation: $\ddot{x} - \alpha(1 - x^2)\dot{x} + x = 0, \quad \alpha > 0 \quad (E_1)$

Assume $x(t) = x_0(t) + \alpha x_1(t) + \alpha^2 x_2(t) \quad (E_2)$

$$\omega_0^2 = 1 = \omega^2 - \alpha \omega_1 - \alpha^2 \omega_2 \quad (E_3)$$

where $\omega_0^2 = 1 = \text{coefficient of } x \text{ in } E_0. (E_1).$

Substitution of (E_2) and (E_3) into (E_1) gives

$$\begin{aligned} \alpha^0 [\ddot{x}_0 + \omega^2 x_0] + \alpha^1 [\ddot{x}_1 - \dot{x}_0 + \dot{x}_0 x_0^2 - \omega_1 x_0 + \omega^2 x_1] \\ + \alpha^2 [\ddot{x}_2 - \dot{x}_1 + \dot{x}_1 x_0^2 + 2 x_0 \dot{x}_0 x_1 - \omega_2 x_0 - \omega_1 x_1 + \omega^2 x_2] \\ + \alpha^3 [\dots] + \dots = 0 \end{aligned} \quad (E_4)$$

Setting coefficient of α^0 in (E_4) to zero, we obtain

$$\ddot{x}_0 + \omega^2 x_0 = 0, \quad \text{i.e.,} \quad x_0(t) = A_1 \cos \omega t + A_2 \sin \omega t \quad (E_5)$$

Assuming the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$, we get $A_1 = A$ and $A_2 = 0$. Thus (E_5) reduces to

$$x_0(t) = A \cos \omega t \quad (E_6)$$

Setting coefficient of α^1 to zero, in E_4 ,

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 &= \dot{x}_0 - \dot{x}_0 x_0^2 + \omega_1 x_0 \\ &= -A\omega \sin \omega t + A^3 \omega \sin \omega t \cos^2 \omega t + \omega_1 A \cos \omega t \\ &= \left(-A\omega + \frac{1}{4} A^3 \omega\right) \sin \omega t + \omega_1 A \cos \omega t + \frac{A^3 \omega}{4} \sin 3\omega t \end{aligned} \quad (E_7)$$

The coefficients of $\sin \omega t$ and $\cos \omega t$ must be zero in E_7 to avoid secular terms. This gives

$$A = \pm 2, \quad \omega_1 = 0 \quad (E_8)$$

Thus the particular solution of (E_7) can be expressed as

$$x_1(t) = A_3 \sin 3\omega t + A_4 \cos 3\omega t \quad (E_9)$$

Substitution of (E_8) and (E_9) into E_7 gives

$$A_3 = \frac{1}{32} \frac{A^3}{\omega} \quad \text{and} \quad A_4 = 0 \quad (E_{10})$$

$$\text{Thus} \quad x_1(t) = \frac{1}{32} \frac{A^3}{\omega} \sin 3\omega t \quad (E_{11})$$

Finally, setting coefficient of α^2 in (E_4) to zero, we get

$$\ddot{x}_2 + \omega^2 x_2 = \dot{x}_1 - \dot{x}_1 x_0^2 - 2 x_0 \dot{x}_0 x_1 + \omega_2 x_0 + \omega_1 x_1 \quad (E_{12})$$

Substitution of (E_{11}) , (E_6) and (E_8) into (E_{12}) leads to

$$\begin{aligned} \ddot{x}_2 + \omega^2 x_2 &= \frac{3}{32} A^3 \cos 3\omega t - \left(\frac{3}{32} A^3 \cos 3\omega t\right) A^2 \cos^2 \omega t \\ &\quad - 2(A \cos \omega t)(-A\omega \sin \omega t) \left(\frac{A^3}{32\omega} \sin 3\omega t\right) + \omega_2 A \cos \omega t \\ &= \left(-\frac{3}{128} A^5 + \frac{1}{64} A^5 + A\omega_2\right) \cos \omega t + \left(\frac{3}{32} A^3 - \frac{3}{64} A^5\right) \cos 3\omega t \\ &\quad + \left(-\frac{3}{128} A^5 - \frac{1}{64} A^5\right) \cos 5\omega t \end{aligned} \quad (E_{13})$$

To avoid secular terms, the coefficient of $\cos \omega t$ in (E_{13}) must be zero. This gives $\omega_2 = \frac{1}{128} A^4$ (E₁₄)

With this, and using $A=2$, E_2 , (E_{13}) reduces to

$$\ddot{x}_2 + \omega^2 x_2 = -\frac{3}{4} \cos 3\omega t - \frac{5}{4} \cos 5\omega t \quad (E_{15})$$

$$\text{Assuming } x_2(t) = A_5 \cos 3\omega t + A_6 \cos 5\omega t \quad (E_{16})$$

We find, from E_2 , (E_{15}) ,

$$A_5 = \frac{3}{32} \cdot \frac{1}{\omega^2}, \quad A_6 = \frac{5}{96} \cdot \frac{1}{\omega^2} \quad (E_{17})$$

$$\therefore x_2(t) = \frac{3}{32\omega^2} \cos 3\omega t + \frac{5}{96\omega^2} \cos 5\omega t \quad (E_{18})$$

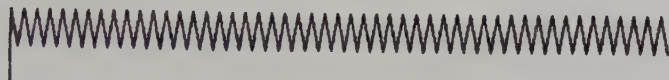
Thus the complete solution, E_2 , (E_2) and (E_3) , become

$$x(t) = 2 \cos \omega t + \frac{\alpha}{4\omega} \sin 3\omega t + \frac{3\alpha^2}{32\omega^2} \cos 3\omega t + \frac{5\alpha^2}{96\omega^2} \cos 5\omega t$$

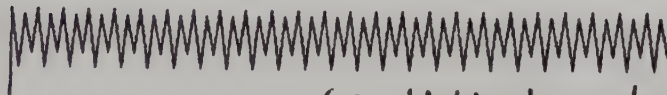
and $\omega^2 = 1 + \frac{\alpha^2}{8}.$

13.35

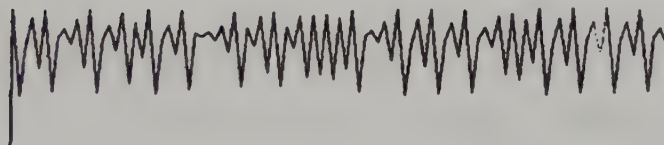
With $k = 3.25$, the sequence of values generated oscillate between two distinct values. With $k = 3.5$, the sequence of values generated oscillate between four distinct values. With $k = 3.75$, the sequence of values generated do not show any apparent periodicity (i.e., exhibit chaotic behavior). The resulting plots are shown below.



$k = 3.25$ (2 distinct values)



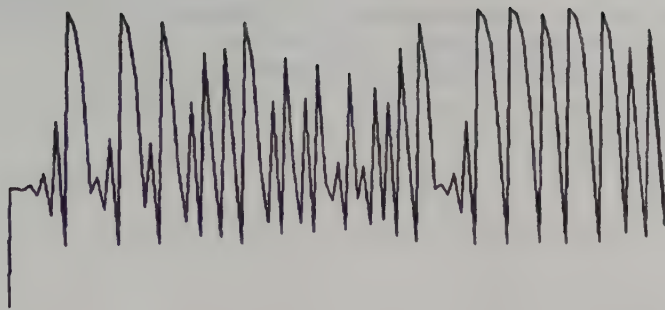
$k = 3.50$ (4 distinct values)



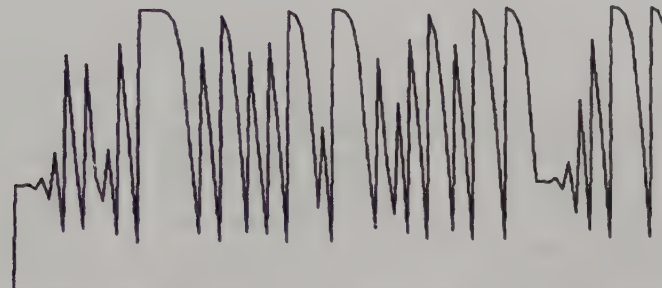
$k = 3.75$ (chaotic)

13.36

By taking two different initial values of x_1 which apparently do not differ much from one another, completely different sequences of values are generated. The sequences are shown plotted in the following figure.



$$x_1 = 1.002$$



$$x_1 = 1.003$$

13.37

Equations of motion:

$$\dot{\vec{x}}(t) = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} = \begin{Bmatrix} x_2(t) \\ [1.8 \cos 0.4t - 0.5 x_2 - x_1 - 1.2 x_1^3] \end{Bmatrix} = \vec{F}$$

The main program which calls RK4 and subroutine FUN along with the output are given below:

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C =====
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION TIME(100), X(100, 2), XX(2), F(2), YI(2), YJ(2), YK(2), YL(2),
2  UU(2)
  XX(1)=0.0
  XX(2)=0.0
  NEQ=2
  NSTEP=100
  DT=0.05
  T=0.0
  PRINT 10
10  FORMAT (//, 3X, 5H I , 10H , TIME(I), 7X, 5H X(1), 12X, 5H X(2), /)
  DO 40 I=1, NSTEP
    CALL RK4 (T, DT, NEQ, XX, F, YI, YJ, YK, YL, UU)
    TIME(I)=T
    DO 20 J=1, NEQ
20  X(I, J)=XX(J)
  PRINT 30, I, TIME(I), (X(I, J), J=1, NEQ)
30  FORMAT (2X, I5, F10.4, 2X, E15.8, 2X, E15.8)
40  CONTINUE
  STOP
  END

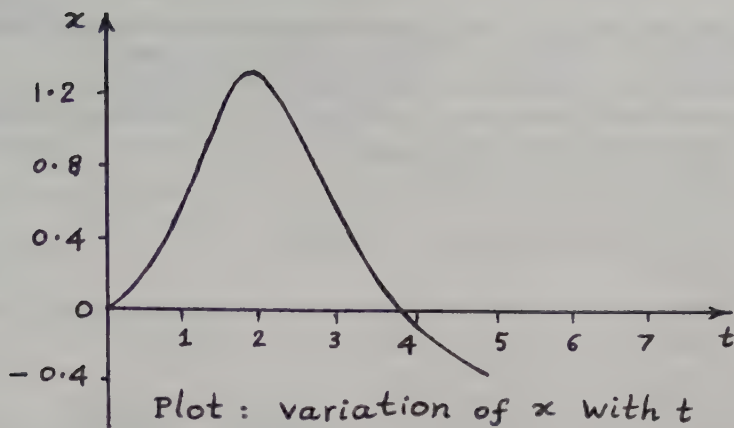
```

```

C =====
C SUBROUTINE FUN FOR USE IN THE SUBROUTINE RK4
C THIS SUBROUTINE CHANGES FROM PROBLEM TO PROBLEM
C =====
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F(2)=1.8*COS(0.4*T)-0.5*X(2)-X(1)-1.2*(X(1)**3)
      RETURN
      END

```

I	TIME(I)	X(1)	X(2)
1	0.0500	0.22308242E-02	0.88841319E-01
2	0.1000	0.88433102E-02	0.17523426E+00
3	0.1500	0.19710280E-01	0.25899023E+00
4	0.2000	0.34695394E-01	0.33993125E+00
5	0.2500	0.53653665E-01	0.41788858E+00
6	0.3000	0.76431811E-01	0.49270076E+00
7	0.3500	0.10286868E+00	0.56421006E+00
8	0.4000	0.13279516E+00	0.63225830E+00
9	0.4500	0.16603410E+00	0.69668114E+00
10	0.5000	0.20239991E+00	0.75730175E+00
11	0.5500	0.24169767E+00	0.81392360E+00
12	0.6000	0.28372198E+00	0.86632329E+00
13	0.6500	0.32825541E+00	0.91424310E+00
14	0.7000	0.37506670E+00	0.95738453E+00
15	0.7500	0.42390853E+00	0.99540269E+00
16	0.8000	0.47451508E+00	0.10279036E+01
17	0.8500	0.52659953E+00	0.10544405E+01
18	0.9000	0.57985157E+00	0.10745211E+01
19	0.9500	0.63393515E+00	0.10876102E+01
20	1.0000	0.68848670E+00	0.10931435E+01
.			
.			
.			
.			
86	4.2999	-0.19073796E+00	-0.27927017E+00
87	4.3499	-0.20462143E+00	-0.27624547E+00
88	4.3999	-0.21837986E+00	-0.27426249E+00
89	4.4499	-0.23206437E+00	-0.27328062E+00
90	4.4999	-0.24572390E+00	-0.27325594E+00
91	4.5499	-0.25940514E+00	-0.27414125E+00
92	4.5999	-0.27315235E+00	-0.27588624E+00
93	4.6499	-0.28700721E+00	-0.27843720E+00
94	4.6999	-0.30100858E+00	-0.28173721E+00
95	4.7499	-0.31519240E+00	-0.28572613E+00
96	4.7999	-0.32959157E+00	-0.29034048E+00
97	4.8499	-0.34423572E+00	-0.29551363E+00
98	4.8999	-0.35915107E+00	-0.30117553E+00
99	4.9499	-0.37436020E+00	-0.30725288E+00
100	4.9999	-0.38988203E+00	-0.31366915E+00



13.38

$$m l^2 \ddot{\theta} + m g l \left(\theta - \frac{1}{6} \theta^3 \right) = 0 \quad \text{i.e.} \quad \ddot{\theta} + \frac{g}{2} - \frac{1}{12} \theta^3 = 0 \quad \text{for} \quad \frac{g}{l} = 0.5$$

$$\vec{F} = \begin{Bmatrix} x_2 \\ -\frac{x_1}{2} + \frac{1}{12} x_1^3 \end{Bmatrix} \quad \text{where} \quad x_1 = \theta \quad \text{and} \quad x_2 = \dot{\theta}$$

Since the natural time period of the linear system is $\frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{0.5}} = 8.8858$, Δt is taken as 0.25 in the present case. Using

$\theta(t=0) = 0.7854$ rad and $\dot{\theta}(t=0) = 0$, the Runge-Kutta method is used to find the solution. The main program which calls RK4 and the subroutine FUN alongwith the output are given below.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C =====
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
  DIMENSION TIME(100),X(100,2),XX(2),F(2),YI(2),YJ(2),YK(2),YL(2),
2  UU(2)
  XX(1)=0.7854
  XX(2)=0.0
  NEQ=2
  NSTEP=100
  DT=0.25
  T=0.0
  WRITE (15,10)
10  FORMAT (//,3X,5H I ,10H TIME(I),7X,5H X(1),12X,5H X(2),/)
  DO 40 I=1,NSTEP
    CALL RK4 (T,DT,NEQ,XX,F,YI,YJ,YK,YL,UU)
    TIME(I)=T
    DO 20 J=1,NEQ
20  X(I,J)=XX(J)
  WRITE (15,30) I,TIME(I),(X(I,J),J=1,NEQ)
30  FORMAT (2X,15,F10.4,2X,E15.8,2X,E15.8)
40  CONTINUE
  STOP
  END

```

```

C =====
C SUBROUTINE FUN FOR USE IN THE SUBROUTINE RK4
C THIS SUBROUTINE CHANGES FROM PROBLEM TO PROBLEM
C =====
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F(2)=-X(1)/2.0+(X(1)**3)/12.0
      RETURN
      END

```

I	TIME(I)	X(1)	X(2)
1	0.2500	0.77440965E+00	-0.87762989E-01
2	0.5000	0.74167883E+00	-0.17359795E+00
3	0.7500	0.68793911E+00	-0.25548738E+00
4	1.0000	0.61444390E+00	-0.33127007E+00
5	1.2500	0.52300471E+00	-0.39865014E+00
6	1.5000	0.41601548E+00	-0.45528558E+00
7	1.7500	0.29644495E+00	-0.49895233E+00
8	2.0000	0.16778104E+00	-0.52775872E+00
9	2.2500	0.33919662E-01	-0.54036677E+00
10	2.5000	-0.10099708E+00	-0.53616732E+00
11	2.7500	-0.23277587E+00	-0.51536494E+00
12	3.0000	-0.35737443E+00	-0.47894892E+00
13	3.2500	-0.47108278E+00	-0.42855906E+00
14	3.5000	-0.57066125E+00	-0.36628109E+00
15	3.7500	-0.65342385E+00	-0.29442233E+00
16	4.0000	-0.71726823E+00	-0.21531790E+00
17	4.2500	-0.76066524E+00	-0.13120258E+00
18	4.5000	-0.78262591E+00	-0.44162579E-01
19	4.7500	-0.78266639E+00	0.43838933E-01
20	5.0000	-0.76078576E+00	0.13088597E+00
⋮			
94	23.5000	-0.76101488E+00	0.13025595E+00
95	23.7500	-0.71784961E+00	0.21441360E+00
96	24.0000	-0.65422338E+00	0.29358405E+00
97	24.2500	-0.57165933E+00	0.36553413E+00
98	24.5000	-0.47225332E+00	0.42792925E+00
99	24.7500	-0.35868505E+00	0.47846088E+00
100	25.0000	-0.23418844E+00	0.51503974E+00

13.39 For $\ddot{x} + a^2 F(x) = 0$,
$$\tau = \frac{2\sqrt{2}}{a} \int_0^{x_0} \frac{d\xi}{\left\{ \int_{\xi}^{x_0} F(\eta) \cdot d\eta \right\}^{1/2}} \quad (E.1)$$

Here $\ddot{\theta} + \frac{g}{l} \left(\theta - \frac{\theta^3}{6} \right) = 0$; $\frac{g}{l} = 0.5 \Rightarrow a = \sqrt{0.5}$; $F(\theta) = \theta - \frac{\theta^3}{6}$;
 $\theta_0 = 0.7854$ rad.

Simpson's rule is used for integration; note that one integral is embedded inside a second integral in (E.1). According to Simpson's rule,

$$\int_a^b f(x) \cdot dx = \frac{h}{3} \left\{ f(a) + 2 \sum_{i=1}^{\left(\frac{N}{2}-1\right)} f_{2i+1} + 4 \sum_{i=1}^{\left(\frac{N}{2}\right)} f_{2i} + f(b) \right\}$$

where $h = \frac{b-a}{N}$, $N = \text{even number of equal intervals}$ and
 $f_i = f(x = x_i = a + \{i-1\} \{ \frac{b-a}{N} \})$.

The program listing and the result are given below.

```

C =====
C
C PROBLEM 13.39
C
C =====
      AA=0.0
      BB=0.7854
      NN=40
      HH=(BB-AA)/FLOAT(NN)
      CALL FFINT (AA,FA)
      BB1=BB-(BB-AA)/100.0
      CALL FFINT (BB1,FB)
      CC=BB-HH
      CALL FFINT (CC,FC)
      TAU=HH*(FA+FB+4.0*FC)/3.0
      KK=NN/2
      KKM1=KK-1
      DO 10 JJ=1,KKM1
      XX=AA+HH*FLOAT(2*JJ-1)
      CALL FFINT (XX,FX)
      XXHH=XX+HH
      CALL FFINT (XXHH,FXH)
10    TAU=TAU+HH*(4.0*FX+2.0*FXH)/3.0
      CONTINUE
      TAU=4.0*TAU
      PRINT 20, TAU
20    FORMAT (2X,E15.8)
      STOP
      END
C =====
C SUBROUTINE FFINT
C =====
      SUBROUTINE FFINT (X,Y)
      A=X
      B=0.7854
      N=40
      CALL SIM (A,B,N,FINT)
      Y=1.0/SQRT(FINT)
      RETURN
      END
C =====
C SUBROUTINE SIM
C =====
      SUBROUTINE SIM (A,B,N,FINT)
      H=(B-A)/FLOAT(N)
      FINT=H*(F(A)+F(B)+4.0*F(B-H))/3.0
      K=N/2
      KM1=K-1
      DO 10 J=1,KM1
      X=A+H*FLOAT(2*J-1)

```

```

      FINT=FINT+H*(4.0*F(X)+2.0*F(X+H))/3.0
10    CONTINUE
      RETURN
      END

```

```

C =====
C FUNCTION F
C =====
      FUNCTION F(X)
      F=X-(X**3)/6.0
      RETURN
      END

```

RESULT: $\tau = 8.7627392$

This can be compared with the value of τ of a linear pendulum which is equal to $\frac{2\pi}{\omega} = 2\pi/\sqrt{\frac{g}{l}} = 2\pi/\sqrt{0.5} = 8.857867$.

13.40 (a) Equation of motion

$$(M - m_0 t) \ddot{x} + 10 \dot{x} + 20 x^2 + 8 \times 10^5 x + 6000 x^3 = m_0 v$$

$$\text{i.e., } (2000 - 10t) \ddot{x} + 10 \dot{x} + 20 x^2 + 8 \times 10^5 x + 6000 x^3 = (10)(2000) = 2 \times 10^4 \quad (E_1)$$

(b) let $x_1 = x$, $x_2 = \frac{dx_1}{dt} = \frac{dx}{dt}$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \vec{F} = \begin{Bmatrix} x_2 \\ (2 \times 10^4 - 10x_2 - 20x_2^2 - 8 \times 10^5 x_1 - 6000 x_2^3)/(2000 - 10t) \end{Bmatrix}$$

Initial conditions: assume $x_1(0) = x_2(0) = 0$.

The main program, which calls RK4 and subroutine FUN, along with the output are given below.

```

C =====
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C =====
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(200), X(200, 2), XX(2), F(2), YI(2), YJ(2), YK(2), YL(2), YI(2),
2    UU(2)
      XX(1)=0.0
      XX(2)=0.0
      NEQ=2
      NSTEP=200
      DT=0.1
      T=0.0
      PRINT 10
10    FORMAT (//, 3X, 5H I , 10H TIME(I), 7X, 5H X(1), 12X, 5H X(2), /)
      DO 40 I=1, NSTEP
      CALL RK4 (T, DT, NEQ, XX, F, YI, YJ, YK, YL, UU)
      TIME(I)=T
      DO 20 J=1, NEQ
20    X(I, J)=XX(J)

```



```

30 PRINT 30, I, TIME(I), (X(I,J), J=1, NEQ)
40 FORMAT (2X, I5, F10. 4, 2X, E15. 8, 2X, E15. 8)
CONTINUE
STOP
END

```

```

C =====
C SUBROUTINE FUN FOR USE IN THE SUBROUTINE RK4
C THIS SUBROUTINE CHANGES FROM PROBLEM TO PROBLEM
C =====

```

```

SUBROUTINE FUN (X, F, N, T)
  DIMENSION X(N), F(N)
  F(1)=X(2)
  F(2)=(20000. 0-10. 0*X(2)-20. 0*(X(2)**2)
2 -800000. 0*X(1)-6000. 0*(X(1)**3))/(2000. 0-10. 0*T)
  RETURN
END

```

I	TIME(I)	X(1)	X(2)
1	0. 1000	0. 33320766E-01	0. 33307534E+00
2	0. 2000	0. 33306178E-01	-0. 22198296E+00
3	0. 3000	0. 14845021E-01	-0. 36448002E-01
4	0. 4000	0. 27181853E-01	0. 14729625E+00
5	0. 5000	0. 29156089E-01	-0. 78316689E-01
6	0. 6000	0. 21012224E-01	-0. 29013485E-01
7	0. 7000	0. 25375746E-01	0. 62707484E-01
8	0. 8000	0. 26947644E-01	-0. 26040435E-01
9	0. 9000	0. 23485284E-01	-0. 17102312E-01
10	1. 0000	0. 24944551E-01	0. 25843836E-01
.			
.			
.			
195	19. 5003	0. 24999883E-01	0. 47695581E-09
196	19. 6003	0. 24999883E-01	0. 54962435E-09
197	19. 7003	0. 24999883E-01	0. 62233330E-09
198	19. 8003	0. 24999883E-01	0. 69508244E-09
199	19. 9003	0. 24999883E-01	0. 76787199E-09
200	20. 0003	0. 24999883E-01	0. 84070195E-09

13.41

van der Pol's equation: $\ddot{x} - \alpha(1-x^2)\dot{x} + x = 0$ (E₁)

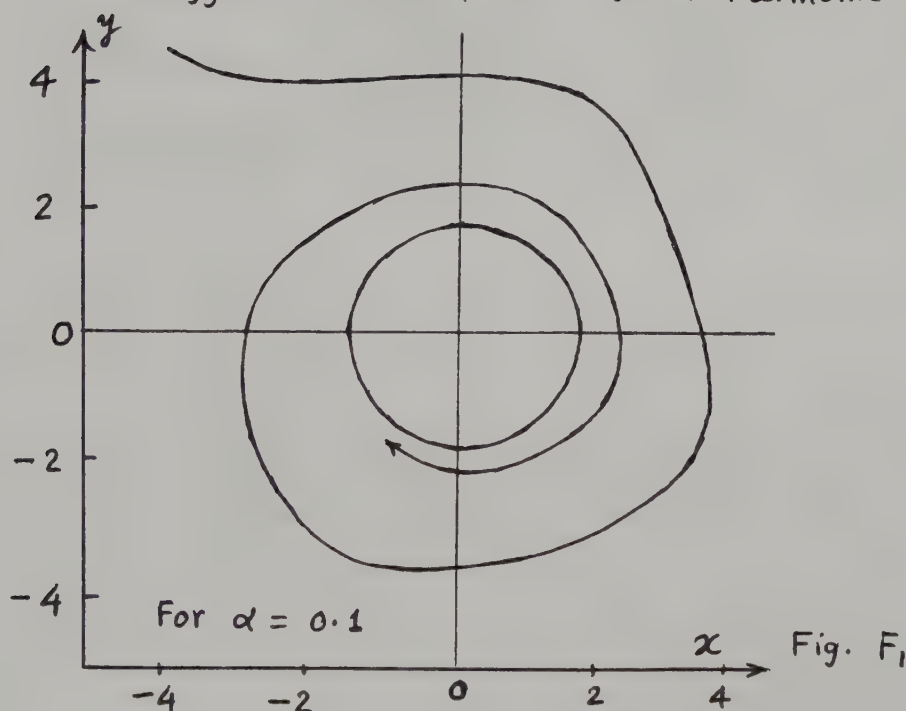
i.e.,
$$\left\{ \begin{array}{l} \frac{dx}{dt} \\ \frac{dy}{dt} \end{array} \right\} = \left\{ \begin{array}{l} y \\ \alpha(1-x^2)y - x \end{array} \right\}$$
 (E₂)

(a) Equation for the phase plane trajectory is

$$\frac{dy}{dx} = \frac{\alpha(1-x^2)y - x}{y} \quad (E_3)$$

The trajectories corresponding to $\alpha = 0.1$, $\alpha = 1.0$ and $\alpha = 10$ are shown in Figs. F₁, F₂ and F₃, respectively. It can be seen, from these figures, that irrespective of the initial point, the trajectories converge to closed curves. These closed curves denote periodic motions with constant amplitude. In addition, as the value of α increases, the closed curve deviates more and more from a circle.

(b) If (E₁) is solved numerically, the variation of the solution $x(t)$ can be plotted against the t -axis. The result will be as shown in Figs. F₄, F₅ and F₆ for $\alpha = 0.1$, 1.0 and 10 , respectively. It can be seen from these figures, that as the value of α is increased, the variation of x with respect to t differs more and more from harmonic motion.



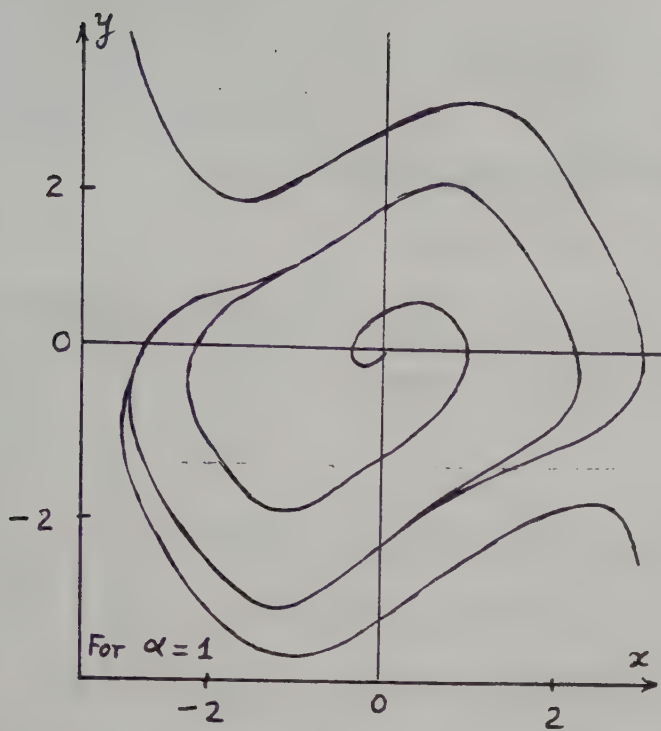


Fig. F₂

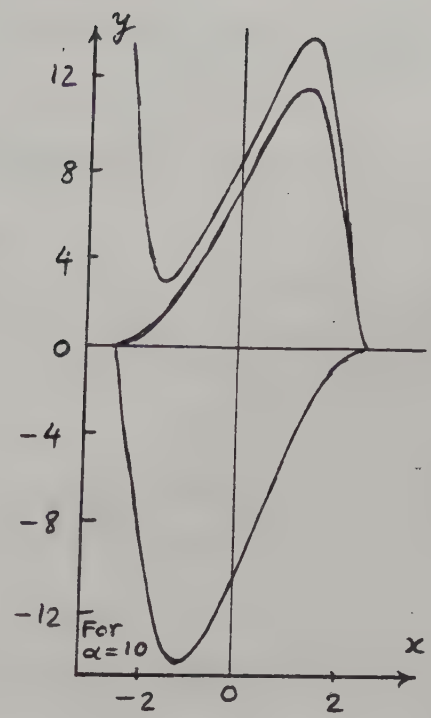


Fig. F₃

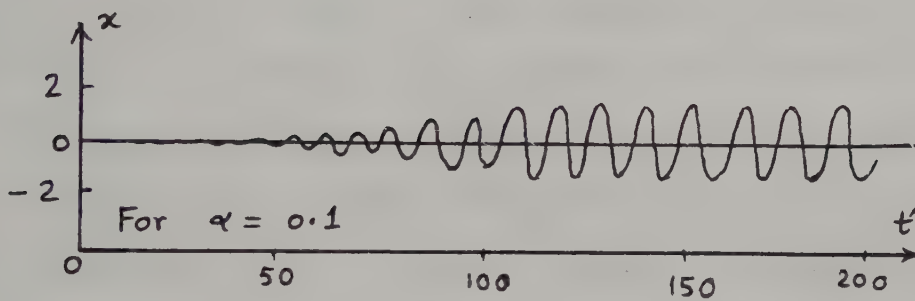


Fig. F₄

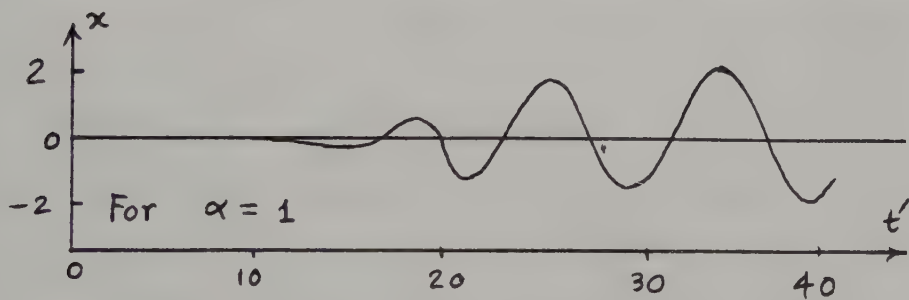


Fig. F₅

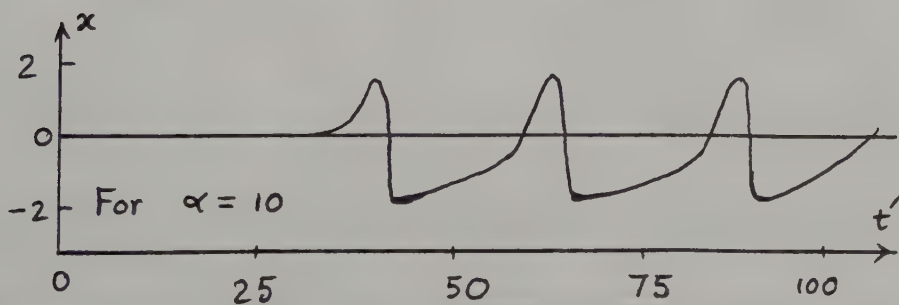


Fig. F₆

13.42 Equations of motion:

$$m \ddot{x} + k_{11}(x - l_1 \theta) + k_{12}(x - l_1 \theta)^3 + k_{21}(x + l_2 \theta) + k_{22}(x + l_2 \theta)^3 = 0$$

$$J_0 \ddot{\theta} - k_{11}(x - l_1 \theta)l_1 - k_{12}(x - l_1 \theta)^3 l_1 + k_{21}(x + l_2 \theta)l_2 + k_{22}(x + l_2 \theta)^3 l_2 = 0$$

Using $x_1 = x$, $x_2 = \theta$, $x_3 = \frac{dx}{dt} = \dot{x}$, and $x_4 = \frac{d\theta}{dt} = \dot{\theta}$, the equations of motion can be expressed as

$$\dot{\vec{X}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \vec{F} \equiv \begin{Bmatrix} x_3 \\ x_4 \\ -\frac{1}{m} [k_{11}(x_1 - l_1 x_2) + k_{12}(x_1 - l_1 x_2)^3 + k_{21}(x_1 + l_2 x_2) + k_{22}(x_1 + l_2 x_2)^3] \\ \frac{1}{J_0} [k_{11}(x_1 - l_1 x_2)l_1 + k_{12}(x_1 - l_1 x_2)^3 l_1 - k_{21}(x_1 + l_2 x_2)l_2 - k_{22}(x_1 + l_2 x_2)^3 l_2] \end{Bmatrix}$$

Data:

$$m = 1000 \text{ kg}, \quad J_0 = 2500 \text{ kg-m}^2, \quad l_1 = 1 \text{ m}, \quad l_2 = 1.5 \text{ m},$$

$$k_{11} = 40000, \quad k_{12} = 10000, \quad k_{21} = 50000, \quad k_{22} = 5000.$$

Runge-Kutta method can be used to solve the equations of motion using the initial conditions $x(0) = \dot{x}(0) = \theta(0) = \dot{\theta}(0) = 0$.

Chapter 14

Random Vibration

14.1

$$p(x) = \begin{cases} k \left(1 - \frac{x}{30}\right) & ; \quad 20 \leq x \leq 30 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Normalization:

$$\int_{-\infty}^{\infty} p(x) dx = k \int_{20}^{30} \left(1 - \frac{x}{30}\right) dx = k \left(x - \frac{x^2}{60}\right)_{20}^{30} = 1$$

$$\Rightarrow k = 0.6$$

$$P(x \geq 28) = \int_{28}^{\infty} p(x) dx = k \int_{28}^{30} \left(1 - \frac{x}{30}\right) dx = k \left(x - \frac{x^2}{60}\right)_{28}^{30}$$

$$= k \left(\frac{4}{60}\right) = 0.6 \left(\frac{4}{60}\right) = 0.04$$

14.2

$$p(x, y) = \begin{cases} \frac{xy}{9} & ; \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 3 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Normalization:

$$\int_{x=0}^2 \int_{y=0}^3 p(x, y) dx dy = \int_{x=0}^2 \frac{x}{9} dx \int_{y=0}^3 y dy = \int_{x=0}^2 \frac{x}{9} dx \left(\frac{y^2}{2}\right)_0^3$$

$$= \frac{1}{2} \int_{x=0}^2 x dx = \frac{1}{2} \left(\frac{x^2}{2}\right)_0^2 = 1 \quad (\text{satisfied})$$

(a) Marginal density functions

$$p_x(x) = \int_{y=0}^3 p(x, y) dy = \frac{x}{9} \int_{y=0}^3 y dy = \frac{x}{2}$$

$$p_y(y) = \int_{x=0}^2 p(x, y) dx = \frac{y}{9} \int_{x=0}^2 x dx = \frac{2}{9} y$$

$$(b) \mu_x = \bar{x} = \int_0^2 p(x) \cdot x dx = \frac{1}{2} \int_0^2 x^2 dx = \frac{1}{2} \left(\frac{x^3}{3}\right)_0^2 = \frac{4}{3}$$

$$\mu_y = \bar{y} = \int_0^3 p(y) \cdot y dy = \frac{2}{9} \int_0^3 y^2 dy = \frac{2}{9} \left(\frac{y^3}{3}\right)_0^3 = 2$$

$$\begin{aligned}\sigma_x^2 &= E[(x - \mu_x)^2] = \int_0^2 (x - \mu_x)^2 p(x) dx = \int_0^2 (x - \frac{4}{3})^2 \frac{x}{2} dx \\ &= \frac{1}{2} \left(\frac{x^4}{4} + \frac{16}{9} \frac{x^2}{2} - \frac{8}{3} \frac{x^3}{3} \right)_0^2 = \frac{2}{9}\end{aligned}$$

$$\sigma_x = 0.4714$$

$$\begin{aligned}\sigma_y^2 &= E[(y - \mu_y)^2] = \int_0^3 (y - \mu_y)^2 p(y) dy = \int_0^3 (y - 2)^2 \frac{2y}{9} dy \\ &= \frac{2}{9} \left(\frac{y^4}{4} + 2y^2 - \frac{4}{3} y^3 \right)_0^3 = \frac{1}{2}\end{aligned}$$

$$\sigma_y = 0.7071$$

$$\begin{aligned}(c) \quad \sigma_{x,y} &= E[(x - \mu_x)(y - \mu_y)] = \int_{x=0}^2 \int_{y=0}^3 (x - \frac{4}{3})(y - 2) \frac{xy}{9} dx dy \\ &= \int_{x=0}^2 (x - \frac{4}{3}) \frac{x}{9} dx \left(\frac{y^3}{3} - 2 \cdot \frac{y^2}{2} \right)_0^3 = 0\end{aligned}$$

$$\rho_{x,y} = 0$$

$$14.3 \quad E[x] = \bar{x} = \int_{-\infty}^{\infty} x \cdot p(x) dx = \int_0^2 0.5x dx = 1.0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p(x) dx = \int_0^2 0.5x^2 dx = 1.3333$$

$$\begin{aligned}\sigma_x^2 &= \int_{-\infty}^{\infty} (x - \bar{x})^2 p(x) dx = \int_0^2 (x - 1)^2 (0.5) dx = 0.5 \int_0^2 (x^2 - 2x + 1) dx \\ &= 0.3333\end{aligned}$$

$$\therefore \sigma_x = 0.5773$$

$$14.4 \quad z = x + y, \quad E[z^2] = E[(x+y)^2] = E[x^2 + y^2 + 2xy] \\ = E[x^2] + E[y^2] + 2E[xy]$$

Since x and y are independent, $E[xy] = E[x] \cdot E[y]$

$$E[z^2] = E[x^2] + E[y^2] + 2E[x] \cdot E[y]$$

$$14.5 \quad R_x(\tau) = 20 + \frac{5}{1 + 3\tau^2}$$

$$E[x^2] = R(0) = 20 + 5 = 25$$

$$14.6 \quad R_x(\tau) = A \cos \omega \tau$$

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

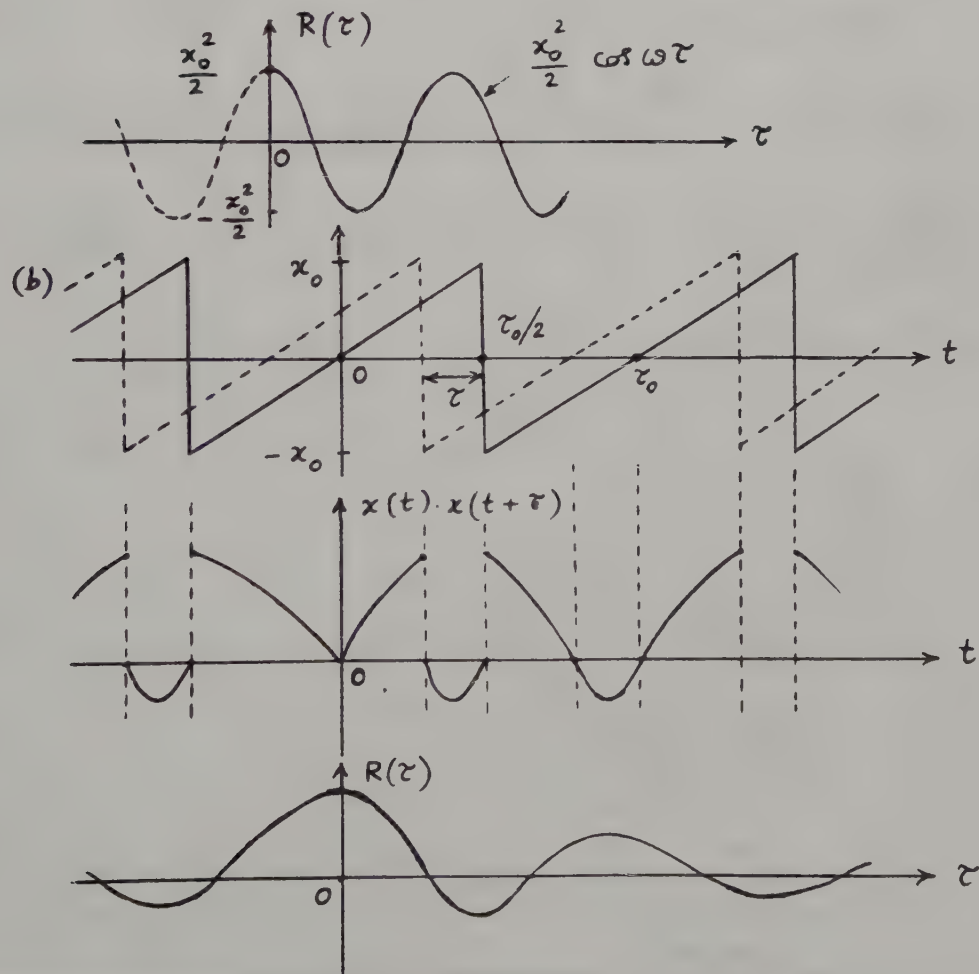
$$\begin{aligned}
&= \frac{A}{2\pi} \int_{-\pi/2\omega}^{\pi/2\omega} \cos \omega \tau (\cos \omega \tau - i \sin \omega \tau) d\tau \\
&= \frac{A}{2\pi} \int_{-\pi/2\omega}^{\pi/2\omega} (\cos^2 \omega \tau - i \frac{1}{2} \sin 2\omega \tau) d\tau \\
&= \frac{A}{2\pi} \left\{ \left[\frac{\tau}{2} + \frac{1}{4\omega} \sin 2\omega \tau \right]_{-\pi/2\omega}^{\pi/2\omega} - \frac{i}{2} \left(-\frac{\cos 2\omega \tau}{2\omega} \right)_{-\pi/2\omega}^{\pi/2\omega} \right\} \\
&= \frac{A}{4\omega}
\end{aligned}$$

14.7

(a) $x(t) = x_0 \sin \omega t$, $x(t+\tau) = x_0 \sin \omega(t+\tau)$

$$x(t) x(t+\tau) = x_0^2 (\sin^2 \omega t \cos \omega \tau + \sin \omega t \cos \omega t \sin \omega \tau)$$

$$\begin{aligned}
R(\tau) &= \lim_{T \rightarrow \infty} \frac{x_0^2}{T} \int_{-T/2}^{T/2} \left[\left(\frac{1 - \cos 2\omega t}{2} \right) \cos \omega \tau + \frac{\sin 2\omega t}{2} \sin \omega \tau \right] dt \\
&= \lim_{T \rightarrow \infty} \frac{x_0^2}{T} \left[\frac{T}{2} \cos \omega \tau + (0) \sin \omega \tau \right] = \frac{x_0^2}{2} \cos \omega \tau
\end{aligned}$$



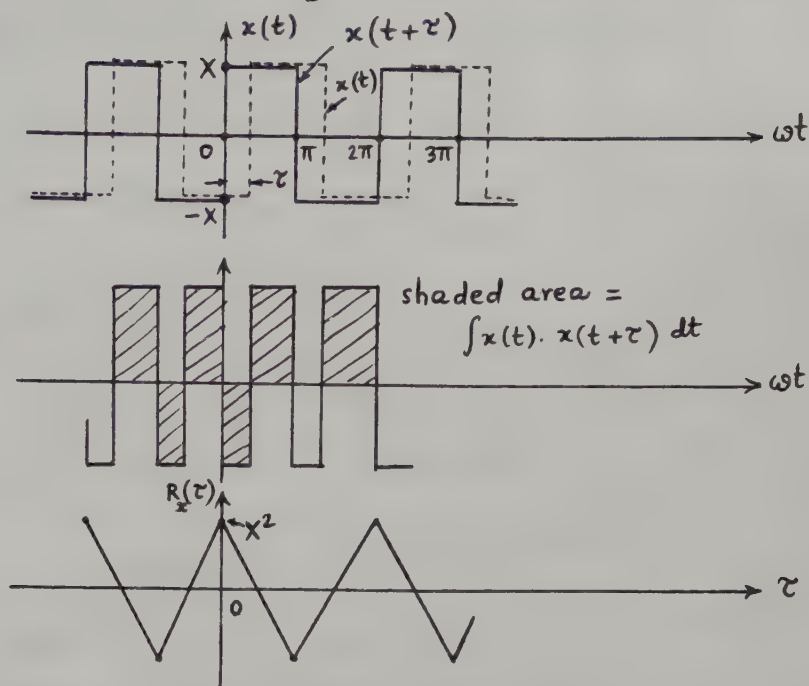
$$14.8 \quad x(t) = \begin{cases} (2x_0 t/\tau) & ; \quad 0 \leq t \leq \tau/2 \\ (2x_0 t/\tau) - 2x_0 & ; \quad \tau/2 \leq t \leq \tau \end{cases}$$

$$\begin{aligned} c_n &= \frac{1}{\tau} \int_0^{\tau} x(t) e^{-in\omega t} dt = \frac{1}{\tau} \left[\int_0^{\tau/2} \frac{2x_0 t}{\tau} e^{-in\omega t} dt \right. \\ &\quad \left. + \int_{\tau/2}^{\tau} \left(\frac{2x_0 t}{\tau} - 2x_0 \right) e^{-in\omega t} dt \right] \\ &= \frac{2x_0}{\tau^2} \int_0^{\tau/2} t e^{-in\omega t} dt + \frac{2x_0}{\tau^2} \int_{\tau/2}^{\tau} t e^{-in\omega t} dt - \frac{2x_0}{\tau} \int_{\tau/2}^{\tau} e^{-in\omega t} dt \\ &= \frac{2x_0}{\tau^2} \left(\frac{1}{n^2 \omega^2} \right) \left[e^{-in\omega t} (in\omega t) + e^{-in\omega t} \right]_0^{\tau/2} \\ &\quad + \frac{2x_0}{\tau^2} \left(\frac{1}{n^2 \omega^2} \right) \left[e^{-in\omega t} (in\omega t) + e^{-in\omega t} \right]_{\tau/2}^{\tau} \\ &\quad - \frac{2x_0}{\tau} \left(\frac{1}{-in\omega} \right) \left(e^{-in\omega t} \right)_{\tau/2}^{\tau} \\ &= \frac{2x_0}{n^2 \pi^2} \left[in\pi e^{-in\pi} + e^{-in\pi} - 1 \right] - \frac{2x_0 i}{n\pi} \left[e^{-in\pi} - e^{-in\frac{\pi}{2}} \right] \\ &= \frac{2x_0}{n^2 \pi^2} \cos n\pi + i \frac{2x_0}{n\pi} \cos \frac{n\pi}{2} + \frac{2x_0}{n\pi} \sin \frac{n\pi}{2} - \frac{2x_0}{n^2 \pi^2} \\ &= \frac{2x_0}{n^2 \pi^2} \left[(-1)^n - 1 \right] + i \underbrace{\frac{2x_0}{n\pi} (-1)^{\frac{n}{2}}}_{n = \text{even}} + \underbrace{\frac{2x_0}{n\pi} (-1)^{\frac{n-1}{2}}}_{n = \text{odd}} \end{aligned}$$

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \\ &= \sum_{n=-\infty}^{\infty} \frac{2x_0}{n^2 \pi^2} \left[(-1)^n - 1 \right] e^{in\omega t} \\ &\quad + i \frac{2x_0}{\pi} \sum_{\substack{n=-\infty \\ (n = \text{even})}}^{\infty} (-1)^{\frac{n}{2}} \frac{1}{n} e^{in\omega t} \\ &\quad + \frac{2x_0}{\pi} \sum_{\substack{n=-\infty \\ (n = \text{odd})}}^{\infty} (-1)^{\frac{n-1}{2}} \frac{1}{n} e^{in\omega t} \end{aligned}$$

14.9

For $x(t) = X \sin \omega t$, $R_x(\tau) = \frac{X^2}{2} \cos \omega \tau$ [from Problem 14.7]
 For square wave:



14.10

$$p(t) = \begin{cases} \lambda e^{-\lambda t} & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$

$$(i) P(t) = \int_{-\infty}^t p(t') dt' = \lambda \int_0^t e^{-\lambda t'} dt' = 1 - e^{-\lambda t}$$

$$(ii) \bar{T} = \int_{-\infty}^{\infty} t p(t) dt = \lambda \int_0^{\infty} t e^{-\lambda t} dt = \frac{\lambda}{(-\lambda)^2} [e^{-\lambda t} (-\lambda t - 1)]_0^{\infty} \\ = \frac{1}{\lambda}$$

$$(iii) \sigma_T^2 = \int_0^{\infty} (t - \frac{1}{\lambda})^2 p(t) dt \\ = \lambda \int_0^{\infty} t^2 e^{-\lambda t} dt + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda t} dt - 2 \int_0^{\infty} t e^{-\lambda t} dt \\ = \lambda \left[-\frac{t^2}{\lambda} e^{-\lambda t} - \frac{2t}{\lambda^2} e^{-\lambda t} - \frac{2}{\lambda^3} e^{-\lambda t} \right]_0^{\infty} \\ - \frac{1}{\lambda^2} (e^{-\lambda t})_0^{\infty} + \frac{2}{\lambda^2} (\lambda t e^{-\lambda t} + e^{-\lambda t})_0^{\infty} = \frac{1}{\lambda^2}$$

$$\sigma_T = \frac{1}{\lambda}$$

14.11 $x(t) = x_0 \sin \frac{\pi t}{2}$

$$\langle x(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x_0 \sin \frac{\pi t}{2} dt = \lim_{T \rightarrow \infty} \frac{1}{T} \left(\frac{-x_0 \cos \frac{\pi t}{2}}{\frac{\pi}{2}} \right) \Big|_{-T/2}^{T/2}$$

$$= \lim_{T \rightarrow \infty} -\frac{x_0}{T} \left(\frac{2}{\pi} \right) \left[\cos \frac{\pi T}{4} - \cos \left(-\frac{\pi T}{4} \right) \right] = 0$$

$$\langle x^2(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x_0^2 \sin^2 \frac{\pi t}{2} dt$$

$$= \lim_{T \rightarrow \infty} \frac{x_0^2}{T} \int_{-T/2}^{T/2} \left\{ \frac{1}{2} - \frac{1}{2} \cos \pi t \right\} dt = \lim_{T \rightarrow \infty} \frac{x_0^2}{2T} \left[T - \frac{2}{\pi} \sin \frac{\pi T}{2} \right]$$

$$= \lim_{T \rightarrow \infty} \left(\frac{x_0^2}{2} - \frac{x_0^2}{2} \left\{ \frac{\sin \frac{\pi T}{2}}{\frac{\pi T}{2}} \right\} \right) = \frac{x_0^2}{2}$$

14.12 $\omega = 2\pi(1800)/60 = 188.496 \text{ rad/sec}$

$k = (\bar{k}, \sigma_k) = (2.25 \times 10^6, 0.225 \times 10^6) \text{ N/m}$
(normally distributed)



$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{100}} = \frac{\sqrt{k}}{10} \text{ rad/sec}$$

$$P(\omega_n \geq \omega) = P[\omega_n^2 \geq \omega^2]$$

$$= P[k \geq 100(188.496)^2] = P[k \geq 3553074.202] \quad (E_1)$$

Defining standard normal variate z as $z = \left(\frac{k - \bar{k}}{\sigma_k} \right)$,

Eg. (E₁) can be rewritten as

$$P[\omega_n > \omega] = P\left[\frac{k - \bar{k}}{\sigma_k} \geq \frac{3.5531 \times 10^6 - 2.25 \times 10^6}{0.225 \times 10^6} \right]$$

$$= P[z \geq 5.7916] = 0.3316 \times 10^{-8}$$

from standard normal distribution tables

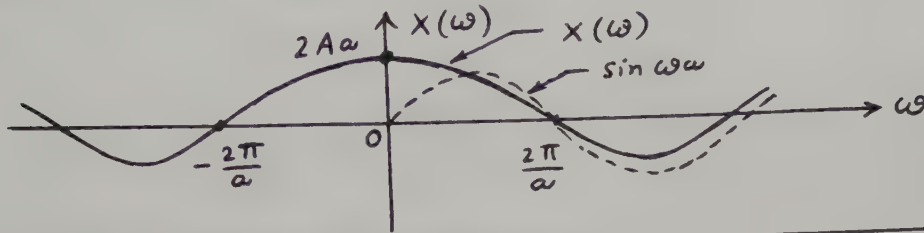
[see, for example, Ref. 14.5]

14.13 $x(t) = \begin{cases} A & ; -a \leq t \leq a \\ 0 & ; \text{elsewhere} \end{cases}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt = A \int_{-a}^a e^{-i\omega t} dt = \frac{A}{-i\omega} \left(e^{-i\omega t} \right)_{-a}^a$$

$$\begin{aligned}
 &= \frac{Ai}{\omega} (e^{-i\omega a} - e^{i\omega a}) \\
 &= \frac{Ai}{\omega} (\cos \omega a - i \sin \omega a - \cos \omega a - i \sin \omega a) \\
 &= \frac{2Aa}{\omega a} \sin \omega a
 \end{aligned} \tag{E_1}$$

Eq. (E₁) shows that $X(\omega) = 2Aa$ as $\omega \rightarrow 0$.

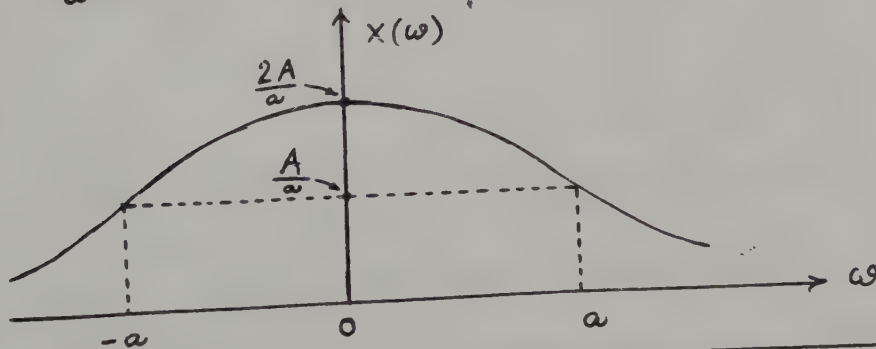


14.14 $x(t) = \begin{cases} A e^{-at} & ; t \geq 0 \\ 0 & ; \text{elsewhere} \end{cases}$

$$\begin{aligned}
 X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt = A \int_0^{\infty} e^{-(a+i\omega)t} dt \\
 &= \frac{-A}{(a+i\omega)} \left[e^{-(a+i\omega)t} \right]_0^{\infty} = \frac{A}{a+i\omega} = \frac{Aa}{a^2+\omega^2} - i \frac{A\omega}{a^2+\omega^2}
 \end{aligned}$$

14.15 $x(t) = A e^{-a|t|}$

$$\begin{aligned}
 X(\omega) &= A \int_{-\infty}^0 e^{at} e^{-i\omega t} dt + A \int_0^{\infty} e^{-at} e^{-i\omega t} dt \\
 &= \frac{A}{-(i\omega-a)} \left[e^{-(i\omega-a)t} \right]_{-\infty}^0 - \frac{A}{(a+i\omega)} \left[e^{-(a+i\omega)t} \right]_0^{\infty} \\
 &= \frac{A}{a-i\omega} + \frac{A}{a+i\omega} = \frac{A(a+i\omega)}{a^2+\omega^2} + \frac{A(a-i\omega)}{a^2+\omega^2} = \frac{2Aa}{a^2+\omega^2}
 \end{aligned}$$

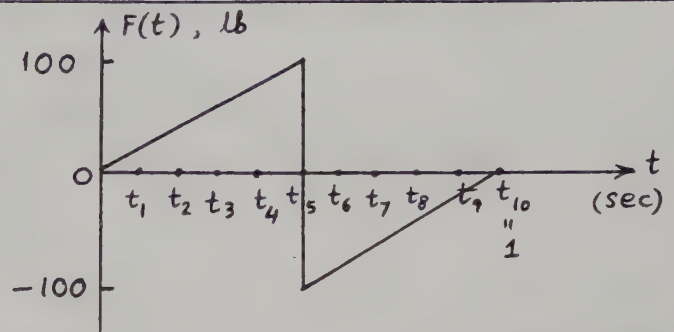


14.16 $x(t) = \delta(t-a)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t-a) e^{-i\omega t} dt = e^{-i\omega a} = \cos \omega a - i \sin \omega a$$

since, by definition, the Dirac delta function is zero every where except at $t=a$. At $t=a$, $e^{-i\omega t} = e^{-i\omega a}$.

14.17



j	0	1	2	3	4	5	6	7	8	9	10
$t_j(\text{sec})$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$F_j = F(t_j), \text{lb}$	0	20	40	60	80	100	-80	-60	-40	-20	0

(i) Spectrum of $F(t)$:

From Eq. (14.46),
$$c_n = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} F(t) e^{-in\omega_0 t} dt \quad (E_1)$$

where $\tau = \text{time period} = \frac{2\pi}{\omega_0}$ and $\omega_0 = \text{fundamental frequency}$, $\tau = 1.0 \text{ sec}$, $\omega_0 = 2\pi \text{ rad/sec}$.

Since the integration in Eq. (E₁) is over one-time period, c_n can be expressed as

$$c_n = \frac{1}{\Delta t N} \sum_{j=1}^N F(t_j) \cdot e^{-in(\frac{2\pi}{\tau})t_j} \cdot \Delta t \quad \text{where } \frac{t_j}{\tau} = \frac{j}{N}$$

$$\begin{aligned} \therefore c_n &= \frac{1}{N} \sum_{j=1}^N F_j e^{-i(\frac{2\pi n j}{N})} \\ &= \frac{1}{N} \sum_{j=1}^N F_j \left\{ \cos \frac{2\pi n j}{N} - i \sin \frac{2\pi n j}{N} \right\} \\ &= \text{real}(c_n) + i \text{imag}(c_n) \end{aligned} \quad (E_2)$$

Eq. (E₂) gives the following results for $N=10$:

n	Real (c_n)	Imag (c_n)	Spectrum of $F(t) = c_n ^2$ $= \text{Real}(c_n)^2 + \text{Imag}(c_n)^2$
0	10	0	100
1	-10	-30.7768	1047.21
2	10	13.7636	289.44
3	-10	-7.2652	152.79
4	10	3.2489	111.56
5	-10	0.0003	100.00
6	10	-3.2497	111.56
7	-10	7.2658	152.79
8	10	-13.7648	289.45
9	-10	30.7775	1047.21

(ii) Mean square value of $F(t)$:

Using Eq. (14.49),

$$\overline{F^2(t)} = \sum_{n=0}^{N-1} |c_n|^2 = \sum_{n=0}^9 |c_n|^2 = 3400.00$$

14.18

$$R_x(\tau) = a e^{-b|\tau|}$$

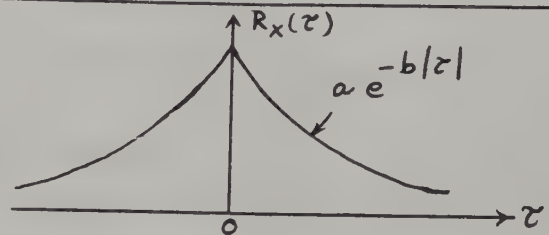
$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^0 a e^{b\tau} e^{-i\omega\tau} d\tau$$

$$+ \frac{1}{2\pi} \int_0^{\infty} a e^{-b\tau} e^{-i\omega\tau} d\tau$$

$$= \left(\frac{a}{2\pi}\right) \frac{1}{-(i\omega-b)} \left[e^{-(i\omega-b)\tau} \right]_{-\infty}^0 + \left(\frac{a}{2\pi}\right) \frac{1}{-(i\omega+b)} \left[e^{-(i\omega+b)\tau} \right]_0^{\infty}$$

$$= \frac{-a}{2\pi(i\omega-b)} + \frac{a}{2\pi(i\omega+b)} = \frac{ab}{\pi(b^2 + \omega^2)}$$

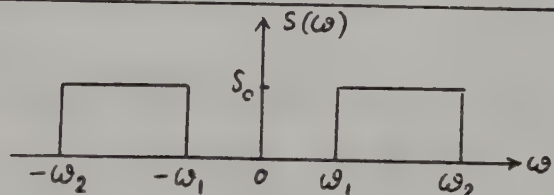


14.19

$$R(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \quad \dots (E_1)$$

Since $S(\omega) = S_0$ is symmetric and real, we can neglect the imaginary component in (E_1) and write $R(\tau)$ as

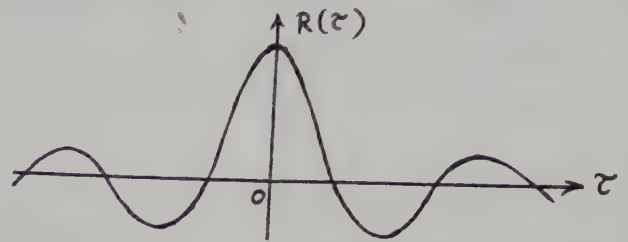
$$R(\tau) = 2S_0 \int_{\omega_1}^{\omega_2} \cos \omega\tau \cdot d\omega = \frac{2S_0}{\tau} (\sin \omega_2\tau - \sin \omega_1\tau) \quad \dots (E_2)$$



$R(\tau)$ as $\tau \rightarrow 0$ is given by

$$\lim_{\tau \rightarrow 0} \left[2 S_0 \left(\frac{\omega_2 \sin \omega_2 \tau}{\omega_2 \tau} \right) - 2 S_0 \left(\frac{\omega_1 \sin \omega_1 \tau}{\omega_1 \tau} \right) \right] = 2 S_0 (\omega_2 - \omega_1) \dots (E_3)$$

The variation of $R(\tau)$ is shown in the figure.



14.20

From Eq. (14.60),

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau) e^{-i\omega\tau} d\tau = \frac{1}{2\pi} \sigma_x^2 \int_{-\infty}^{\infty} e^{-\alpha|\nu\tau|} \cos \beta \nu \tau e^{-i\omega\tau} d\tau$$

$$\text{But } \cos \beta \nu \tau = \frac{1}{2} (e^{i\beta \nu \tau} + e^{-i\beta \nu \tau})$$

$$\begin{aligned} S_x(\omega) &= \frac{\sigma_x^2}{4\pi} \int_{-\infty}^{\infty} \left\{ e^{-\alpha|\nu\tau|} e^{-i\omega\tau} (e^{i\beta \nu \tau} + e^{-i\beta \nu \tau}) d\tau \right\} \\ &= \frac{\sigma_x^2}{4\pi} \left[\int_0^{\infty} e^{(-\alpha\nu - i\omega + i\beta\nu)\tau} d\tau + \int_0^{\infty} e^{(-\alpha\nu - i\omega - i\beta\nu)\tau} d\tau \right. \\ &\quad \left. + \int_{-\infty}^0 e^{(\alpha\nu - i\omega + i\beta\nu)\tau} d\tau + \int_{-\infty}^0 e^{(\alpha\nu - i\omega - i\beta\nu)\tau} d\tau \right] \\ &= \frac{\sigma_x^2}{2\pi} \left[\frac{\alpha\nu}{\alpha^2\nu^2 + (\omega - \beta\nu)^2} + \frac{\alpha\nu}{\alpha^2\nu^2 + (\omega + \beta\nu)^2} \right] \\ &= \frac{\alpha\nu\sigma_x^2}{\pi} \left[\frac{\omega^2 + \beta^2\nu^2 + \alpha^2\nu^2}{\{\alpha^2\nu^2 + (\omega - \beta\nu)^2\} \{\alpha^2\nu^2 + (\omega + \beta\nu)^2\}} \right] \end{aligned}$$

For asphalt surface:

$$\sigma_x = 1.1, \quad \alpha = 0.2, \quad \beta = 0.4$$

$$\begin{aligned} S_x(\omega) &= \frac{(0.2)\nu(1.21)}{\pi} \left[\frac{\omega^2 + 0.16\nu^2 + 0.04\nu^2}{\{0.04\nu^2 + (\omega - 0.4\nu)^2\} \{0.04\nu^2 + (\omega + 0.4\nu)^2\}} \right] \\ &= 0.07703\nu \left[\frac{\omega^2 + 0.2\nu^2}{\{0.04\nu^2 + (\omega - 0.4\nu)^2\} \{0.04\nu^2 + (\omega + 0.4\nu)^2\}} \right] \end{aligned}$$

For paved surface:

$$\sigma_x = 1.6, \quad \alpha = 0.3, \quad \beta = 0.6$$

$$S_x(\omega) = \frac{(0.3)\nu(2.56)}{\pi} \left[\frac{\omega^2 + 0.36\nu^2 + 0.09\nu^2}{\{0.09\nu^2 + (\omega - 0.6\nu)^2\} \{0.09\nu^2 + (\omega + 0.6\nu)^2\}} \right]$$

$$= 0.24446 v \left[\frac{\omega^2 + 0.45 v^2}{\{0.09 v^2 + (\omega - 0.6 v)^2\} \{0.09 v^2 + (\omega + 0.6 v)^2\}} \right]$$

For gravel surface:

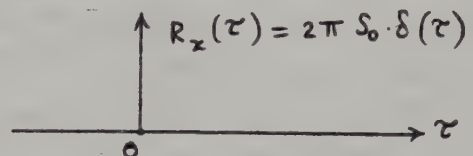
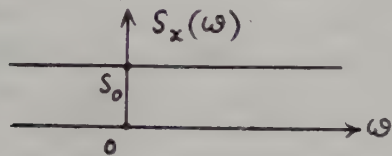
$$\sigma_x = 1.8, \quad \alpha = 0.5, \quad \beta = 0.9$$

$$S_x(\omega) = \frac{(0.5)v(3.24)}{\pi} \left[\frac{\omega^2 + 0.81 v^2 + 0.25 v^2}{\{0.25 v^2 + (\omega - 0.9 v)^2\} \{0.25 v^2 + (\omega + 0.9 v)^2\}} \right]$$

$$= 0.51566 v \left[\frac{\omega^2 + 1.06 v^2}{\{0.25 v^2 + (\omega - 0.9 v)^2\} \{0.25 v^2 + (\omega + 0.9 v)^2\}} \right]$$

14.21 From Eq. (14.61), $R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega) e^{i\omega\tau} d\omega = S_0 \int_{-\infty}^{\infty} e^{i\omega\tau} d\omega$
 $= 2\pi S_0 \delta(\tau)$

Where $\delta(\tau)$ is the Dirac delta function.



14.22 $S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-i\omega\tau} d\tau$ or $S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-i2\pi f\tau} d\tau$ --- (E₁)

and

$$R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) e^{i\omega\tau} d\omega \quad \text{or} \quad R_x(\tau) = \int_{-\infty}^{\infty} S_x(f) e^{i2\pi f\tau} df$$
 --- (E₂)

where $S_x(f)$ is the two-sided power spectral density function for $-\infty < f < \infty$. Noting that

$S_x(f) = S_x(-f)$ and $R_x(\tau) = R_x(-\tau)$, (E₁) and (E₂) can be written as

$$S_x(f) = 2 \int_0^{\infty} R_x(\tau) \cdot \{\cos 2\pi f\tau - i \sin 2\pi f\tau\} d\tau \quad \text{for } -\infty < f < \infty$$
 --- (E₃)

and

$$R_x(\tau) = 2 \int_0^{\infty} S_x(f) \cdot \{\cos 2\pi f\tau + i \sin 2\pi f\tau\} d\tau \quad \text{for } -\infty < f < \infty$$
 --- (E₄)

For a real random function $x(t)$, $S_x(f)$ and $R_x(\tau)$ are real-valued functions and hence, by neglecting the imaginary parts in (E₃) and (E₄), we obtain

$$S_x(f) = 2 \int_0^{\infty} R_x(\tau) \cdot \cos 2\pi f\tau \cdot d\tau \quad \text{for } -\infty < f < \infty$$
 --- (E₅)

and

$$R_x(\tau) = 2 \int_0^{\infty} S_x(f) \cdot \cos 2\pi f \tau \cdot df \quad \text{for } -\infty < \tau < \infty \dots (E_6)$$

If $S(f)$ denotes the one-sided power spectral density function,

$$S(f) = 2 S_x(f) \quad \text{and} \quad R(\tau) = R_x(\tau), \quad \text{Eg. (E}_5\text{) and (E}_6\text{) become}$$

$$S(f) = 4 \int_0^{\infty} R(\tau) \cos 2\pi f \tau \cdot d\tau \quad \text{for } 0 \leq f < \infty \dots (E_7)$$

and

$$R(\tau) = \int_0^{\infty} S(f) \cos 2\pi f \tau \cdot df \quad \text{for } 0 \leq \tau < \infty \dots (E_8)$$

14.23

Mean square value of a single d.o.f. system is given by Eq. (14.95):

$$E[y^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega) d\omega \equiv \int_{-\infty}^{\infty} A(\omega) d\omega \quad (E_1)$$

where

$$|H(\omega)| = \frac{1}{(k - m\omega^2)^2 + c^2\omega^2} \quad (E_2)$$

Let $-\omega^{(1)}$ and $+\omega^{(2)}$ denote finite lower and upper bounds to be used for integration in Eq. (E₁) [$\omega^{(1)}$ and $\omega^{(2)}$ should be sufficiently large].

We can use trapezoidal rule of integration for simplicity. In this method, we divide the interval $(\omega^{(2)} + \omega^{(1)})$ into $(n-1)$ equal divisions so that

$$\left. \begin{aligned} \omega_1 &= -\omega^{(1)} \\ \omega_2 &= -\omega_1 + \Delta\omega \\ \omega_3 &= -\omega_1 + 2\Delta\omega \\ &\vdots \\ \omega_n &= -\omega_1 + (n-1)\Delta\omega = \omega^{(2)} \end{aligned} \right\} \quad (E_3)$$

Then

$$\begin{aligned} E[y^2] &= \int_{-\omega^{(1)}}^{\omega^{(2)}} A(\omega) d\omega = \sum_{p=1}^{n-1} [A(\omega_p) + A(\omega_{p+1})] \frac{\Delta\omega}{2} \\ &= [A(-\omega^{(1)}) + A(\omega^{(2)})] \frac{\Delta\omega}{2} + \sum_{p=2}^{n-1} A(\omega_p) \cdot \Delta\omega \dots (E_4) \end{aligned}$$

The computer program listing is given below. In this program, the following notation is used:

$$OM1 = \omega^{(1)}, \quad OM2 = \omega^{(2)}, \quad OMP = \omega_p, \quad SX = S_x(\omega),$$

$$HOM = |H(\omega)|^2, \quad SUM = E[y^2].$$


```

REAL M, K, C
DATA M, K, C /... /
DATA OM1, OM2 /... /
N = ...
N1 = N - 1
DOM = (OM2 + OM1) / REAL (N1)
OM11 = -OM1
CALL PSD (OM11, SX1, HOM1)
CALL PSD (OM2, SX2, HOM2)
SUM = [(SX1/HOM1) + (SX2/HOM2)] * DOM / 2.0
DO 11 I = 2, N1
  OMP = -OM1 + REAL (I-1) * DOM
  CALL PSD (OMP, SXP, HOMP)
11 SUM = SUM + (SXP/HOMP) * DOM
PRINT 12, SUM
12 FORMAT (...)
STOP
END

C SUBROUTINE TO EVALUATE POWER SPECTRAL DENSITY AND HOM
SUBROUTINE PSD (OM, SX, HOM)
  SX = ...
  HOM = ...
  RETURN
END

```

14.24 $m = 2000/386.4 = 5.1760 \text{ lb-sec}^2/\text{in}$
 $k = 4 \times 10^4 \text{ lb/in}, \quad c = 1200 \text{ lb-in/sec}$

Mean square response of the machine is given by Eq. (14.95):

$$E[y^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega) d\omega = \sum_{n=0}^{N-1} \left| \frac{1}{-m\omega_n^2 + ic\omega_n + k} \right|^2 |c_n|^2$$

where $\omega_n = n\omega_0 = n\left(\frac{2\pi}{\tau}\right) = 2\pi n \text{ rad/sec}$ ----- (E₁)

and $|c_n|^2$ are given in the solution of problem 14.17.

Eq. (E₁) can be rewritten as

$$E[y^2] = \sum_{n=0}^{N-1} \frac{|c_n|^2}{(k - m\omega_n^2)^2 + c^2\omega_n^2} \quad \text{--- (E}_2\text{)}$$

Computations:

n	$\omega_n = 2\pi n$ rad/sec	$(k - m\omega_n^2)^2 + c^2\omega_n^2$	$ c_n ^2$ lb ²	$\left\{ \frac{ c_n ^2}{(k - m\omega_n^2)^2 + c^2\omega_n^2} \right\}$ in ²
0	0	16×10^8	100.00	6.25×10^{-8}
1	2π	16.4054×10^8	1047.21	63.8331×10^{-8}

2	4π	17.6268×10^8	289.44	16.4205×10^{-8}
3	6π	19.6790×10^8	152.79	7.7641×10^{-8}
4	8π	22.5872×10^8	111.56	4.9391×10^{-8}
5	10π	26.3865×10^8	100.00	3.7898×10^{-8}
6	12π	31.1218×10^8	111.56	3.5846×10^{-8}
7	14π	36.8485×10^8	152.79	4.1464×10^{-8}
8	16π	43.6315×10^8	289.45	6.6340×10^{-8}
9	18π	51.5461×10^8	1047.21	20.3160×10^{-8}

Thus E_y . (E_2) gives the mean square value of the response as

$$E[y^2] = 137.6777 \times 10^{-8} \text{ in}^2$$

14.25 Equation of motion is

$$m\ddot{x} + c\dot{x} = F(t) \quad (E_1)$$

Let $y = \dot{x}$ = velocity. Then (E_1) becomes

$$m\dot{y} + cy = F(t) \quad (E_2)$$

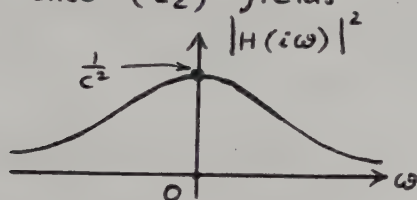
Let the excitation be

$$F(t) = F_0 e^{i\omega t}$$

and the velocity response be

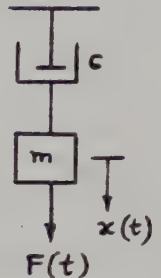
$$y(t) = Y e^{i\omega t}$$

Hence (E_2) yields $m i\omega Y e^{i\omega t} + c Y e^{i\omega t} = F_0 e^{i\omega t}$



$$\text{or } \frac{Y}{Y_0} = \frac{1}{i\omega m + c} \equiv H(i\omega)$$

$$|H(i\omega)|^2 = \frac{1}{m^2\omega^2 + c^2}$$

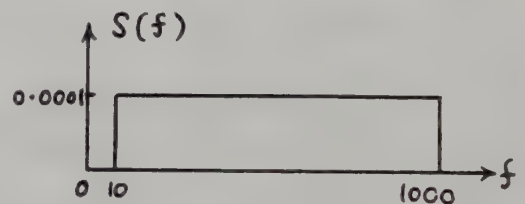


14.26 $S(f) = 0.0001 \text{ m}^2/\text{cycle per second}$ for $10 \text{ Hz} \leq f \leq 1000 \text{ Hz}$

$$\bar{x}^2 = (0.0001)(1000 - 10) = 0.099 \text{ m}^2$$

$$\text{RMS value} = \sqrt{\bar{x}^2} = 0.3146 \text{ m}$$

$$\sigma = \sqrt{\bar{x}^2 - (\bar{x})^2} = \sqrt{0.099 - 0.0025} = 0.3106 \text{ m}$$



14.27
$$\int_{t=-\tau/2}^{\tau/2} \cos(n-m)\omega_0 t \, dt = \frac{1}{(n-m)\omega_0} \left[\sin(n-m)\omega_0 t \right]_{t=-\tau/2}^{t=\tau/2}$$

$\frac{\tau}{2} = \frac{\pi}{\omega_0}$
 $t = -\frac{\tau}{2} = -\frac{\pi}{\omega_0}$

$$= \frac{1}{(n-m)\omega_0} [\sin(n-m)\pi + \sin(n-m)\pi] = 0 \text{ for all } m \neq n$$

since $\tau = \frac{2\pi}{\omega_0}$.

$$\int_{-\tau/2}^{\tau/2} \sin(n-m)\omega_0 t \cdot dt = \frac{1}{-(n-m)\omega_0} \left[\cos(n-m)\omega_0 t \right]_{t=-\pi/\omega_0}^{t=\pi/\omega_0}$$

$$= \frac{-1}{(n-m)\omega_0} [\cos(n-m)\pi - \cos(n-m)\pi] = 0 \quad \text{for all } m \neq n$$

$$\int_{-\tau/2}^{\tau/2} \cos 0 \cdot dt = \left(t \right)_{-\tau/2}^{\tau/2} = \tau \quad \text{for } m=n$$

$$\int_{-\tau/2}^{\tau/2} \sin 0 \cdot dt = 0 \quad \text{for } m=n$$

$\therefore$ Eq. (14.45) becomes

$$\int_{-\tau/2}^{\tau/2} x(t) e^{-in\omega_0 t} dt = c_n \tau \Rightarrow c_n = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \cdot e^{-in\omega_0 t} dt$$

----(14.46)

14.28

This system can be modeled as a single d.o.f. system with random base excitation. The equations of motion are given by:

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{x} \quad (E_1)$$

where $z = y - x$.

The frequency response function of the system can be derived as follows:

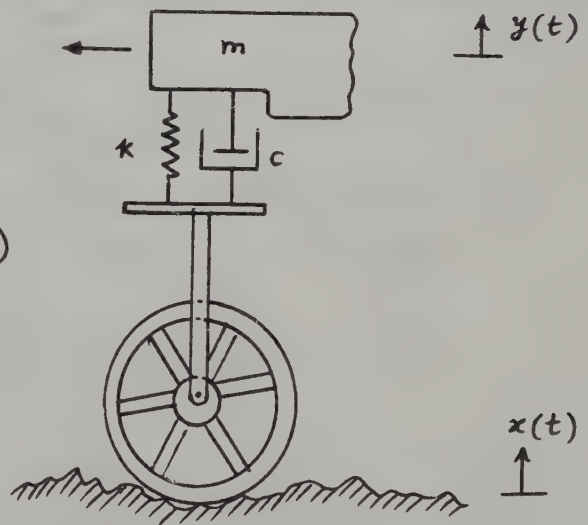
$$\left. \begin{aligned} \text{Let } x(t) &= e^{i\omega t} \\ z(t) &= H(\omega) e^{i\omega t} \end{aligned} \right\} \quad (E_2)$$

Substitution of (E_2) into (E_1) gives

$$(-\omega^2 m + ic\omega + k) H(\omega) e^{i\omega t} = m\omega^2 e^{i\omega t}$$

$$\text{i.e., } H(\omega) = \frac{m\omega^2}{-m\omega^2 + k + ic\omega} = \frac{\omega^2}{(\omega_n^2 - \omega^2) + i2\zeta\omega\omega_n}$$

$$\text{i.e., } |H(\omega)|^2 = \frac{\omega^4}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega^2\omega_n^2} \quad (E_3)$$



The power spectral density of the response $z(t)$ is given by

$$S_z(\omega) = |H(\omega)|^2 S_x(\omega) \quad (E_4)$$

In the present case, $S_x(\omega) = S_0$ and Eq. (E₄) reduces to

$$S_z(\omega) = S_0 |H(\omega)|^2 \quad (E_5)$$

The mean square value of the relative displacement of the mass can be found as

$$\begin{aligned} E[z^2] &= \int_{-\infty}^{\infty} S_z(\omega) \cdot d\omega = S_0 \int_{-\infty}^{\infty} \left| \frac{\omega^2}{-\omega^2 + \omega_n^2 + \frac{i\omega c}{m}} \right|^2 \cdot d\omega \\ &= S_0 \omega^4 \pi \left\{ \frac{\left(\frac{1}{\omega_n^2}\right)}{\left(\frac{c}{m}\right)} \right\} = \frac{\pi S_0 \omega^4}{2 \zeta \omega_n^3} \quad (E_6) \end{aligned}$$

14.29 $\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{1}{m} F$ --- (E₁)

This gives $S_x(\omega) = \frac{1}{m^2} \frac{S_F(\omega)}{\{(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2\}}$ --- (E₂)

$$E[x^2] = \frac{1}{m^2} \int_{-\infty}^{\infty} \frac{S_F(\omega) d\omega}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2} \quad \text{--- (E}_3\text{)}$$

For small damping, (E₃) becomes

$$E[x^2] \simeq \frac{1}{m^2} S_F(\omega_n) \cdot \frac{\pi}{2\zeta\omega_n^3} \quad \text{--- (E}_4\text{)}$$

Here $S_F(\omega) = A^2 \frac{1 + \left(\frac{L\omega}{v}\right)^2}{\left\{1 + \left(\frac{L\omega}{v}\right)^2\right\}^2 \left(1 + \frac{\pi\omega c}{v}\right)}$ --- (E₅)

$$\therefore E[x^2] = \frac{\pi A^2}{2\zeta\omega_n^3 m^2} \left\{ \frac{1 + \left(\frac{L\omega_n}{v}\right)^2}{\left(1 + \frac{L^2\omega_n^2}{v^2}\right)^2 \left(1 + \frac{\pi\omega_n c}{v}\right)} \right\} \quad \text{--- (E}_6\text{)}$$

14.30 $\omega_1 = \text{undamped natural frequency} = \sqrt{k_{eq}/m_{eq}}$
 or $k_{eq} = \omega_1^2 m_{eq}$ (E₁)

$\omega_2 = \text{damped natural frequency} = \omega_n \sqrt{1 - \zeta^2}$

$$= \omega_1 \sqrt{1 - \zeta^2} = \sqrt{\frac{k_{eq}}{m_{eq}}} \cdot \sqrt{1 - \zeta^2} = \sqrt{\frac{k_{eq}}{m_{eq}}} \cdot \sqrt{\left(1 - \frac{c_{eq}^2}{4 k_{eq} m_{eq}}\right)}$$

$$\omega_2^2 = \frac{k_{eq}}{m_{eq}} \left(1 - \frac{c_{eq}^2}{4 k_{eq} m_{eq}} \right) = \omega_1^2 - \frac{c_{eq}^2}{4 m_{eq}^2}$$

$$\text{or} \quad \left(\frac{c_{eq}}{2 m_{eq}} \right)^2 = \omega_1^2 - \omega_2^2$$

$$\text{or} \quad c_{eq} = 2 m_{eq} \sqrt{\omega_1^2 - \omega_2^2} \quad (E_2)$$

Mean square value of the displacement of the wing (m_{eq})

$$E[y^2] = \frac{\pi S_0}{k_{eq} c_{eq}} = \delta \quad \text{or} \quad c_{eq} = \frac{\pi S_0}{\delta k_{eq}} = \delta. \quad (E_3)$$

Eqs. (E₁) and (E₃) give

$$c_{eq} = \frac{\pi S_0}{\delta \omega_1^2 m_{eq}} \quad (E_4)$$

Equating (E₂) and (E₄),

$$c_{eq} = 2 m_{eq} \sqrt{\omega_1^2 - \omega_2^2} = \frac{\pi S_0}{\delta \omega_1^2 m_{eq}}$$

$$\therefore m_{eq} = \left[\frac{\pi S_0}{2 \delta \omega_1^2 (\omega_1^2 - \omega_2^2)^{\frac{1}{2}}} \right]^{\frac{1}{2}} \quad (E_5)$$

Eqs. (E₁) and (E₂), in view of (E₅), yield

$$k_{eq} = \left[\frac{\pi S_0 \omega_1^2}{2 \delta (\omega_1^2 - \omega_2^2)^{\frac{1}{2}}} \right]^{\frac{1}{2}} \quad (E_6)$$

and

$$c_{eq} = 2 (\omega_1^2 - \omega_2^2)^{\frac{1}{2}} \left[\frac{\pi S_0}{2 \delta \omega_1^2 (\omega_1^2 - \omega_2^2)^{\frac{1}{2}}} \right]^{\frac{1}{2}}$$

$$= \left[\frac{2 \pi S_0 (\omega_1^2 - \omega_2^2)^{\frac{1}{2}}}{\delta \omega_1^2} \right]^{\frac{1}{2}} \quad (E_7)$$

14.31 In case of structural damping, the uncoupled equations of motion are given by

$$\ddot{q}_i(t) + (1 + i \beta) \omega_i^2 q_i(t) = Q_i(t) ; i = 1, 2, \dots, n \quad (1)$$

where β denotes the structural damping coefficient. The mean square values of $x_i(t)$ are given by Eq. (14.113):

$$\overline{x_i^2(t)} = \sum_{r=1}^n \left(X_i^{(r)} \right)^2 \frac{N_r^2}{\omega_r^4} \int_{-\infty}^{\infty} |H_r(\omega)|^2 S_r(\omega) d\omega \quad (2)$$

where, from Eq. (3.106),

$$|H_r(\omega)| = \frac{1}{\left\{1 - \frac{\omega^2}{\omega_n^2}\right\}^2 + \beta^2} \quad (3)$$

For $\beta \ll 1$, Eq. (2) can be approximated as

$$\begin{aligned} \overline{x_i^2(t)} &\approx \sum_{r=1}^n \left(X_i^{(r)}\right)^2 \frac{N_r^2}{\omega_r^4} S_r(\omega_r) \int_{-\infty}^{\infty} |H_r(\omega)|^2 d\omega \\ &\approx \sum_{r=1}^n \left(X_i^{(r)}\right)^2 \frac{N_r^2}{\omega_r^4} S_r(\omega_r) \frac{\pi \omega_r}{\beta} \end{aligned} \quad (4)$$

Using the computational details of Example 14.7, we can obtain (using $g = 0.01$ in place of $2 \zeta_r = 0.04$):

$$\overline{z_1^2(t)} = 0.0021253 \text{ m}^2 \quad (5)$$

$$\overline{z_2^2(t)} = 0.0055983 \text{ m}^2 \quad (6)$$

$$\overline{z_3^2(t)} = 0.0086582 \text{ m}^2 \quad (7)$$

14.32

$$S(\omega) = \frac{1}{4 + \omega^2} \frac{\text{m}^2/\text{s}^4}{\text{rad/sec}} ; \quad \zeta_i = 0.02$$

Since the natural frequencies (rad/sec) are given by $\omega_1 = 14.0734$, $\omega_2 = 39.4368$, and $\omega_3 = 57.0001$, we can approximate $S(\omega_r)$ for use in Eq. (14.115) as

$$S_r(\omega_1) = \frac{1}{4 + 14.0734^2} = 0.0049490 \quad (1)$$

$$S_r(\omega_2) = \frac{1}{4 + 39.4368^2} = 0.0006413 \quad (2)$$

$$S_r(\omega_3) = \frac{1}{4 + 57.0001^2} = 0.0003074 \quad (3)$$

and hence the mean square values of the relative displacements of the various floors can be computed as (using Eq. (E.18) of Example 14.7):

$$\begin{aligned} \overline{z_1^2(t)} &= \frac{\pi}{2 \zeta_r} \left[\left(Z_1^{(1)}\right)^2 \frac{N_1^2}{\omega_1^3} S(\omega_1) + \left(Z_1^{(2)}\right)^2 \frac{N_2^2}{\omega_2^3} S(\omega_2) + \left(Z_1^{(3)}\right)^2 \frac{N_3^2}{\omega_3^3} S(\omega_3) \right] \\ &= \frac{\pi}{0.04} [0.0001058 (0.004949) + 0.0000243 (0.0006413) + 0.0000052 (0.0003074)] \\ &= 42.4744 (10^{-6}) \text{ m}^2 \\ \overline{z_2^2(t)} &= \frac{\pi}{2 \zeta_r} \left[\left(Z_2^{(1)}\right)^2 \frac{N_1^2}{\omega_1^3} S(\omega_1) + \left(Z_2^{(2)}\right)^2 \frac{N_2^2}{\omega_2^3} S(\omega_2) + \left(Z_2^{(3)}\right)^2 \frac{N_3^2}{\omega_3^3} S(\omega_3) \right] \end{aligned} \quad (4)$$

$$= \frac{\pi}{0.04} [0.0003436 (0.004949) + 0.0000048 (0.0006413) + 0.0000080 (0.0003074)]$$

$$= 133.9971 (10^{-6}) \text{ m}^2$$

(5)

$$\overline{z_3^2(t)} = \frac{\pi}{2 \zeta_r} \left[\left(Z_3^{(1)} \right)^2 \frac{N_1^2}{\omega_1^3} S(\omega_1) + \left(Z_3^{(2)} \right)^2 \frac{N_2^2}{\omega_2^3} S(\omega_2) + \left(Z_3^{(3)} \right)^2 \frac{N_3^2}{\omega_3^3} S(\omega_3) \right]$$

$$= \frac{\pi}{0.04} [0.0005340 (0.004949) + 0.0000156 (0.0006413) + 0.0000016 (0.0003074)]$$

$$= 208.3902 (10^{-6}) \text{ m}^2$$

(6)

14.33

Let

d = inner diameter of column

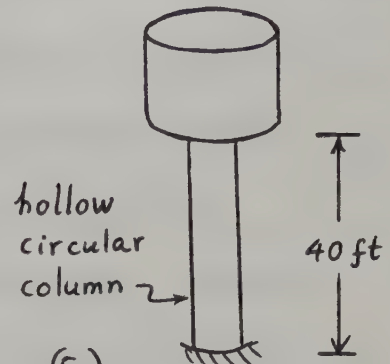
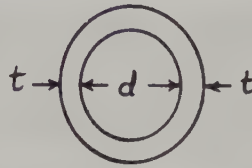
t = wall thickness

l = height = 480"

$E = 30 \times 10^6 \text{ psi}$

k = stiffness of column (cantilever)

$$= \frac{3EI}{l^3} = \frac{3(30 \times 10^6)}{(480)^3} \frac{\pi}{64} [(d+t)^4 - d^4] \text{ ---- } (E_1)$$

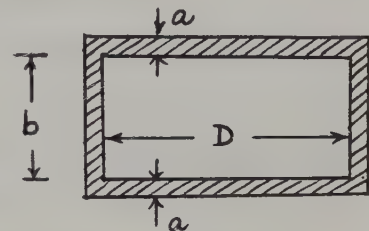


Let tank be a thin-walled cylindrical vessel with

D = mean diameter

a = thickness of shell (top & bottom)

b = axial length of shell.



$$\text{water volume} = 10\,000 \text{ gallons} = 231 \times 10^4 \text{ in}^3$$

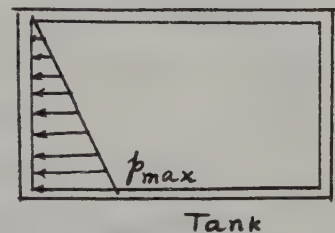
$$\text{volume of tank} = \frac{\pi D^2}{4} b = 231 \times 10^4 \text{ in}^3 \text{ ---- } (E_2)$$

$$\text{Max. pressure in tank} = p_{\max} = \gamma h$$

Where γ = weight density of

Water = 62.4 lb/in³

$$= 0.0361 \text{ lb/in}^3$$



$$\text{Max. tangential stress in the tank} = \frac{p_{\max} D}{2a} \text{ ---- } (E_3)$$

$$\text{Let permissible stress in tank} = \sigma_p = \frac{\sigma_y}{2} = 15\,000 \text{ psi ---- } (E_4)$$

(using a factor of safety of 2)

Equating (E₃) and (E₄),

$$15000 = \frac{0.0361 h D}{2a}$$

---- (E₅)

$$\text{Weight of empty steel tank} = \gamma_s \pi D a b$$

$$\text{where } \gamma_s = \text{weight density of steel} = 0.283 \text{ lb/in}^3$$

$$m_t = \text{mass of empty steel tank} = \frac{0.283}{386.4} \pi D a b$$

$$= 2.3009 \times 10^{-3} D a b \quad \text{---- (E6)}$$

$$m_w = \text{mass of water}$$

$$= (231 \times 10^4) \left(\frac{0.0361}{386.4} \right) = 215.8 \frac{\text{lb} \cdot \text{sec}^2}{\text{in}} \quad \text{---- (E7)}$$

$$\text{Natural frequency of empty tank} = \sqrt{m_t/k} > 6.2832 \quad \text{---- (E8)}$$

$$\text{Natural frequency of full tank} = \sqrt{\frac{m_t + m_w}{k}} > 6.2832 \quad \text{---- (E9)}$$

where m_t , m_w and k are given by (E6), (E7) and (E1).

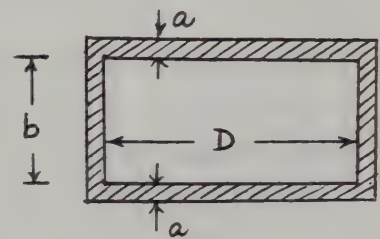
Due to ground acceleration with power spectral density $S(\omega)$, the mean square value of relative displacement of

Let tank be a thin-walled cylindrical vessel with

D = mean diameter

a = thickness of shell
(top & bottom)

b = axial length of shell.



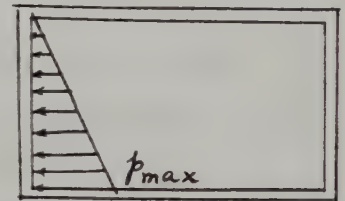
$$\text{Water volume} = 10\,000 \text{ gallons} = 231 \times 10^4 \text{ in}^3$$

$$\text{Volume of tank} = \frac{\pi D^2}{4} b = 231 \times 10^4 \text{ in}^3 \quad \text{---- (E2)}$$

$$\text{Max. pressure in tank} = p_{\max} = \gamma h$$

$$\text{where } \gamma = \text{weight density of water} = 62.4 \text{ lb/in}^3$$

$$= 0.0361 \text{ lb/in}^3$$



$$\text{Max. tangential stress in the tank} = \frac{p_{\max} D}{2a} \quad \text{---- (E3)}$$

$$\text{Let permissible stress in tank} = \sigma_p = \frac{\sigma_y}{2} = 15\,000 \text{ psi} \quad \text{---- (E4)}$$

(Using a factor of safety of 2)

Equating (E3) and (E4),

$$15\,000 = \frac{0.0361 h D}{2a} \quad \text{---- (E5)}$$

$$\text{Weight of empty steel tank} = \gamma_s \pi D a b$$

$$\text{where } \gamma_s = \text{weight density of steel} = 0.283 \text{ lb/in}^3$$

$$m_t = \text{mass of empty steel tank} = \frac{0.283}{386.4} \pi D a b$$

$$= 2.3009 \times 10^{-3} D a b \quad \text{---- (E}_6\text{)}$$

$$m_w = \text{mass of water}$$

$$= (231 \times 10^4) \left(\frac{0.0361}{386.4} \right) = 215.8 \frac{\text{lb-sec}^2}{\text{in}} \quad \text{---- (E}_7\text{)}$$

$$\text{Natural frequency of empty tank} = \sqrt{m_t/k} > 6.2832 \quad \text{---- (E}_8\text{)}$$

$$\text{Natural frequency of full tank} = \sqrt{\frac{m_t + m_w}{k}} > 6.2832 \quad \text{---- (E}_9\text{)}$$

$$\text{where } m_t, m_w \text{ and } k \text{ are given by (E}_6\text{), (E}_7\text{) and (E}_1\text{).}$$

Due to ground acceleration with power spectral density $S(\omega)$, the mean square value of relative displacement of

tank is given by Eq. (E₁₂) of Example (14.6):

$$E[z^2] = \frac{S_0 \pi m^2}{k c} \quad \text{---- (E}_{10}\text{)}$$

$$\text{where } c = 0.1 \quad c_c = 0.2 \sqrt{k m} \quad \& \quad S_0 = 0.0002 / \text{m}^2/\text{cycle/sec.}$$

When empty,

$$E[z^2] = \frac{\pi S_0 m_t^2}{k (0.2 \sqrt{k m_t})} \leq 16 \text{ in}^2 \quad \text{---- (E}_{11}\text{)}$$

and when full,

$$E[z^2] = \frac{\pi S_0 (m_t + m_w)^2}{k (0.2 \sqrt{k (m_t + m_w)})} \leq 16 \text{ in}^2 \quad \text{---- (E}_{12}\text{)}$$

$\therefore$ We need to find the values of d, t, D, a and b to satisfy the inequalities (E₈), (E₉), (E₁₁) and (E₁₂).

(1) $\frac{W}{W_0} = \frac{1}{1 + \frac{W_0}{W_0}} = \frac{1}{2}$

(2) $\frac{W}{W_0} = \frac{1}{1 + \frac{W_0}{W_0}} = \frac{1}{2}$

(3) $\frac{W}{W_0} = \frac{1}{1 + \frac{W_0}{W_0}} = \frac{1}{2}$

(4) $\frac{W}{W_0} = \frac{1}{1 + \frac{W_0}{W_0}} = \frac{1}{2}$

(5) $\frac{W}{W_0} = \frac{1}{1 + \frac{W_0}{W_0}} = \frac{1}{2}$

(6) $\frac{W}{W_0} = \frac{1}{1 + \frac{W_0}{W_0}} = \frac{1}{2}$

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(12) $\frac{W}{W_0} = \frac{1}{1 + \frac{W_0}{W_0}} = \frac{1}{2}$

(13) $\frac{W}{W_0} = \frac{1}{1 + \frac{W_0}{W_0}} = \frac{1}{2}$

(14) $\frac{W}{W_0} = \frac{1}{1 + \frac{W_0}{W_0}} = \frac{1}{2}$

(15) $\frac{W}{W_0} = \frac{1}{1 + \frac{W_0}{W_0}} = \frac{1}{2}$

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SCU0144993

ISBN 0-201-52687-5



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P9-AHY-666